Complex Nonlinear Lagrangian for the Hasegawa–Mima Equation

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Abstract. The Hasegawa–Mima equation is the simplest nonlinear single-field model equation that captures the essence of drift wave dynamics. Like the Schrödinger equation it is first order in time. However its coefficients are real, so if the potential φ is initially real it remains real. However, by embedding φ in the space of complex functions a simple Lagrangian is found from which the Hasegawa–Mima equation may be derived from Hamilton's Principle. This Lagrangian is used to derive an action conservation equation which agrees with that of Biskamp and Horton.

1. Introduction

Recent interest in zonal flow generation via the mechanism of modulational instability of plasma drift waves [1, 2] motivates revisiting Whitham's [3, 4] averaged-Lagrangian approach, which provides an elegant and efficient method for deriving equations describing wave-mean flow interactions [5].

The one-field Hasegawa–Mima (HM) equation [6, 7] provides a simple theoretical starting point for describing the nonlinear interaction of drift waves and zonal flows and is appropriate for describing some experimental régimes [8, 9]. The same equation also describes Rossby wave turbulence in planetary flows [7].

A Lagrangian for drift waves in the linearized approximation has been found by Mattor and Diamond [10], but how to extend this to the nonlinear Hasegawa–Mima equation has not been clear. In this paper we present a complex-field Lagrangian whose Euler–Lagrange equation reduces to the HM equation in the invariant subspace of real functions.

The HM equation is an equation for the evolution in time, t, of the electrostatic potential $\varphi(x, y, t)$. Here x and y are Cartesian coordinates describing position in a cross-sectional domain Ω of a plasma with a strong magnetic field predominantly in the z-direction, $\hat{\mathbf{z}}$ is the unit vector in the z-direction, ω_{c} is the cyclotron frequency and $n_0(x, y)$ is the background electron density, assumed independent of poloidal angle. In slab geometry Ω is taken to be rectangular and y to be the poloidal direction, so that n_0 is a function only of x. If in toroidal geometry we take Ω to be a circular domain and work in polar coordinates, then n_0 is taken to be a function of $r \equiv (x^2 + y^2)^{1/2}$.

2. Lagrangian

We consider the action integral

$$
\begin{aligned}
S \;=\; & \int_{t_1}^{t_2} dt \int_\Omega d^2x\; \mathcal{L}(x, y, \varphi^*, \varphi, \varphi_t^*, \varphi_t, \boldsymbol{\nabla}\varphi^*, \boldsymbol{\nabla}\varphi, \\
& \boldsymbol{\nabla}\varphi_t^*, \boldsymbol{\nabla}\varphi_t, \nabla^2\varphi, \nabla^2\varphi^*, \nabla^2\varphi)\;, \qquad (1)
\end{aligned}
$$

where * denotes the complex conjugate, $\varphi_t \equiv \partial\varphi/\partial t$, $\boldsymbol{\nabla}\varphi \equiv \hat{\mathbf{x}}\partial\varphi/\partial x + \hat{\mathbf{y}}\partial\varphi/\partial y$, and $\mathcal{L}$ is the Lagrangian density

$$
\begin{aligned}
\mathcal{L} \;=\; & \frac{i}{2}(\varphi^*\varphi_t - \varphi_t^*\varphi) + \frac{i}{2}(\boldsymbol{\nabla}\varphi^*\boldsymbol{\cdot}\boldsymbol{\nabla}\varphi_t - \boldsymbol{\nabla}\varphi_t^*\boldsymbol{\cdot}\boldsymbol{\nabla}\varphi) \\
& + \frac{i}{2}(\varphi^*\boldsymbol{\nabla}\varphi - \varphi\boldsymbol{\nabla}\varphi^*)\boldsymbol{\cdot}\hat{\mathbf{z}}\times\boldsymbol{\nabla}\ln\frac{\omega_{\rm c}}{n_0} \\
& - \frac{i}{2}\boldsymbol{\nabla}\varphi^*\boldsymbol{\cdot}\boldsymbol{\nabla}\varphi\times\hat{\mathbf{z}}\;\nabla^2(\varphi^* + \varphi)\;, \qquad (2)
\end{aligned}
$$

the first term being similar to that of the Lagrangian for the Schrödinger equation [11].

