

# Nonlinear Zonal Dynamics of Drift and Drift-Alfvén Turbulences in Tokamak Plasmas

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**Abstract.** The present work addresses the issue of identifying the major nonlinear physics processes which may regulate drift and drift-Alfvén turbulence using a weak turbulence approach. Within this framework, based upon the nonlinear gyrokinetic equation for both electrons and ions, we present an analytic theory for nonlinear zonal dynamics described in terms of two axisymmetric potentials,  $\delta\phi_z$  and  $\delta A_{\parallel z}$ , which spatially depend only on a (magnetic) flux coordinate. Spontaneous excitation of zonal flows by electrostatic drift microinstabilities is demonstrated both analytically and by direct 3D gyrokinetic simulations. Direct comparisons indicate good agreement between analytic expressions of the zonal flow growth rate and numerical simulation results for Ion Temperature Gradient (ITG) driven modes. Analogously, we show that zonal flows may be spontaneously excited by drift-Alfvén turbulence, in the form of modulational instability of the radial envelope of the mode as well, whereas, in general, excitations of zonal currents are possible but they have little feedback on the turbulence itself.

## 1. Introduction

In recent years, there has been increasing attention devoted to exploring nonlinear dynamics of zonal flow [1] associated with electrostatic drift-type turbulence [2, 3, 4]. On the other hand, despite it being well known how electrostatic drift modes couple to the electromagnetic shear Alfvén wave as the plasma  $\beta$  (or  $R_0\beta'$ ) increases [5, 6, 7], little effort has been devoted so far to investigating nonlinear zonal dynamics of drift-Alfvén turbulence.

The present work addresses the issue of identifying the major nonlinear physics processes which may regulate drift and drift-Alfvén turbulence using a weak turbulence approach. Within this framework, based upon the nonlinear gyrokinetic equation [8] for both electrons and ions, we present an analytic theory for nonlinear zonal dynamics described in terms of two axisymmetric potentials,  $\delta\phi_z$  and  $\delta A_{\parallel z}$ , which spatially depend only on a (magnetic) flux coordinate. Physically,  $\delta\phi_z$  is associated with zonal flow formation, while  $\delta A_{\parallel z}$  corresponds to zonal currents  $\delta j_{\parallel z} = -(c/4\pi)\nabla_{\perp}^2\delta A_{\parallel z}$ . The introduction of a *zonal vector potential*,  $\delta A_{\parallel z}$ , is one of the characteristic differences of the electromagnetic with respect to the electrostatic case.

Zonal potentials are characterized by time variations on typical scales which are long compared to the characteristic ones of the drift-Alfvén instabilities. This specific ordering of time scales, which formally requires proximity to the marginal stability such that the linear growth rate is smaller than the mode frequency, will be exploited for explicitly manipulating formal expressions in the theoretical analysis. In contrast to other approaches, however, which also assume slow radial variations of the zonal fields ( $k_z^{-1}$ ) with respect to the typical spatial scale of the background turbulence ( $k_r^{-1}$ ), we generally take  $k_z \approx k_r$ , although we still assume  $|\partial_r k_z/k_z^2| \ll 1$  for consistency of our eikonal approach. In this respect our work is the generalization of Ref. [9], which demonstrated that zonal flows can be *spontaneously excited* by electrostatic drift turbulence and that these are characterized by  $k_z \approx k_r$  (FIG. 1). In the present

work, we show that zonal flows in toroidal equilibria can be *spontaneously excited* via *modulations* of the radial structure (*envelope*) of a single- $n$  coherent drift-wave, with  $n$  the toroidal mode number. In this framework, the turbulent state and the nonlinear couplings among different  $n$ 's will manifest only via zonal dynamics. Similarly to Ref. [9], the present theory is strictly applicable to toroidal plasma equilibria, where poloidal asymmetry forces each mode to be (at least in the linear limit) the superposition of many poloidal harmonics  $m$ , characterized by the same  $n$ . In this respect, the present theoretical analysis is a systematic treatment of the radial mode structure (envelope) of zonal fields and drift turbulence in the general electromagnetic case, including slow time evolutions and accounting for linear (toroidal) and nonlinear mode couplings on the same footing. More specifically, we demonstrate that zonal flows ( $\delta\phi_z$ ) are due to *charge separation effects* associated with both finite ion Larmor radius and finite ion orbit width effects (magnetic curvature), whereas zonal currents ( $\delta A_{\parallel z}$ ) are due to parallel electron pressure imbalance (cf. also Ref. [10]).

