Equilibria with Stochastic Regions Having Non-zero Pressure Gradient

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Abstract. The nature of plasma equilibria in a magnetic field with stochastic regions is examined. We show that the magnetic differential equation along chaotic field line trajectories that determines the equilibrium pressure-driven currents in the stochastic regions can be cast in a form similar to various nonlinear equations for a turbulent plasma, allowing application of the mathematical methods of statistical turbulence theory. In particular, resonance broadening theory has been applied to obtain a solution. Two difficulties must be surmounted in applying resonance-broadening theory in the context of 3D equilibria: 1) Resonance broadening theory makes use of causality, but causality does not hold in the context of equilibrium calculations; 2) The equilibrium solution in a torus must be periodic in the two angular variables, unlike the time-dependent equations to which resonance broadening theory is usually applied. In addition, we must also deal with the issue that a plasma having finite pressure gradients in stochastic regions cannot satisfy the MHD equilibrium equations. There is an extensive literature on the theory of plasma transport in the presence of stochastic magnetic field lines, and our work addresses the issue of the nature of the equilibrium solution in that context. Equilibria with stochastic regions are important for understanding fusion plasma confinement in tokamaks with ergodic limiters or resistive wall modes, or with nonaxisymmetric fields imposed for stabilizing ELMs. They are also of interest in contemporary stellarator experiments at their highest achievable values of beta, where there is evidence of the formation of a large region of stochastic field lines in the outer region of the plasma, with a finite pressure gradient in that region. The solution for the current in the stochastic region has been incorporated into the PIES 3D Equilibrium code, and has been applied to the calculation of equilibria for the W7AS stellarator. The calculated equilibrium solutions are consistent with the experimental observations, including a strong dependence of the achievable beta on the current in the divertor control coils, differences in pressure profiles between different shots in the regions calculated to be stochastic, and consistency with the Rechester-Rosenbluth estimate for the contribution of the field line stochasticity to energy transport.

I. Introduction

The work described in this paper was initially motivated by a puzzling observation on the W7AS stellarator experiment. Figure 1 shows the maximum achievable value of $\langle\beta\rangle$ in a set

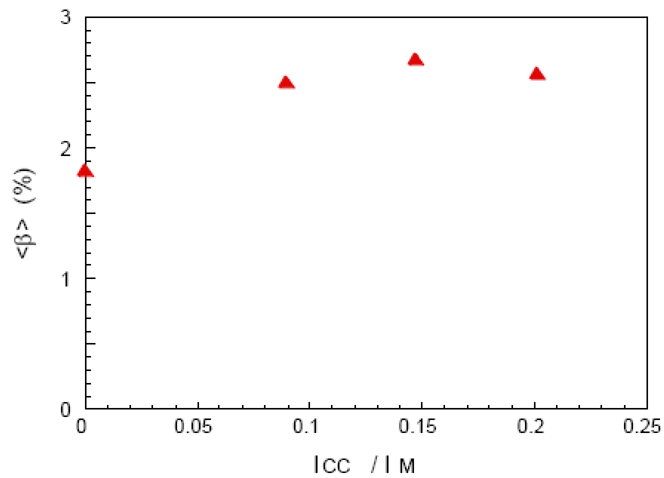

FIG. 1. Variation of peak-$\langle\beta\rangle$ versus the divertor control-coil current I_{CC} normalized by the modular coil current, for B=1.25 T, PNB = 2.8 MW absorbed and ι_{vac}= 0.44.

of shots as a function of the current in the divertor control coils. These coils affect the resonant component of the magnetic field near the plasma edge. It was found that the achievable ⟨β⟩ could be increased by about 50% by energizing these coils. These intriguing results motivated calculations using the PIES code, a 3D equilibrium code that can handle islands and stochastic regions.

Figure 2 shows the results of a set of PIES calculations. The code finds a region of stochastic field lines which increases in width as the value of ⟨β⟩ is increased. The width of the stochastic region is strongly affected by the current in the divertor control coils, and this provides a plausible explanation for the observed affect on the achievable ⟨β⟩. There was no method available for directly verifying the existence of the stochastic region in the W7AS experiments. The calculated equilibria were, however, consistent with the experimental observations, including an observed difference in the pressure profiles in the region predicted to be stochastic, and a Rechester-Rosenbluth estimate of the enhancement in transport due to field line stochasticity.[1] (But the Rechester-Rosenbluth thermal diffusivity scales as $T_e^{5/2}$, so the error bars are large.)

The W7AS diagnostics indicated that there was a non-zero pressure gradient in the region calculated to be stochastic. In order to do the calculations for W7AS, it was necessary to develop a theory for equilibria in stochastic regions. That is the subject of this paper.