Hamilton's Principle is the requirement that S be stationary for arbitrary variations of φ and φ^* (except at the initial and final times t_1 and t_2, and on the boundary $\partial\Omega$, where φ and $\boldsymbol{\nabla}\varphi$ and their complex conjugates are held fixed). The field and its complex conjugate are variably independently because the real and imaginary parts are independent. This principle implies the general Euler–Lagrange equation

$$
\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\varphi_t^*} + \boldsymbol{\nabla}\boldsymbol{\cdot}\frac{\partial\mathcal{L}}{\partial\boldsymbol{\nabla}\varphi^*} - \frac{\partial\mathcal{L}}{\partial\varphi^*} - \frac{\partial}{\partial t}\boldsymbol{\nabla}\boldsymbol{\cdot}\frac{\partial\mathcal{L}}{\partial\boldsymbol{\nabla}\varphi_t^*} - \nabla^2\frac{\partial\mathcal{L}}{\partial\nabla^2\varphi^*} = 0\;, \qquad (3)
$$

With our specific form of the Lagrangian density this leads (on dividing through by i) to

$$
\frac{\partial}{\partial t}(\nabla^2\varphi - \varphi) - \boldsymbol{\nabla}\varphi\times\hat{\mathbf{z}}\boldsymbol{\cdot}\boldsymbol{\nabla}\left(\nabla^2{\rm Re}\varphi + \ln\frac{\omega_{\rm c}}{n_0}\right) + \frac{1}{2}\nabla^2(\boldsymbol{\nabla}\varphi^*\boldsymbol{\cdot}\boldsymbol{\nabla}\varphi\times\hat{\mathbf{z}}) = 0\;. \qquad (4)
$$

The last term vanishes when φ is real, thus recovering the Hasegawa–Mima equation.

3. Action conservation

Considering the case of a drift wave in the form of a coherent wavetrain we use a nonlinear Wentzel-Kramers-Brillouin (WKB) trial function of the form

$$
\begin{aligned}
\varphi \;=\; & \epsilon^{-1}\Phi(\epsilon x, \epsilon y, \epsilon t) + \sum_{\pm}\{\alpha A_1^{\pm}(\epsilon x, \epsilon y, \epsilon t)\exp[\pm{\rm i}\epsilon^{-1}\theta_{\pm}(\epsilon x, \epsilon y, \epsilon t)] \\
& + \alpha^2 A_2^{\pm}(\epsilon x, \epsilon y, \epsilon t)\exp[\pm 2{\rm i}\epsilon^{-1}\theta_{\pm}(\epsilon x, \epsilon y, \epsilon t)] + O(\alpha^3)\}\;, \qquad (5)
\end{aligned}
$$

where α is an expansion parameter for the amplitude, which we here distinguish from the WKB expansion parameter ϵ. The phases θ_+ and θ_-, the slowly varying fundamental amplitudes $A_1^{\pm}$, and the second harmonic amplitudes $A_2^{\pm}$ are all treated as independently variable when applying Hamilton's principle. The slowly varying frequencies and wave vectors are defined as $\omega_{\pm} \equiv -\partial_t(\theta_{\pm}/\epsilon) = O(1)$ and $\mathbf{k}_{\pm} \equiv \boldsymbol{\nabla}(\theta_{\pm}/\epsilon) = O(1)$, respectively.

We have also included a slowly varying $O(1)$ background potential Φ to represent zonal flows.