Spontaneous excitation of zonal flows by electrostatic drift microinstabilities is demonstrated both analytically and by direct 3D gyrokinetic simulations [9]. Direct comparisons indicate good agreement between analytic expressions of the zonal flow growth rate and numerical simulation results for ITG modes. Analogously, we show that zonal flows may be *spontaneously excited* by drift-Alfvén turbulence, in the form of *modulational instability* of the radial envelope of the mode as well. From the analytic expression for the growth rate of the spontaneously excited zonal flows ( $\delta\phi_z$ ) we show how no flow generation is expected for a *pure shear Alfvén wave*, due to the peculiar nature of the *Alfvénic state*. Meanwhile, we also demonstrate that in general zonal currents are also excited but they have negligible effect on the turbulence itself. The general results obtained within this theoretical model are also applied to Alfvénic oscillations; such as the Kinetic Alfvén Waves and the more recently discussed Alfvén ITG (AITG) [6] mode.

## 2. Theoretical Model

Here, we strictly follow Ref. [9] and assume a low- $\beta$  ( $\beta = 8\pi/B^2$ ) toroidal equilibrium with major radius  $R_0$  and minor radius  $a$ , with typically  $R_0/a = 1/\epsilon \gg 1$ . For simplicity, we also take the case of shifted circular magnetic flux surfaces. In this case, we can describe drift wave dynamics in terms of two scalar fields: the scalar potential  $\delta\phi$  and the parallel vector potential  $\delta A_{\parallel}$  fluctuations. For both fluctuating fields, as stated in the Introduction, we describe the nonlinear dynamic evolution in terms of a four-mode coupling scheme, *i.e.*, each electromagnetic fluctuation is taken to be coherent and composed of a single  $n \neq 0$  drift wave ( $\delta\phi_d, \delta A_{\parallel d}$ ) and a zonal perturbation ( $\delta\phi_z, \delta A_{\parallel z}$ ); *e.g.*, for scalar potential fluctuations we take

$$\begin{aligned}
\delta\phi_d &= \delta\phi_0 + \delta\phi_+ + \delta\phi_- \\
\delta\phi_0 &= e^{i \int n\theta_k dq + in\varphi} \sum_m e^{-im\vartheta} \phi_0(nq - m) + c.c. \ , \\
\delta\phi_{\pm} &= \begin{pmatrix} e^{i \int n\theta_k dq} \\ e^{-i \int n\theta_k^* dq} \end{pmatrix} e^{\pm in\varphi + i \int k_z dr} \sum_m e^{\mp im\vartheta} \phi_{\pm}(nq - m) + c.c. \ , \\
\delta\phi_z &= e^{i \int k_z dr} \phi_z + c.c. \ ,
\end{aligned} \tag{1}$$

where  $(r, \varphi, \vartheta)$  are toroidal coordinates, and an analogue decomposition is assumed for fluctuating parallel vector potentials. Here,  $\theta_k$  is the eikonal describing the radial structure of the drift wave radial envelope and  $q$  is the safety factor. Thus, Eq. (1) suggests that zonal fields may be

actually considered as radial modulations of the drift wave envelope, while the  $(\pm)$  modes are simply upper and lower sidebands due to zonal fields modulations of the drift wave [9]. Furthermore, we have adopted the convention that, in the expressions involving the  $\pm$  sidebands, the first row in a two component array will refer to the  $+$  while the second row will refer to the  $-$  sideband. The same notation will be used throughout.

We first derive nonlinear equations for zonal fields from the quasineutrality condition and parallel Ampère's law. Here, we just report the final results of such derivations in the small ion Larmor radius ( $\rho_{Li}$ ) limit: details will be given elsewhere. Contrary to the electrostatic limit, where the electron response to an  $n \neq 0$  perturbation is adiabatic and, thus, only ions contribute to the nonlinear dynamics, electron nonlinearities are important in the general electromagnetic case. Assuming  $k_{\perp}^2 \rho_{Li}^2 \ll 1$ , the nonlinear coupling coefficients are formally of the Hasegawa-Mima type and the quasineutrality condition reads:

$$\begin{aligned} \partial_t \chi_{iz} \delta \phi_z &= \frac{c}{B} k_{\theta} k_z k_z^2 \rho_{Li}^2 \left[ \left( \alpha_0 - \left| \frac{k_{\parallel} v_A}{\omega_0} \right|^2 \right) \langle \langle |\Psi_0|^2 \rangle \rangle + 2\alpha_0 \text{Re} \langle \langle (\Phi_0 - \Psi_0)^* \Psi_0 \rangle \rangle \right. \\ &\quad \left. + \alpha_0 \langle \langle |\Phi_0 - \Psi_0|^2 \rangle \rangle \right] (A_0^* A_+ - A_0 A_-) . \end{aligned} \quad (2)$$

