# On Dependence of Thermal Transport on the Safety Factor $q$ in Tokamaks

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**Abstract**. Mixing length estimates of the anomalous ion and electron thermal diffusivities caused by the ITG and ETG mode, respectively, have revealed that both diffusivities are proportional to the safety factor $q$. In the case of ITG mode, the maximum $\chi_i$ occurs at long wavelengths where coupling to the ion acoustic mode is dominant while ETG driven $\chi_e$ peaks at wavelengths comparable to the electron skin depth.

## 1. Introduction

Several observations have been reported on the dependence of the thermal diffusivity $\chi$ in tokamaks on the safety factor $q$. Petty et al. have found that in the DIII-D tokamak, both the ion and electron thermal diffusivities increase almost linearly with the safety factor $q$ [1]. In gyro-fluid simulations of tokamak transport due to the ion temperature gradient (ITG) mode, the relationship $\chi_i \propto q$ has also been observed [2]. In a recent study on the electron temperature gradient (ETG) mode without imposing charge neutrality, the electron thermal diffusivity has been found to scale linearly with $q$, $\chi_e \propto q$ [3]. In an earlier study [4], it was shown that the electron thermal diffusivity in the form

$$\chi_e \simeq \frac{qc_s}{R}\left(\frac{c}{\omega_{pe}}\right)^2, \quad (\eta_e > 1)\,, \tag{1}$$

could reproduce the $\chi_e$ profiles in various tokamaks under vastly different discharge conditions. Here, $c_s$ is the ion acoustic speed, $c/\omega_{pe}$ is the collisionless electron skin depth and $\eta_e$ is the electron temperature gradient relative to the density gradient. Eq. (1) differs qualitatively from the diffusivity proposed by Ohkawa [5],

$$\chi_{e,\,Ohkawa} \simeq \frac{v_{Te}}{qR}\left(\frac{c}{\omega_{pe}}\right)^2, \tag{2}$$

which is *inversely* proportional to $q$.

In this paper, results of kinetic, integral equation based ballooning stability analysis of both ITG and ETG modes will be presented. The kinetic, electromagnetic integral equation code [6] is able to handle short wavelength modes in the regime $k \simeq k_{De}$ (Debye wavenumber) where charge neutrality does not hold. The linear increase with the safety factor $q$ is observed for both ITG driven $\chi_i$ and ETG driven $\chi_e$ and it is attributed to the coupling to the ion sound mode in the case of ITG mode and to the skin size drift mode in the case of ETG mode. When $s$ (magnetic shear) $\simeq 1$, and $r/R = 0.1$, the ion thermal diffusivity is well represented by

$$\chi_i = 0.21q\sqrt{\frac{R}{L_{Ti}}}\frac{v_{Ti}\rho^2}{L_{Ti}},$$

and the electron thermal diffusivity by

$$\chi_e \simeq \frac{q v_{Te}}{L_{Te}} \left( \frac{c}{\omega_{pe}} \right)^2 \sqrt{\beta_e}.$$

## 2. ITG Mode

In the case of the ITG mode, such $q$ dependence of the ion thermal diffusivity ($\chi_i \propto q$) has not been revealed in the standard drift mode analysis in which the coupling to the ion acoustic mode is usually ignored and main attention has been paid to shorter wavelength, fastest growing mode with a growth rate significantly larger than the ion acoustic frequency. However, peaking of the diffusivity based on the mixing length estimate,

$$\chi \simeq \frac{\gamma}{k_\perp^2}, \tag{3}$$

does not coincides with the peaking of the growth rate $\gamma$. Rather, the maximum diffusivity occurs at a longer wavelength close to the instability threshold. In the case of ITG mode, the instability is deactivated when the growth rate approaches the ion transit (or acoustic) frequency $\omega_{Ti} = v_{Ti}/qR$. Since the growth rate is approximately given by $\gamma \simeq \sqrt{\eta_i \omega_* \omega_D}$, the threshold $k_\perp \rho$ becomes inversely proportional to $q$,

$$(k_\perp \rho)^2 \propto \frac{1}{q^2} \frac{L_{Ti}}{R},$$

and the following diffusivity emerges,

$$\chi_i \propto q \frac{v_{Ti} \rho^2}{L_{Ti}} f \left( \frac{R}{L_{Ti}} \right), \tag{4}$$

where the function $f$ is the growth rate normalized by the ion transit speed, $\gamma = f\left(R/L_{Ti}\right) v_{Ti}/qR$. As will be shown, analysis based on a kinetic ballooning integral equation code confirms the linear dependence of $\chi_i \propto q$ and the function $f$ is approximately given by $f = \sqrt{R/L_{Ti}}$.

