# Is the Mercier Criterion Relevant to Stellarator Stability?

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**Abstract**. Local flattening of the pressure profile at the resonant surfaces may significantly change the stellarator stability properties. This flattening may be an intrinsic consequence of the three-dimensional nature of the equilibrium and may invalidate the local stability criteria often used in stellarator design.

## 1. Introduction

There are many reasons to suspect that pressure profiles with zero gradients at the rational surfaces may be the relevant profiles for stellarators. The existence of a three-dimensional (3-D) toroidal equilibrium is still an unresolved mathematical problem [1]. Of course, numerical solutions of the equilibrium magnetohydrodynamic (MHD) equations are commonly calculated; however, they may just be weak solutions of these equations [2]. The 3-D equilibrium may have magnetic structures at the rational surfaces that increase transport in those regions, therefore decreasing gradients. Other arguments can be made for such a pressure profile. From the perspective of having smooth particle fluxes in a 3-D equilibrium, Boozer [3] suggested that the pressure-gradient should be zero at the singular surfaces. In dynamical calculations of equilibria unstable to resistive interchanges, we have seen the formation of these flat spots at the resonant surfaces even for very low values of beta. In those calculations, we have observed a delicate interplay between resistive and ideal interchange modes. The first causes the local flattening of the pressure profile, which then causes a modification of the stability threshold for the ideal modes. Finally, in experiments, high-resolution electron temperature and density measurements in TJ–II [4] show the existence of multiple structures that may be related to the resonant surfaces. Therefore, it is reasonable to assume that the pressure profiles in stellarators have a complex structure with zero-gradient at each rational surface. The size of these flat spots can be very small, but even in such cases there are important consequences for stellarator stability.

## 2. Linear Stability Properties of Low-*n* Interchange Modes for Pressure Profiles with Zero Gradient at the Singular Surfaces

Interchange modes extend uniformly along the magnetic field lines. They are flutelike instabilities. Therefore, for these instabilities it is possible to average over the toroidal magnetic field modulation induced by the helical windings. Using the Greene and Johnson formalism [5] and assuming a straight helical system, the averaged equilibrium magnetic field geometry has cylindrical symmetry. In such a system, we consider magnetic configurations with bad curvature and rotational transform with shear. For the numerical calculations, we have the form of the averaged curvature in a helically symmetric system:

$$\frac{d\Omega}{dr} = \varepsilon^2 M\left(4r\iota + r^2\iota'\right) \quad . \tag{1}$$

Here, $\varepsilon$ is the inverse aspect ratio, $M$ is the number of toroidal field periods, and $\iota$ is the rotational transform. For the calculation presented in this paper, we have used $\iota = 0.32\left(1 + 2.2r^2 - 0.46r^4 + 2.5r^6\right)$. For a smooth pressure profile, such as $p_0(r) = \bar{p}_0\left(1 - r^2\right)$, the linear growth rate of an interchange mode has a dependence with beta as illustrated in Fig. 1 (continuous line). For these instabilities, the eigenfunction is sharply localized and symmetric with respect to the singular surfaces. We investigate the change in these stability properties when we consider a pressure profile, $p(r)$, which is like $p_0(r)$ but with zero gradient at the resonant surfaces. To obtain such a profile, we modify the smooth pressure profile

$$p\,(r) = p_0(r) - \sum_m \left(dp/dr\right)\Big|_{r=r_m} \left[\left(r^2 - r_m^2\right)/\left(2r_m\right)\right] \exp\left[-\frac{\left(r - r_m\right)^2}{2W_m^2}\right], \tag{2}$$

where $W_m$ is a measure of the size of the flat spot. Even for very small values of $W_m$, there is a qualitative change of the stability properties described previously. For instance, Fig. 1 shows the linear growth of the instability (broken line) after modifying the pressure profile as described by Eq. (2) with $W_m/a = 0.004$. The result is that the instability threshold has increased by more than 60% and the form of the eigenfunction has changed. For this profile and at low beta, the localized interchange instability branch is stable, and two other types of modes are now possible (Fig. 2). Their eigenfunctions are mirror-symmetric with respect to the resonant surface, and which one has the larger growth rate depends on details of the profiles and the exact location of the flattening. Some of these changes in the stability properties of cylindrical plasmas have already been discussed elsewhere [6]-[9]. In Figs. 1 and 2, the linear stability results are for the ($m$ = 6; $n$ = 3) mode with the resonant surface located at r/a ≈ 0.50. The lower-$m$ ($m$ < 3) radially symmetric interchange modes may require a larger size flat spot for stabilization for a fixed beta value [6]. Note that the modification of the pressure profile by local flat spots with a width of $W_m/a = 0.004$ is hardly noticeable, and it would require very high resolution diagnostics to detect it in an experiment. The linear stability calculations also require very high resolution. Here, we have used a radial grid of $\Delta r/a = 4 \times 10^{-5}$.