Here, we have introduced the notations  $\chi_{iz} \simeq 1.6q^2 \epsilon^{-1/2} k_z^2 \rho_{Li}^2$  [11],  $\alpha_0 \equiv 1 + \delta P_{Li0}/(ne\delta\phi_0)$ ,  $\mathbf{b} \cdot \nabla \delta\psi \equiv -(1/c)\partial_t \delta A_{\parallel}$ ,  $\Phi_0$  indicates the symmetric Fourier Transform into ballooning space of  $\phi_0$  in Eq. (1) and similar notations are used for the other scalar fields. Furthermore, we have omitted for simplicity collisional damping of  $\delta\phi_z$  [12],  $\langle \langle \dots \rangle \rangle$  stands for integration over ballooning space,  $|k_{\parallel}|^2 \langle \langle |\Psi_0|^2 \rangle \rangle \equiv \langle \langle |\partial_{\theta} \Psi_0 / qR_0|^2 \rangle \rangle$ ,  $\theta$  is the ‘‘angle-like’’ coordinate in ballooning space, and  $A_0$  and  $A_{\pm}$  indicate the amplitude of radial envelopes of the drift wave and sidebands at the current radial position. Similarly, from the parallel Ampère's law we obtain

$$\begin{aligned} \partial_t \delta A_{\parallel z} &= \frac{c}{B} k_{\theta} k_z k_z^2 \delta_e^2 \left\langle \left\langle \left( 1 - \alpha_0 \left| \frac{\omega_0}{k_{\parallel} v_A} \right|^2 \right) \frac{\omega_0}{k_{\parallel} c} \left[ |\Psi_0|^2 \right. \right. \right. \\ &\quad \left. \left. \left. + 2\text{Re} \left( \frac{\omega_0}{k_{\parallel} c} (\Phi_0 - \Psi_0)^* \Psi_0 \right) \right] \right\rangle \right\rangle (A_0^* A_+ - A_0 A_-) . \end{aligned} \quad (3)$$

Here, the presence of  $\delta_e^2 = c^2/\omega_{pe}^2$  is a consequence of the strong shielding effect of parallel electron current on the electron collisionless skin depth. Furthermore, the *forced* response in  $\delta A_{\parallel z}$  has been neglected since it is of order  $(\omega_z/\omega_0)(k_z/k_{\perp})/(1 - \omega_0^2 \alpha_0/k_{\parallel}^2 v_A^2)$  with respect to the spontaneously excited component of Eq. (3). A direct comparison of Eqs. (2) and (3) indicates that both zonal fields may be spontaneously excited except for a *pure shear Alfvén wave*, for which  $\omega_0^2 = k_{\parallel}^2 v_A^2$ ,  $\alpha_0 = 1$  and  $\Phi_0 = \Psi_0$ . In general, however, zonal flows can be efficiently excited via  $\delta\phi_z$ , whereas zonal currents (or poloidal magnetic fields) are strongly reduced because of electron shielding on scale lengths larger than  $\delta_e$ . We also note that, typically,

$$\frac{\omega_0}{k_{\parallel} c} \delta A_{\parallel z} \approx \frac{\omega_0^2}{k_{\parallel}^2 v_A^2} k_z^2 \delta_e^2 \delta \phi_z \ll \delta \phi_z ,$$

which will make it possible to neglect the effect of  $\delta A_{\parallel z}$  below.

The drift wave nonlinear equations are the quasineutrality condition

$$\frac{ne^2}{T_i} \left( 1 + \frac{T_i}{T_e} \right) \delta \phi_k = \langle e J_0(\gamma) \overline{\delta H_i} \rangle_k - \langle e \overline{\delta H_e} \rangle_k , \quad (4)$$

and the vorticity equation

$$\begin{aligned}
& B\partial_\ell \left( k_\perp^2 \frac{\partial_\ell \delta\psi_k}{B} \right) + \frac{\omega^2 k_\perp^2}{v_A^2 b_i} \left[ \left( 1 - \frac{\omega_{*ni}}{\omega} \right) (1 - \Gamma_0(b_i)) - \frac{\omega_{*Ti}}{\omega} b_i (\Gamma_0(b_i) - \Gamma_1(b_i)) \right] \delta\phi_k \\
&= \frac{4\pi}{c^2} \sum_{e,i} \langle e\omega\omega_d J_0 \overline{\delta H} \rangle_k + \frac{\mathbf{b} \cdot (\mathbf{k}'_\perp \times \mathbf{k}''_\perp)}{cB} \partial_t \left( \delta A_{\parallel, k'} \nabla_\perp^2 \delta A_{\parallel, k''} \right)_k \\
&+ \frac{4\pi}{c^2} \partial_t \left\langle e \frac{c}{B} \mathbf{b} \cdot (\mathbf{k}''_\perp \times \mathbf{k}'_\perp) (J_0(\gamma) J_0(\gamma') - J_0(\gamma'')) \delta L_{k'} \overline{\delta H}_{ik''} \right\rangle_k . \tag{5}
\end{aligned}$$