In the spirit of Whitham [3, 4] we substitute Eq. (5) into Eq. (2), retaining only leading order terms in ϵ. Rapidly varying terms such as $\exp \pm i(\theta_+ + \theta_-)/\epsilon$ make an exponentially small contribution to the action, Eq. (1), by the Riemann–Lebesgue lemma [5]. We can thus safely discard such terms and replace $\mathcal{L}$ by the *averaged Lagrangian density*

$$
\begin{aligned}
\bar{\mathcal{L}} &= (1+k_+^2)(\omega_+ - \mathbf{k}_+\cdot\mathbf{V})|A_+|^2 - (1+k_-^2)(\omega_- - \mathbf{k}_+\cdot\mathbf{V})|A_-|^2 \\
&\quad -(\mathbf{k}_+|A_+|^2 - \mathbf{k}_-|A_-|^2)\cdot\hat{\mathbf{z}}\times\boldsymbol{\nabla}\ln\frac{\omega_\mathrm{c}}{n_0}, \qquad (6)
\end{aligned}
$$

where $\mathbf{V}$ is the (nondimensionalized) background $\mathbf{E}\times\mathbf{B}$ drift

$$
\mathbf{V} \equiv \hat{\mathbf{z}}\times\boldsymbol{\nabla}\mathrm{Re}\Phi. \qquad (7)
$$

Variation of the amplitudes in the Lagrangian density Eq. (6):

$$
\frac{\partial\bar{\mathcal{L}}}{\partial|A_\pm|} = 0, \qquad (8)
$$

reproduces the dispersion relation of drift waves [7]:

$$
\omega_\pm - \mathbf{k}_\pm\cdot\mathbf{V} = \frac{\mathbf{k}_\pm\cdot\hat{\mathbf{z}}\times\boldsymbol{\nabla}\ln\frac{\omega_\mathrm{c}}{n_0}}{1+k_\pm^2}. \qquad (9)
$$

The averaged Lagrangian density does not have an explicit dependence on $\theta_\pm$. Therefore the Euler–Lagrange equation for variation with respect to θ_+ is in the form of a conservation equation

$$
\frac{\partial}{\partial t}\left(\frac{\partial\bar{\mathcal{L}}}{\partial\omega_\pm}\right) + \boldsymbol{\nabla}\cdot\frac{\partial\bar{\mathcal{L}}}{\partial\mathbf{k}_\pm} = 0\ , \qquad (10)
$$

where N is the *wave action density*

$$
N = \frac{\partial\bar{\mathcal{L}}}{\partial\omega_\pm} = (1+k_\pm^2)|A_\pm|^2\ . \qquad (11)
$$

We note that this definition of the wave action density differs by a factor of ω from that found by Mattor and Diamond [10] and Brizard [12], who use Lagrangians derived for linearized drift waves. However, it agrees with the result of Biskamp and Horton [13], who do not use the Whitham variational method and conclude that wave energy and action are equal for drift waves.

Our result thus shows that the discrepancy between the two wave action expressions in the literature does not arise from the use of the variational method *per se*, but from the form of the Lagrangian adopted. For time-independent background quantities, both wave energy and action are conserved, so the conservation laws derived are not inconsistent. Rather, it is the *naming* of the conserved quantities that is in question in that case.

When the background quantities *are* time-dependent, however, energy can be exchanged between the wave and background subsystems [5] and thus wave energy is not conserved but wave action remains a conserved quantity. Thus the full resolution of the discrepancy requires an analysis of the time-varying-background case. The time variation we have allowed in the background flow potential Φ is sufficient to expose the problem, even if n_0 and ω_c are constant in time.

4. Conclusion

The Lagrangian we have found has the attraction of being very simple, yet, unlike those in Refs. [10, 12], it is nonlinear. As it leads to the Hasegawa–Mima equation, in the case of real initial conditions, it is as physically correct as this equation and can thus be used, for instance, to derive mode-coupling equations. However, the action-conservation discrepancy suggests caution should be adopted in using it for deriving wave-background interactions such as zonal flow generation.

The fact that the Lagrangians of Mattor and Diamond and of Brizard are derived from physical Lagrangians, rather than constructed *ad hoc* as ours was, makes it more likely that their form for the wave action is correct. However, further work remains to be done to resolve this issue.

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