An analytical solution for the linear stability problem with modes like the ones in Fig. 2 can be found by dividing the minor radius in the three regions. One region is between the magnetic axis and the singular surface. In this region, we assume a constant rotational transform, $\iota_0$, and parabolic pressure profile. A second region of width $W$ is centered at the singular surface, and we assume that the profiles have zero pressure gradient and small magnetic shear. In the outer region, the values of the parameters do not matter because the eigenfunction is taken to be zero. Matching the solutions in these three regions, one obtains an eigenvalue condition that gives the following form for the linear growth rate:

$$\gamma^2 = m^2\left[\left(\frac{r_s - W/2}{r_0 j'_{ms}}\right)^2 D_s - \left(\frac{n}{m} - \iota_0\right)^2\right]. \tag{3}$$

Here, $D_s = 4\beta_0 M\iota(0)$, $j'_{ms}$ is the sth zero of the derivative of the Bessel function of order $m$, and W is the width of the flat spot at the rational surface. Near the critical point, the calculated growth rate is very sensitive to the value of each of the two terms in the rhs of Eq. (3). Because we had assumed that the rotational transform is constant within the inside region, it is difficult to determine the proper value to use for $\iota_0$. An evaluation of $\iota_0$ will be given elsewhere. However, in comparison with the numerical results, we found it useful to plot $\gamma^2$ versus $\left(r_s - W/2\right)^2$. The numerical results are well described by a straight line, and the value of the coefficient of $(r_s - W/2)^2$ (Fig. 3) agrees with the value calculated from the analytical result, $C = D_s (m/j'_{ms} r_0)^2$.

These changes on stability properties caused by the pressure profiles with flat spots at the resonant surfaces are not limited to cylindrical geometry. By using the averaging method for a realistic stellarator geometry and including the toroidal couplings in the model, one obtains similar results [8, 10].

## 3. Asymptotic Behavior and Beta Critical

We can see from Eq. (3) that the growth-rate dependence with *m* is quite different from the dependence of the local ideal interchange instability. For the latter, the linear growth rate is very weakly dependent on *m*. In the case of pressure profiles with local flattenings, the high-*m* modes are strongly suppressed (Fig. 4). Hence, the asymptotic stability criteria derived for *m* cannot give any information on the stability of the modes described by Eq. (3). In contrast, those criteria became undefined in the case of a zero pressure gradient in each rational surface. Therefore, for those pressure profiles, the asymptotic local stability criteria cannot be applied.

From Eq. (3) we can calculate the corresponding beta critical for these new instabilities:

$$\beta_0^c = \frac{1}{4M\iota_0}\left(\frac{r_0 j'_{ms}}{r_s}\right)^2 \left(\frac{n}{m} - \iota_0\right)^2 . \tag{4}$$

If we take the smooth profile $p_0(r)$ corresponding to the pressure profile with flat spots $p(r)$, we can calculate the beta critical given by the Suydam criterion,

$$\beta_{0S}^c = \frac{1}{16M\iota_0}\left(\frac{r_0}{r_s}\right)^2 \left(r_s \frac{d\iota}{dr}\right)^2 . \tag{5}$$

If we take $\iota_0$ to be the rotational transform at the magnetic axis, then $r_s (d\iota/dr) \approx (n/m - \iota_0)$. We can compare the real beta critical, Eq. (4), with the Suydam beta critical, $\beta_{0S}^c$, and we obtain $\beta_0^c \approx (2 j'_{m1})^2 \beta_{0S}^c$. That is, the real beta critical may be an order of magnitude higher than $\beta_{0S}^c$.

Therefore, local stability criteria such as the Suydam criterion are not relevant for such pressure profiles, although it can be used as a measure of stability once it is conveniently renormalized.