In Eqs. (4) and (5), the subscript  $k$  stands for 0 or  $\pm$  depending on whether the drift wave or its sidebands are considered, simple angular brackets  $\langle \dots \rangle$  denote velocity space integration,  $\gamma \equiv k_\perp v_\perp / \omega_{ci}$ ,  $J_0$  is the Bessel function of zero order,  $\partial_\ell \equiv \mathbf{b} \cdot \nabla$ ,  $b_i = k_\perp^2 \rho_{Li}^2$ ,  $\Gamma_{0,1}(b_i) \equiv I_{0,1}(b_i) \exp(-b_i)$ ,  $\omega_{*ni}$  and  $\omega_{*Ti}$  are the ion diamagnetic frequencies associated with - respectively - density and temperature gradients,  $\omega_d$  is the magnetic drift frequency,  $\mathbf{k} = \mathbf{k}' + \mathbf{k}''$ ,  $\delta L_k \equiv \delta\phi_k - (v_\parallel/c) \delta A_{\parallel k}$  and the fluctuating particle distribution functions have been decomposed in adiabatic and nonadiabatic responses as

$$\delta F = \frac{e}{m} \delta\phi \frac{\partial}{\partial v^2/2} F_0 + \sum_{\mathbf{k}_\perp} \exp(-i\mathbf{k}_\perp \cdot \mathbf{v} \times \mathbf{b}/\omega_c) \overline{\delta H}_k . \tag{6}$$

The nonadiabatic response of the particle distribution function,  $\overline{\delta H}$ , is obtained from the nonlinear gyrokinetic equation [8]:

$$\begin{aligned}
& \left( \partial_t + v_\parallel \partial_\ell + i\omega_d \right)_k \overline{\delta H}_k = i \frac{e}{m} Q F_0 J_0(\gamma) \delta L_k - \frac{c}{B} \mathbf{b} \cdot (\mathbf{k}''_\perp \times \mathbf{k}'_\perp) J_0(\gamma') \delta L_{k'} \overline{\delta H}_{k''} , \\
& Q F_0 = \omega_k \frac{\partial F_0}{\partial v^2/2} + \mathbf{k} \cdot \frac{\hat{\mathbf{b}} \times \nabla}{\omega_c} F_0 . \tag{7}
\end{aligned}$$

In Eq. (7), the linear response  $\propto Q F_0$  and the ‘‘generalized’’  $\mathbf{E} \times \mathbf{B}$  nonlinearity (in the guiding center moving frame  $\delta\phi \rightarrow \delta\phi - (v_\parallel/c) \delta A_{\parallel}$ ) are readily recognized.

Equations (4) and (5) are further simplified when we decompose the linear particle response to the fluctuating fields as [13]:

$$\overline{\delta H}^{LIN} = -\frac{e}{m} J_0(\gamma) \frac{Q F_0}{\omega} \delta\psi + \delta K , \tag{8}$$

where the *linearized* gyrokinetic equation for  $\delta K$  is readily derived from Eq. (7) and may be found in Ref. [13]. It is then readily shown that the quasineutrality condition, Eq. (4), can be cast into the form

$$\begin{aligned}
& \frac{ne^2}{T_i} \left\{ \left( 1 + \frac{T_i}{T_e} \right) (\delta\phi - \delta\psi)_k + \left[ \left( 1 - \frac{\omega_{*ni}}{\omega} \right) (1 - \Gamma_0(b_i)) - \frac{\omega_{*Ti}}{\omega} b_i (\Gamma_0(b_i) - \Gamma_1(b_i)) \right] \right\} \delta\psi_k \\
& - \sum_{e,i} \langle e J_0(\gamma) \delta K \rangle_k = -\frac{i}{\omega_k} \left\langle e \frac{c}{B} \mathbf{b} \cdot (\mathbf{k}''_\perp \times \mathbf{k}'_\perp) (J_0(\gamma) J_0(\gamma') - J_0(\gamma'')) \delta L_{k'} \overline{\delta H}_{ik''} \right\rangle_k \\
& - \frac{i}{\omega_k} \left\langle e \frac{c}{B} \mathbf{b} \cdot (\mathbf{k}''_\perp \times \mathbf{k}'_\perp) \delta\phi_{k'} \overline{\delta H}_{ek''} \right\rangle_k - \langle e \overline{\delta H}_e^{NL} \rangle_k , \tag{9}
\end{aligned}$$

where  $\overline{\delta H}_e^{NL}$  indicates the nonlinear nonadiabatic electron response only, which vanishes in the electrostatic limit, as stated above.

Assuming, now,  $k_{\perp}^2 \rho_{Li}^2 \ll 1$ , consistently with Eqs. (2) and (3), and introducing the notation

$$\delta K = \widehat{\delta K}_{\phi}(\delta\phi - \delta\psi) + \widehat{\delta K}_{\psi}\delta\psi, \quad (10)$$