II. Equilibrium in stochastic regions.

We will use the form of the equilibrium equations that has been adopted for the PIES code:

$$\nabla \times \mathbf{B} = \mathbf{j}(\mathbf{B}), \tag{1}$$

Where **j** is a nonlinear function of **B** determined by,

$$\mathbf{j}_{\perp} = \mathbf{B} \times \nabla p / B^2, \tag{2}$$

and

$$\mathbf{B} \cdot \nabla (j_{\parallel}/\mathrm{B}) = -\nabla \cdot \mathbf{j}_{\perp}. \tag{3}$$

Eq. (2) follows from ∇·**j**=0. These equations can be solved numerically by standard methods,

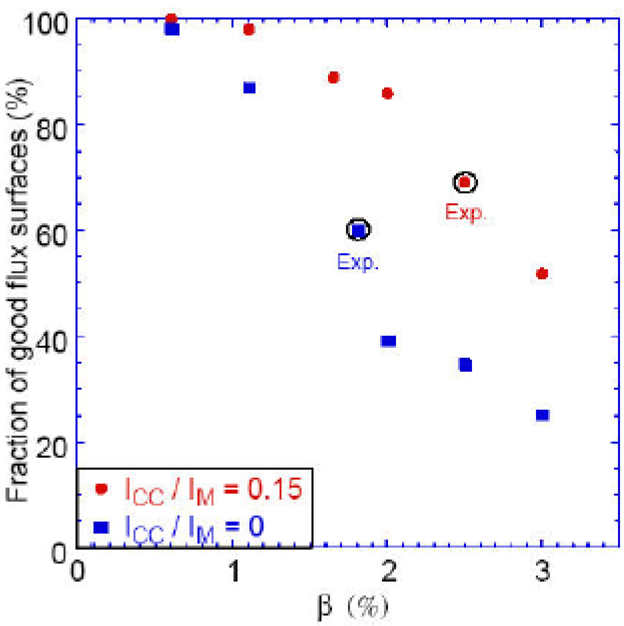

FIG. 2. Fraction of good flux surfaces versus β as predicted by the PIES code for two different values of the control coil current for the W7AS stellarator. The circles indicate the PIES calculations done for the experimentally achieved value of β.

such as Picard iteration or Newton-Krylov.

In solving the equilibrium equations in a stochastic region, the first difficulty we need to face is that an MHD equilibrium cannot have a nonzero pressure gradient in the stochastic region. It follows from $\mathbf{j} \times \mathbf{B} = \nabla p$ that $\mathbf{B}\cdot\nabla p = 0$ in a stochastic region, so that $\nabla p = 0$ in that region. We must retain the small terms that have been discarded in the MHD equilibrium equation,

$$\mathbf{j} \times \mathbf{B} - \boldsymbol{\rho}\mathbf{v} \cdot \boldsymbol{\nabla}\mathbf{v} - \boldsymbol{\nabla} \cdot \boldsymbol{\pi} = \boldsymbol{\nabla} p. \tag{4}$$

It follows from this equation that

$$\mathbf{B} \cdot \nabla \mathrm{p} = -\mathbf{B} \cdot (\,\boldsymbol{\rho}\mathbf{v} \cdot \boldsymbol{\nabla}\mathbf{v} + \boldsymbol{\nabla} \cdot \boldsymbol{\pi}).$$

If the radial diffusion of the field lines is weak, then $\mathbf{B}\cdot\nabla p$ is small, and it can be balanced by the small terms that are neglected in the MHD equilibrium equation.

It also follows from Eq. (4) that

$$\mathbf{j}_\perp = \mathbf{B} \times \frac{\nabla p}{B^2} + \mathbf{B} \times \nabla(\boldsymbol{\rho}\mathbf{v} \cdot \boldsymbol{\nabla}\mathbf{v} + \boldsymbol{\nabla} \cdot \boldsymbol{\pi}).$$

The second term on the right hand side of this equation is small compared to the first and can be neglected. We have to be careful in making this argument, as we will be particularly interested in resonant Fourier components, and we will require that the resonant Fourier components of the neglected term be small relative to the resonant Fourier components of the retained term. This can be verified self-consistently. We thus arrive at the equation

$$\mathbf{j}_\perp \approx \mathbf{B} \times \frac{\nabla p}{B^2}.$$

We can now see the advantage of writing the equilibrium equations in the form of Eqs. (1-3). The parallel and perpendicular components of force balance decouple. The perpendicular components of force balance determine the self-consistent equilibrium field. The parallel component of force balance can be regarded as part of the transport problem rather than the equilibrium problem. We can imagine that a finite pressure gradient along the field lines produces a weak flow velocity along the field lines, and that this leads to weak viscous and convective forces which balance $\mathbf{B}\cdot\nabla p$.

Another advantage of writing the equations in this form is that the field line stochasticity enters only through Eq. (3). We must solve a magnetic differential equation along the chaotic field line trajectories.