Figure 1 shows the ITG mode driven ion thermal diffusivities as functions of $b_s = (k_\theta \rho_s)^2$ when $T_e = T_i$, $L_n/R = 0.2$, $r/R = 0.1$ (destabilizing trapped electrons included), $\eta_i = \eta_e = 2.5$, $s = 1$, $\beta_i = \beta_e = 10^{-4}$ (top), and $10^{-3}$ (bottom). The safety factor $q$ was varied as a parameter between 1 and 4. The low $\beta$ case (top) is essentially electrostatic, and the diffusivity indeed increases with the safety factor being proportional to $q$. When $\beta$ is increased to $\beta_e = \beta_i = 0.1$ % (bottom), the linear increase in $\chi_i$ with the safety factor breaks down. The diffusivity increases more rapidly, $\chi_i \propto q^{3/2}$, until the ballooning parameter $\alpha$ defined by

$$\alpha = \frac{q^2 R}{L_n} \left[ (1+\eta_i)\, \beta_i + (1+\eta_e)\, \beta_e \right],$$

becomes large enough to stabilize the ITG mode [7]. (When $\beta_e = \beta_i = 10^{-3}$ and $q = 4$, the ballooning parameter for the parameters assumed is $\alpha = 0.56$. The ITG mode is stabilized but will be taken over by the ballooning mode. When $s = 1$, the threshold $\alpha$ for the ideal MHD ballooning mode is $\alpha_{MHD} \simeq 0.6$.) This finding may explain the observation that in

H-modes discharges, $\chi_i$ increases with $q$ more rapidly than linearly. The simulations in Ref. 2 were electrostatic without finite $\beta$ effects. In this case, $\chi_i \propto q$ is expected. Fig. 2 shows the dispersion relation of the ITG mode when $q = 2$ (top) and 4 (bottom). The mode frequency and growth rate are normalized by the ion acoustic transit frequency $\omega_s = c_s/qR$. The mode frequency at $b_s$ where $\chi_i$ peaks is indeed close to the ion acoustic frequency, $|\omega| \simeq \omega_s = c_s/qR$. As $q$ increases, the FLR parameter $b_s$ at $\chi_{i\,\mathrm{max}}$ decreases in a manner $b_s \propto 1/q^2$. This can be seen from the threshold of $b_s$ imposed by $\gamma \simeq \sqrt{\eta_i \omega_* \omega_D} \simeq v_{Ti}/qR$, which yields

$$b_s \propto \frac{1}{q^2}\frac{L_{Ti}}{R}.$$

The growth rate is of the order of the ion acoustic frequency $\gamma \simeq f c_s/qR$, and resulting diffusivity is

$$\chi_i \propto q\frac{v_{Ti}\rho^2}{L_{Ti}} f\left(\frac{R}{L_{Ti}}\right).$$

Here the function $f$ is the growth rate normalized by the ion acoustic frequency and near the threshold it is approximately given by $f = \sqrt{R/L_{Ti}}$. The ion thermal diffusivity that has been found by scanning $q$, $\eta_i$, $L_n/R$, over a wide range may be summarized by the following expression

$$\chi_i = 0.21q\sqrt{\frac{R}{L_{Ti}}}\frac{v_{Ti}\rho^2}{L_{Ti}},$$

when $s = 1$, $r/R = 0.1$, and $T_i = T_e$.