Eqs. (5) and (9) for the sidebands in the ballooning space can be rewritten as:

$$\begin{aligned} & \left(1 + \frac{T_i}{T_e} - \sum_{e,i} \langle eJ_0(\gamma) \widehat{\delta K}_{\phi} \rangle_{\pm}\right) A_{\pm} \begin{pmatrix} \Phi_0 - \Psi_0 \\ \Phi_0^* - \Psi_0^* \end{pmatrix} \\ & + \left[ \left(1 - \frac{\omega_{*pi}}{\omega}\right) b_{i\pm} - \sum_{e,i} \langle eJ_0(\gamma) \widehat{\delta K}_{\psi} \rangle_{\pm} \right] A_{\pm} \begin{pmatrix} \Psi_0 \\ \Psi_0^* \end{pmatrix} \\ & = -\frac{i}{\omega_0} \frac{c}{B} \frac{T_i}{T_e} k_{\vartheta} k_z \delta\phi_z \left[ \left(1 + \frac{\omega_{*ni} T_e}{\omega_0 T_i}\right) \begin{pmatrix} A_0 \Psi_0 \\ A_0^* \Psi_0^* \end{pmatrix} - \begin{pmatrix} A_0(\Phi_0 - \Psi_0) \\ A_0^*(\Phi_0^* - \Psi_0^*) \end{pmatrix} \right], \quad (11) \end{aligned}$$

$$\begin{aligned} & \left\{ \partial_{\theta} \left( \frac{k_{\perp}^2}{k_{\vartheta}^2} \partial_{\theta} \right) + \frac{\omega^2}{\omega_A^2} \frac{k_{\perp}^2}{k_{\vartheta}^2} \left[ \left(1 - \frac{\omega_{*pi}}{\omega}\right) - \frac{3}{4} b_i \left(1 - \frac{\omega_{*pi}}{\omega} - \frac{\omega_{*Ti}}{\omega}\right) \right] \right. \\ & \left. - \frac{4\pi q^2 R_0^2}{k_{\vartheta}^2 c^2} \sum_{e,i} \langle e\omega\omega_d J_0 \widehat{\delta K}_{\psi} \rangle \right\}_{\pm} A_{\pm} \begin{pmatrix} \Psi_0 \\ \Psi_0^* \end{pmatrix} + \left\{ \frac{\omega^2}{\omega_A^2} \frac{k_{\perp}^2}{k_{\vartheta}^2} \left[ \left(1 - \frac{\omega_{*pi}}{\omega}\right) \right. \right. \\ & \left. \left. - \frac{3}{4} b_i \left(1 - \frac{\omega_{*pi}}{\omega} - \frac{\omega_{*Ti}}{\omega}\right) \right] - \frac{4\pi q^2 R_0^2}{k_{\vartheta}^2 c^2} \sum_{e,i} \langle e\omega\omega_d J_0 \widehat{\delta K}_{\phi} \rangle \right\}_{\pm} A_{\pm} \begin{pmatrix} \Phi_0 - \Psi_0 \\ \Phi_0^* - \Psi_0^* \end{pmatrix} \\ & = \frac{4\pi i \omega_0}{k_{\vartheta}^2 c^2} \frac{c}{B} \frac{n e^2}{T_i} q^2 R_0^2 k_{\vartheta} k_z \delta\phi_z b_i \begin{pmatrix} A_0 \Phi_0 \\ A_0^* \Phi_0^* \end{pmatrix}. \quad (12) \end{aligned}$$

Equations (2), (11) and (12), together with Eq. (7) are the basis for our analytic investigations described in the next section.

### 3. Some Applications

In the electrostatic limit [9],  $\Psi_0 \rightarrow 0$ , we obtain from Eq. (11)

$$D_{S\pm} A_{\pm} = \frac{i}{\omega_0} \frac{c}{B} \frac{T_i}{T_e} k_{\vartheta} k_z \delta\phi_z \langle |\Phi_0|^2 \rangle \begin{pmatrix} A_0 \\ A_0^* \end{pmatrix}, \quad (13)$$

where

$$D_{S\pm} = \left\langle \left\langle \left(1 + \frac{T_i}{T_e} - \sum_{e,i} \langle eJ_0(\gamma) \widehat{\delta K}_{\phi} \rangle_{\pm}\right) \begin{pmatrix} \Phi_0^2 \\ \Phi_0^{*2} \end{pmatrix} \right\rangle \right\rangle^{-1} \quad (14)$$

and [9]  $D_{S\pm} \simeq i(\partial D_{S0r}/\partial\omega_0)(-i\Delta \pm \Gamma_z \pm \gamma_d)$ ,  $\Delta = (k_z^2/2)(\partial^2 D_{S0r}/\partial k_r^2)/(\partial D_{S0r}/\partial\omega_0)$  is the frequency mismatch,  $k_r = nq'\theta_k$ ,  $\Gamma_z = -i\omega_z$  and  $\gamma_d$  is the sideband damping [9]. Substituting Eq. (13) into Eq. (2), we readily obtain a nonlinear dispersion relation for  $\Gamma_z$ , which, in the  $|\Delta| \ll \gamma_d, \gamma_M$  limit, reads

$$\Gamma_z = -\gamma_d/2 + \left(\gamma_M^2 + \gamma_d^2/4\right)^{1/2}, \quad (15)$$

where  $\gamma_M^2 = (2\alpha_0 \epsilon^{1/2}/1.6q^2)(T_i/T_e)(\omega_0 \partial D_{S0r}/\partial\omega_0)^{-1} k_z^2 \rho_{Li}^2 k_{\vartheta}^2 v_{thi}^2 \langle |eA_0 \Phi_0/T_i|^2 \rangle$ . Including finite zonal flow collisional damping into Eq. (2),  $\nu_z \simeq (1.5\epsilon\tau_{ii})^{-1}$  [12], would have produced