## 4. Conclusions

For stellarator equilibrium with zero-pressure gradient at the rational surfaces, local asymptotic stability criteria cannot be applied. For stellarators, the Mercier criterion has the same problems as the Suydam criterion in cylindrical geometry. It is a local stability criterion that cannot be applied to such pressure profiles. Calculations using the averaged method approach indicate that the stability properties for the low-*m* modes [8, 10] are similar to the case of cylindrical geometry. The local interchange-like modes are stabilized, and the more global eigenfunctions appear as the residual instabilities with the increase of beta. The beta critical also increases more than the one obtained for smooth pressure profiles. These results may explain the apparent violation of this criterion when smooth pressure profiles are used in the calculation of the stability for interpretation of the experimental measurements [11].

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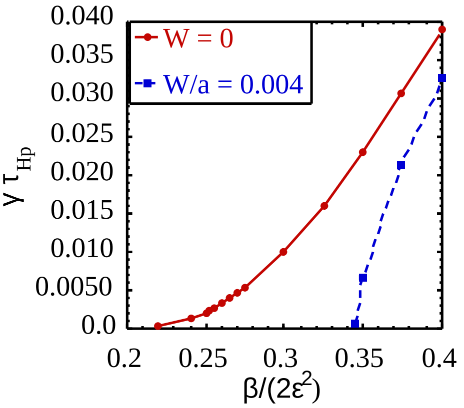


*FIG. 1. Linear growth rate of the m = 6 mode for a parabolic pressure profile.*

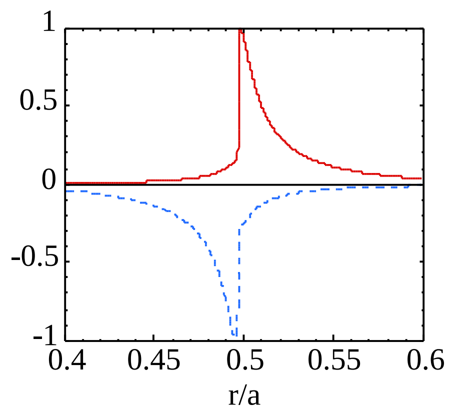


*FIG. 2. Eigenfunctions for the m = 6 mode for the most likely unstable modes once the localized interchange mode has been stabilized.*

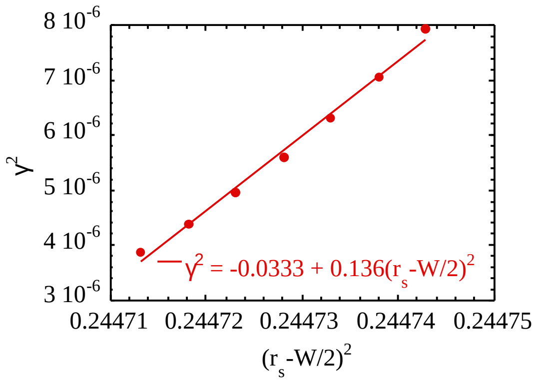


*FIG. 3. Square of the linear growth rate of the m = 6 mode versus* $(r_s - W/2)^2$ *for* $\beta/2\varepsilon^2 = 0.345$. *The slope of the straight-line fit agrees with the prediction of the analytical model.*

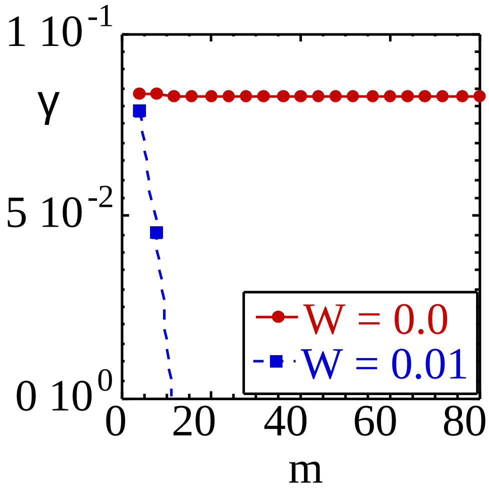


*FIG. 4. Linear growth rate as a function of m at the 4/3 rational surface for a smooth profile (W = 0) and with a flat spot that is 1% of the size of the minor radius (W = 0.01).*