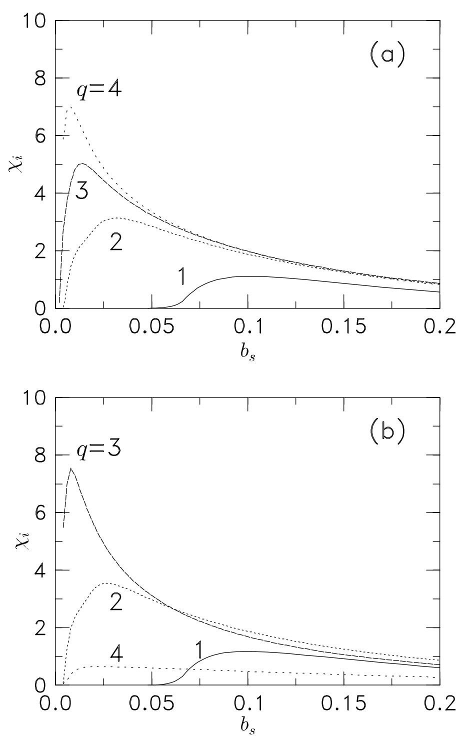


Fig. 1. Ion thermal diffusivity due to the ITG mode normalized by $c_s\rho_s^2/L_n$ vs. $b_s = (k_\theta\rho_s)^2$ when $q$ is varied. (a) $\beta_i = \beta_e = 10^{-4}$, (b) $\beta_i = \beta_e = 10^{-3}$. Common parameters are: $\varepsilon_n = 0.2$, $\varepsilon = 0.1$, $\eta_i = \eta_e = 2$, $s = 1$, $T_i = T_e$, $m_i/m_e = 1836$.

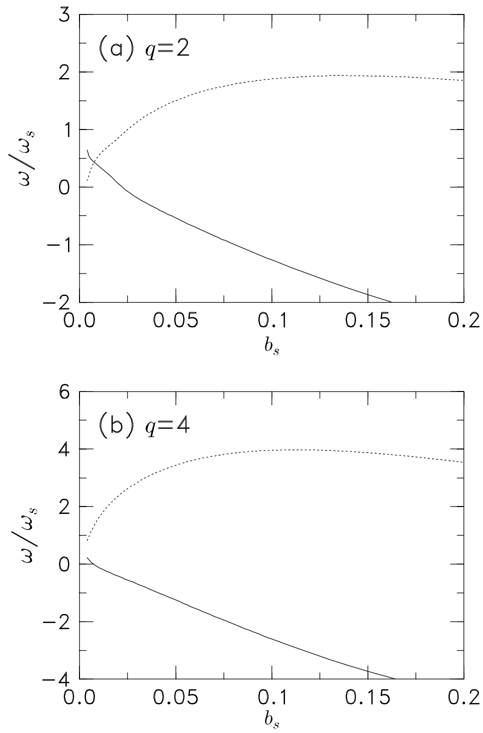


Fig. 2. Mode frequency ($\omega_r/\omega_s$ solid line and growth rate $\gamma/\omega_s$) vs. $b_s$ when $q = 2$ (a) and $q = 4$ (b). $\beta_i = \beta_e = 10^{-4}$, $\varepsilon_n = 0.2$, $\varepsilon = 0.1$, $\eta_i = \eta_e = 2$, $s = 1$, $T_i = T_e$, $m_i/m_e = 1836$.

## 3. ETG Mode

It is generally conjectured that the short wavelength electron temperature gradient (ETG) mode is dual of the long wavelength ion temperature gradient (ITG) mode, since in the former, ions are adiabatic while in the latter, electrons are (except for the destabilizing roles of trapped electrons on the ITG mode [7]). However, there are some basic differences between the two modes. First, in the ETG mode, charge neutrality does not necessarily hold because of short wavelength nature. Second, while the ITG mode can be stabilized by a modest plasma $\beta$ factor through the coupling of electron dynamics to the magnetic perturbation, the ETG mode is quite resilient against finite $\beta$ stabilization which can occur through equilibrium modification only at such a large $\alpha$ (the ballooning parameter) as to cause an effective magnetic drift reversal. If isomorphism between the ETG and ITG modes holds, the mixing length estimate for the electron thermal diffusivity would be of the order of

$$\chi_e \simeq \frac{v_{Te}}{L_n}\rho_e^2, \tag{5}$$

which is smaller than the ion thermal diffusivity due to the ITG mode by approximately a factor of $\sqrt{m_i/m_e}$, and thus would not be relevant to the anomalous electron thermal transport commonly observed in magnetic confinement devices. Here, $v_{Te} = \sqrt{T_e/m_e}$, $L_n$ is the scale length of the density gradient, $\rho_e$ the electron Larmor radius, and $m_i/m_e$ is the ion/electron mass ratio. Such small transport has indeed been observed in a fluid simulation of the ETG mode [8] in which charge neutrality was imposed. In a kinetic simulation without assuming charge neutrality [9, 10], thermal transport significantly larger has been observed. The large transport was attributed to the formation of large scale, radially extended streamers.