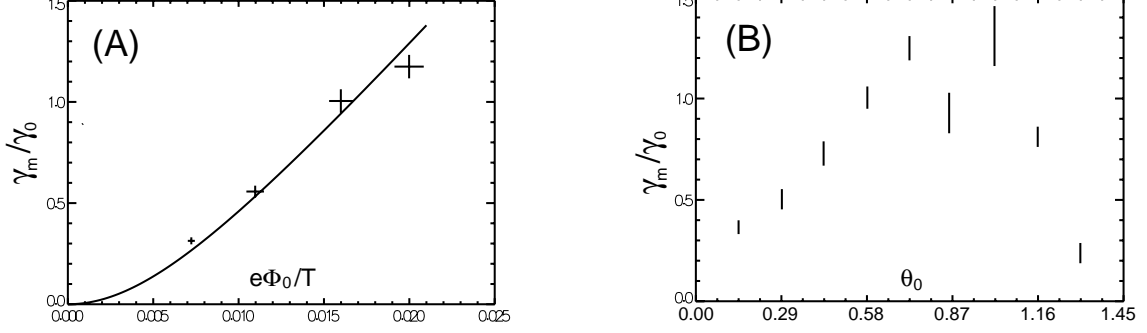


FIG. 1. : From Ref. [9]. (A) Analytic prediction of zonal flow growth rate (normalized to the linear one  $\gamma_0$ ) vs. mode amplitude (solid line) compared with gyrokinetic simulation results. (B) Zonal flow growth rate vs.  $\theta_0 = k_z/nq'$ , for fixed mode amplitude.

a threshold condition  $\gamma_M^2 \geq \nu_z \gamma_d$  on the modulational instability growth rate  $\Gamma_z$  of Eq. (15). In FIG. 1,  $\Gamma_z$  as obtained from Eq. (15) is shown to be in good agreement with the results obtained by direct 3D gyrokinetic simulations [9] of ITG modes, in which  $\gamma_d \simeq 1.5\gamma_0$ ,  $\gamma_0$  being the linear growth rate of the mode. Nonlinear equations for mode amplitudes have been recently derived [9] and they demonstrate saturation of the linearly unstable modes via coupling to the stable envelope sidebands and oscillatory behaviors in the drift-wave intensity and zonal flows [9].

For electromagnetic modes, and more specifically for Alfvénic-type waves, we typically have  $|\Phi_0 - \Psi_0| \ll |\Psi_0|$  in Eqs. (11) and (12). In fact, assuming  $k_{\parallel}^2 q^2 R_0^2 \ll 1$  [6], we have from Eq. (11)

$$\begin{pmatrix} \Phi_0 - \Psi_0 \\ \Phi_0^* - \Psi_0^* \end{pmatrix} A_{\pm} \simeq - \left( \frac{(k_{\parallel}^2 v_A^2 / \omega^2) b_i}{T_i / T_e + \omega_{*ni} / \omega} \right)_{\pm} \begin{pmatrix} \Psi_0 \\ \Psi_0^* \end{pmatrix} A_{\pm} - i \frac{c}{B} \frac{k_{\vartheta} k_z}{\omega_0} \delta \phi_z \begin{pmatrix} A_0 \Psi_0 \\ A_0^* \Psi_0^* \end{pmatrix}, \quad (16)$$

where we recall that  $k_{\parallel}^2$ , in the present treatment, stands for an operator in the ballooning space. Substituting back into Eq. (12), this yields [6]

$$\mathcal{L}_{M\pm} \begin{pmatrix} \Psi_0 \\ \Psi_0^* \end{pmatrix} A_{\pm} = i \frac{\omega_0}{\omega_A^2} \frac{c}{B} k_{\vartheta} k_z \frac{k_{\perp\pm}^2}{k_{\vartheta}^2} \delta \phi_z \left( 1 + \frac{k_{\parallel}^2 v_A^2}{\omega^2} \right)_{\pm} \begin{pmatrix} A_0 \Psi_0 \\ A_0^* \Psi_0^* \end{pmatrix}, \quad (17)$$

$$\mathcal{L}_{M\pm} = \left\{ \partial_{\theta} \left( \frac{k_{\perp}^2}{k_{\vartheta}^2} \partial_{\theta} \right) + \frac{\omega^2}{\omega_A^2} \frac{k_{\perp}^2}{k_{\vartheta}^2} \left[ \left( 1 - \frac{\omega_{*pi}}{\omega} \right) \left( 1 - \frac{(k_{\parallel}^2 v_A^2 / \omega^2) b_i}{T_i / T_e + \omega_{*ni} / \omega} \right) - \frac{3}{4} b_i \left( 1 - \frac{\omega_{*pi}}{\omega} - \frac{\omega_{*Ti}}{\omega} \right) \right] - \frac{4\pi q^2 R_0^2}{k_{\vartheta}^2 c^2} \left( \sum_{e,i} \langle e\omega\omega_d J_0 \delta \widehat{K}_{\psi} \rangle - \frac{(k_{\parallel}^2 v_A^2 / \omega^2) b_i}{T_i / T_e + \omega_{*ni} / \omega} \sum_{e,i} \langle e\omega\omega_d J_0 \delta \widehat{K}_{\phi} \rangle \right) \right\}_{\pm}. \quad (18)$$

Equation (17) can be cast into the form

$$\begin{aligned} D_{M\pm} A_{\pm} &= i \frac{\omega_0}{\omega_A^2} \frac{c}{B} k_{\vartheta} k_z \delta \phi_z \left( 1 + \frac{K_{\parallel}^2 v_A^2}{\omega^2} \right)_{\pm} \begin{pmatrix} A_0 \\ A_0^* \end{pmatrix}, \\ D_{M\pm} &\equiv \left\langle \left\langle \begin{pmatrix} \Psi_0 \\ \Psi_0^* \end{pmatrix} \mathcal{L}_{M\pm} \begin{pmatrix} \Psi_0 \\ \Psi_0^* \end{pmatrix} \right\rangle \right\rangle \left\langle \left\langle \frac{k_{\perp\pm}^2}{k_{\vartheta}^2} \begin{pmatrix} \Psi_0^2 \\ \Psi_0^{2*} \end{pmatrix} \right\rangle \right\rangle^{-1}, \\ K_{\parallel\pm}^2 &\equiv \left\langle \left\langle \begin{pmatrix} \Psi_0 \\ \Psi_0^* \end{pmatrix} \frac{k_{\perp\pm}^2}{k_{\vartheta}^2} k_{\parallel\pm}^2 \begin{pmatrix} \Psi_0 \\ \Psi_0^* \end{pmatrix} \right\rangle \right\rangle \left\langle \left\langle \frac{k_{\perp\pm}^2}{k_{\vartheta}^2} \begin{pmatrix} \Psi_0^2 \\ \Psi_0^{2*} \end{pmatrix} \right\rangle \right\rangle^{-1}. \end{aligned} \quad (19)$$

From Eq. (19) and Eq. (2), it is possible to derive the nonlinear dispersion relation for  $\Gamma_z$ , similar to the electrostatic case. Specifically, using  $D_{M-} = D_{M+}^*$ , we obtain:

$$\begin{aligned} \Gamma_z &= 2k_\theta^2 \rho_{Li}^2 \frac{k_z^2 v_{thi}^2 \omega_0^2}{\omega_0 \omega_A^2} \frac{\epsilon^{1/2}}{1.6q^2} \left\langle \left\langle \left| \frac{eA_0 \Psi_0}{T_i} \right|^2 \right\rangle \right\rangle \frac{\text{Im} \left[ D_{M+} \left( 1 + K_{\parallel}^2 v_A^2 / \omega^2 \right) \right]}{|D_{M+}|^2} \\ &\quad \times \left[ \left( \alpha_0 - \left| \frac{K_{\parallel}^2 v_A^2}{\omega_0^2} \right| \right) - 2\alpha_0 \text{Re} \left( \frac{(K_{\parallel}^2 v_A^2 / \omega^2) K_{\perp}^2 \rho_{Li}^2}{T_i / T_e + \omega_{*ni} / \omega} \right) \right]_{\pm}, \\ K_{\parallel+}^2 K_{\perp+}^2 &\equiv \left\langle \left\langle \Psi_0^* k_{\perp+}^2 k_{\parallel+}^2 \Psi_0 \right\rangle \right\rangle \left\langle \left\langle |\Psi_0|^2 \right\rangle \right\rangle^{-1}. \end{aligned} \quad (20)$$

It is straightforward to further specialize Eq. (20) to the case of KAW, for which  $D_M = -q^2 R_0^2 K_{\parallel}^2 + (\omega^2 / \omega_A^2) (1 - K_{\perp}^2 \rho_{Li}^2 (3/4 + T_e / T_i))$ . In this case  $\alpha_0 = 1$ , and defining

$$\begin{aligned} \hat{\gamma}_M^2 &= 2k_\theta^2 \rho_{Li}^2 k_z^2 v_{thi}^2 \frac{\epsilon^{1/2}}{1.6q^2} \left( \frac{3}{4} - \frac{T_e}{T_i} \right) K_{\perp}^2 \rho_{Li}^2 \left\langle \left\langle \left| \frac{e\Psi_0}{T_i} \right|^2 \right\rangle \right\rangle, \\ \hat{\Delta} &= \left( \frac{3}{4} + \frac{T_e}{T_i} \right) k_z^2 \rho_{Li}^2 \omega_0, \end{aligned} \quad (21)$$

we obtain

$$\Gamma_{z,KAW} \simeq \hat{\gamma}_M \sqrt{1 - \hat{\Delta}^2 / \hat{\gamma}_M^2}. \quad (22)$$

From Eqs. (21) and (22) we see that zonal flows can be spontaneously excited by KAW's and that, as in the electrostatic case, the growth rate  $\Gamma_z$  above threshold scales linearly with the wave amplitude. However, the most important feature of KAW's is that they spontaneously generate zonal flows in their propagating region for  $T_e < (3/4)T_i$  and in their cut-off region for  $T_e > (3/4)T_i$ .