The ETG mode is electrostatic. However, when charge neutrality does not hold, its growth rate becomes dependent on the electron density and thus the electron $\beta$ factor. The $\beta_e$ dependence of the growth rate may be shown qualitatively as follows. Substituting adiabatic ions $n_i = -e\phi n_0/T_i$ and approximate electron density perturbation without electron transit effect,

$$n_e \simeq \frac{\eta_e \omega_{*e}\omega_{De}}{\omega^2} e^{-b_e} I_0\left(b_e\right) \frac{e\phi}{T_e} n_0, \tag{6}$$

in the Poisson's equation $\nabla^2 \phi = -4\pi e \left(n_i - n_e\right)$, we find the growth rate,

$$\gamma\left(b_e\right) = \sqrt{\frac{2T_e/m_e}{L_T R}} \sqrt{\frac{b_e e^{-b_e} I_0\left(b_e\right)}{\tau + \left(b_e/\beta_*\right)}}, \tag{7}$$

where $\eta_e = L_n/L_T$ is the electron temperature gradient parameter, $\omega_{*e} = \frac{v_{Te}}{L_n}\sqrt{b_e}$, $\omega_{De} = \frac{2v_{Te}}{R}\sqrt{b_e}$, $b_e = \left(k_\perp \rho_e\right)^2$, $I_0$ is the modified Bessel function, $\tau = T_e/T_i$, and

$$\beta_* = \beta_e \frac{mc^2}{2T_e} = \left(\frac{\omega_{pe}}{\Omega_e}\right)^2, \tag{8}$$

with $\omega_{pe}$ the electron plasma frequency and $\Omega_e$ the cyclotron frequency. The maximum growth rate can be found by scanning the finite Larmor radius parameter $b_e$. When the maximum growth rate is written as $\gamma_{\max} = \sqrt{2T_e/m_e L_T R} f\left(\beta_*\right)$, the function $f\left(\beta_*\right)$ is approximately proportional to $\sqrt{\beta_*}$ in the regime $\beta_* \lesssim 1$ relevant to tokamaks. It should be noted that in tokamak stability analysis, the FLR parameter $\left(k_\perp \rho_e\right)^2$, the $\beta$ factor and corresponding ballooning parameter $\alpha = q^2 \left(R/L_n\right) \left[(1+\eta_e)\beta_e + (1+\eta_i)\beta_i\right]$ are to be specified. Then, the charge nonneutrality factor $\left(k/k_{De}\right)^2 = \left(k_\perp \rho_e\right)^2 \beta_e \frac{mc^2}{2T_e}$ necessarily involves a normalized temperature.

These qualitative estimates are consistent with the results of stability analysis based on the integral equation code. As explained above, charge neutrality breaks down for typical tokamak discharge parameters and duality between the ITG and ETG modes does not hold anymore [3]. The maximum growth rate, which is of the order of the electron transit frequency $\omega_{Te} = v_{Te}/qR$, does occur at $k \simeq k_{De}$ where $k_{De}$ is the Debye wavenumber. The electron FLR parameter $k_\theta \rho_e$ is not a convenient normalization of the wavenumber in the ETG mode. Also, it is necessary to specify a normalized electron temperature $T_e/mc^2$ in fully electromagnetic gyrokinetic analysis. has to be specified in addition to the numerous other dimensionless parameters in electromagnetic gyrokinetic formulation.

As in the case of ITG mode, the maximum electron thermal transport by the ETG mode occurs at the lower edge of the unstable $k$ spectrum which is in the region of the inverse electron skin depth, $k_\perp \gtrsim \omega_{pe}/c$. The growth rate of the ETG mode is independent of the safety factor $q$. (In Fig. 3 (b), the normalized growth rate $\gamma/\left(v_{Te}/qR\right)$ increases with $q$. The growth rate $\gamma$ itself is thus independent of $q$.) The $q$ dependence of $\chi_e$ due to the ETG mode ($\chi_e \propto q$) stems from the progressive extension of $k$ spectrum toward longer wavelengths as $q$ increases. The normalized wavenumber $\left(ck_\theta/\omega_{pe}\right)^2$ at the most active transport is inversely proportional to $q$. This is shown in Fig. 3 for the case $\beta_e = \beta_i = 0.2\%$,