Another application of Eq. (20) is to AITG modes [6]. In this case, sufficiently close to the *unstable* Alfvén continuum accumulation point,  $D_M = \Lambda^2 + i\Lambda \delta W_f$ , where  $\delta W_f$  is the MHD potential energy associated with the mode and  $\Lambda^2$  is a generalized inertia given by

$$\Lambda^2 = \frac{\omega^2}{\omega_A^2} \left( 1 - \frac{\omega_{*pi}}{\omega} \right) + q^2 \frac{\omega \omega_{ti}}{\omega_A^2} \left[ \left( 1 - \frac{\omega_{*ni}}{\omega} \right) F(\omega / \omega_{ti}) - \frac{\omega_{*Ti}}{\omega} G(\omega / \omega_{ti}) - \frac{N^2(\omega / \omega_{ti})}{D(\omega / \omega_{ti})} \right], \quad (23)$$

and the functions,  $F(x)$ ,  $G(x)$ ,  $N(x)$  and  $D(x)$  with  $x = \omega / \omega_{ti}$ ,  $\omega_{ti} = \sqrt{2} v_{thi} / (qR_0)$ , and using the plasma dispersion function  $Z(x)$ , are defined as [6, 14]

$$\begin{aligned} F(x) &= x \left( x^2 + 3/2 \right) + \left( x^4 + x^2 + 1/2 \right) Z(x), \\ G(x) &= x \left( x^4 + x^2 + 2 \right) + \left( x^6 + x^4/2 + x^2 + 3/4 \right) Z(x), \\ N(x) &= \left( 1 - \frac{\omega_{*ni}}{\omega} \right) \left[ x + \left( 1/2 + x^2 \right) Z(x) \right] - \frac{\omega_{*Ti}}{\omega} \left[ x \left( 1/2 + x^2 \right) + \left( 1/4 + x^4 \right) Z(x) \right], \\ D(x) &= \left( \frac{1}{x} \right) \left( 1 + \frac{T_e}{T_i} \right) + \left( 1 - \frac{\omega_{*ni}}{\omega} \right) Z(x) - \frac{\omega_{*Ti}}{\omega} \left[ x + \left( x^2 - 1/2 \right) Z(x) \right]. \end{aligned} \quad (24)$$

With the new definitions

$$\begin{aligned} \tilde{\gamma}_M^2 &= 2k_\theta^2 \rho_{Li}^2 \left( \frac{k_z^2 v_{thi}^2}{\omega_A^2 \partial \text{Re} \Lambda^2 / \partial \omega_0^2} \right) \frac{\epsilon^{1/2}}{1.6q^2} \left( 1 - \frac{\omega_{*pi}}{\omega_0} - \frac{\omega_A^2}{\omega_0^2} \text{Re} \Lambda^2 \right) \left\langle \left\langle \left| \frac{e\Psi_0}{T_i} \right|^2 \right\rangle \right\rangle, \\ \tilde{\Delta} &= \frac{k_z^2}{2} \frac{\partial^2 \delta W_f}{\partial k_r^2} \bigg/ \frac{\partial}{\partial \omega_0} \text{Re} \Lambda^2, \end{aligned} \quad (25)$$

the zonal flow growth rate induced by AITG is:

$$\Gamma_{z,AITG} = \tilde{\gamma}_M \sqrt{1 - \tilde{\Delta}^2 / \tilde{\gamma}_M^2} . \quad (26)$$

As in the case of KAW, we find a condition for effective excitation of zonal flow by AITG, i.e.,  $\omega_0 > \omega_{*pi}$ , which is the typical case for slightly unstable AITG [14]. Above threshold, also AITG driven zonal flow growth rate scales linearly with the mode amplitude.

#### 4. Conclusions

In the present work, we have demonstrated that zonal flows may be spontaneously generated by a variety of drift and drift-Alfvén turbulences and, above their *spontaneous excitation* threshold, their growth rate typically scales linearly with the mode amplitudes. In the electrostatic limit, good agreement is shown between numerical results from 3D gyrokinetic simulations of ITG and the obtained analytic expression [9]. In the same limit, nonlinear equations for mode amplitudes have been recently derived [9] and they demonstrate saturation of the linearly unstable modes via coupling to the stable envelope sidebands and, as a consequence, oscillatory behaviors in the drift-wave intensity and zonal flows [9]. Similar behaviors can also be expected in the general electromagnetic case, which will be analyzed in the near future.

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