$\varepsilon_n = 0.2$, $\eta_e = 2$, $\eta_i = 1$, $s = 1$, $T_i = T_e = 10$ keV in a hydrogen discharge. The safety factor $q$ is varied between 2 and 4. The diffusivity is normalized by Ohkawa diffusivity as

$$\chi_e \to \chi_e \frac{qR}{v_{Te}} \left(\frac{\omega_{pe}}{c}\right)^2,$$

and $\omega$ and $\gamma$ by $\omega_{Te} = v_{Te}/qR$. The normalized maximum diffusivity found numerically is nearly proportional to $q^2$ and thus the unfolded diffusivity is proportional to $q$.(Note that $\omega_{Te} \propto 1/q$.) The maximum diffusivity occurs in the region $k_\theta^2 \simeq \omega_{pe}^2/c^2$, and $\delta_e = (ck_\theta/\omega_{pe})^2$ at the maximum diffusivity is inversely proportional to $q$ clearly seen in Fig. 3 (a). The thermal transport due to the ETG mode is thus governed by fluctuations in the region of the electron skin depth. As in the case of ITG mode, the maximum growth rate of the ETG mode occurs at much shorter wavelength, $k_\theta \lesssim k_{De}$. The electron thermal diffusivity due to the ETG mode that has been found by extensive parameter scan can be well approximated by

$$\chi_e \simeq 0.1 \frac{q v_{Te}}{L_{Te}} \left(\frac{c}{\omega_{pe}}\right)^2 \sqrt{\beta_e}, \tag{9}$$

where $\beta_e$ is the electron beta factor which destabilizes the ETG mode. (For an extremely low density and thus low $\beta$, the measure of charge nonneutrality $(k/k_{De})^2$ becomes large and the ETG mode tends to be stabilized. The growth rate of the ETG mode is approximately proportional to $\sqrt{\beta_e}$ even though the mode is predominantly electrostatic.) The ratio between Eq. (9) and Ohkawa diffusivity is

$$0.1 q^2 \frac{R}{L_T} \sqrt{\beta_e}, \tag{10}$$

which is of order unity.

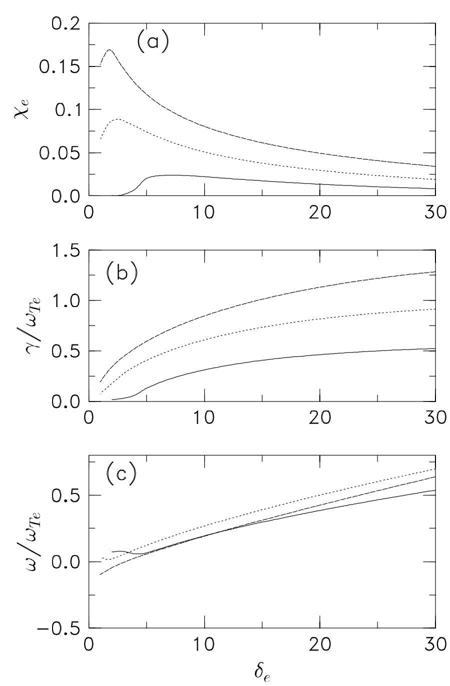


Fig. 3. (a) Electron thermal diffusivity due to the ETG mode normalized by the Ohkawa diffusivity vs. $\delta_e = (ck_\theta/\omega_{pe})^2$ when $\beta_i = \beta_e = 0.002$, $\varepsilon_n = 0.2$, $\eta_i = 1, \eta_e = 2$, $s = 1$, $T_i = T_e = 10$ keV, $m_i/m_e = 1836$. (b) and (c) show corresponding mode frequency $\omega/\omega_{Te}$ and growth rate $\gamma/\omega_{Te}$, $\omega_{Te} = v_{Te}/qR$. $q = 2$ (solid lines), 3 (dotted lines) and 4 (dashed lines).

## 4. Conclusions

In conclusion, it has been shown in terms of linear stability analysis and lowest order mixing length estimates that both long wavelength ITG mode and short wavelength ETG mode yield ion and electron thermal diffusivities approximately proportional to the safety factor $q$. Such dependence has recently been observed in transport analysis of DIII-D tokamak. Sensitive $q$ dependence of $\chi_i$ due to the ITG mode originates from the coupling to the ion acoustic transit mode which dominated transport. The growth rate is of the order of the ion acoustic frequency and thus inversely proportional to $q$. The ion FLR parameter at the maximum transport is proportional to $q^{-2}$, $(k_\theta \rho_s)^2 \propto q^{-2}$, and a resulting mixing length diffusivity is proportional to $q$, $\chi_i \propto q$.

In the case of ETG mode, the lower threshold of the cross-field wavelength occurs at the electron skin depth. The growth rate is independent of $q$. However, $\delta_e = (ck_\theta/\omega_{pe})^2$ at the maximum transport is inversely proportional to $q$ yielding $\chi_e \propto q$.

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