III. **Solution of Magnetic Differential Equations Along Chaotic Field Line Trajectories**

To solve our magnetic differential equation, we will cast it in the same form as some equations that arise in the theory of plasma turbulence.[2] We assume that we can write the field as $\mathbf{B} = \mathbf{B}_0 + \delta\mathbf{B}$, where $\mathbf{B}_0$ has good flux surfaces and $\delta\mathbf{B}$ is a small perturbation that causes the field lines to be stochastic. We work in a magnetic coordinate system for $\mathbf{B}_0$, that is a coordinate system (ψ, θ, ϕ) such that $\mathbf{B}_0\cdot\nabla\psi = 0$ and $\mathbf{B}_0\cdot\nabla\theta \,/\, \mathbf{B}_0\cdot\nabla\phi \equiv \iota(\psi)$ is constant on the flux surfaces of $\mathbf{B}_0$. Letting $\mu \equiv j_\parallel/B$ and $g \equiv -\boldsymbol{\nabla} \cdot \mathbf{j}_\perp/B$, assuming $B_0^\phi >> B_0^\theta$, and assuming that ι is a monotonic function of ψ in the region of interest so that we can adopt it as our radial variable, we get

$$\frac{\partial\mu(\iota,\theta,\phi)}{\partial\phi} + \iota\frac{\partial\mu}{\partial\theta} + \frac{\delta B^{\iota}}{B_0^{\phi}}\frac{\partial\mu}{\partial\iota} + \frac{\delta B^{\theta}}{B_0^{\phi}}\frac{\partial\mu}{\partial\theta} = g. \tag{5}$$

Compare this with the drift-kinetic equation with a strong toroidal field, fluctuating **E**×**B** velocity $\delta\mathbf{V}_E$, neglecting the parallel nonlinearity:

$$\frac{\partial f(x,v_{\|},t)}{\partial t} + v_{\|}\frac{\partial f}{\partial z} + \delta V_{E,x}\frac{\partial f}{\partial x} + \delta V_{E,y}\frac{\partial f}{\partial y} = 0. \tag{6}$$

The similarity in form suggests that we can apply mathematical techniques developed in the context of statistical turbulence theory to the magnetic differential equation. In particular, we are interested in applying resonance broadening theory. Resonance broadening theory allows us to replace terms containing turbulent fields with effective diffusion terms, and it gives an expression for the effective diffusivity in terms of the autocorrelation function of the turbulent variable. It is intuitively plausible that the primary effect of the turbulent fields will be to produce diffusion. Resonance broadening theory places this intuition on a sounder footing, and provides a quantitative measure of the effect.

When we try to apply resonance broadening theory to Eq. (5) we encounter the problem that resonance broadening theory assumes causality. Eq. (6) determines the time-evolution of f, and it therefore satisfies a causality condition. Impulses do not propagate backwards in time. This causality requirement becomes manifest when we replace the turbulent terms by diffusion terms. In that case, time-reversal changes diffusion to anti-diffusion, so that changing the sign of t changes the nature of the solution. However the physics for $j_{\|}$ must look the same whether we integrate backwards or forwards along the field lines. So μ cannot satisfy a diffusion equation.

The causality issue can be addressed by working in terms of Green's functions. We can express the Green's function for μ as a sum of causal and anti-causal Green's functions. The causal Green's function satisfies a diffusion equation. The anti-causal Green's function satisfies an anti-diffusion equation, that is, a diffusion equation with the signs of the diffusion terms reversed.

There remains an issue of periodicity in the torus with respect to θ and ϕ. The solution for μ must satisfy this periodicity condition. Periodic functions are not spatially causal, so the causal and anti-causal Green's functions cannot satisfy this periodicity condition. To solve this problem, we use a ballooning transformation. The causal and anticausal Green's functions are solved for in the infinite covering space of the flux surfaces for $\mathbf{B}_0$, with $\theta \rightarrow \pm\infty, \phi \rightarrow \pm\infty$. Periodic solutions are constructed by using a shifted-sum representation (ballooning representation).

We can obtain an explicit solution in the limit that the radial diffusion of the chaotic magnetic field line trajectories is weak. In that limit, the diffusion coefficients are small, and the diffusion terms in the equations can be neglected , except near rational surfaces, where radial derivatives can be large. This allows us to solve the equation as a boundary layer problem, with boundary layers at the rational surfaces. In the boundary layers, the equation further simplifies because the radial derivatives of the metric elements and of the Jacobian can be neglected relative to the radial derivatives of the Green's function. That allows us to obtain an explicit solution in terms of Airy functions in the boundary layers. Outside the boundary layers, the solution is just that for $\delta\mathbf{B} = 0$.

IV. Summary

To solve for equilibria having nonzero pressure gradients in stochastic regions, we work with the equations in the form $\nabla \times \mathbf{B} = \mathbf{j}(\mathbf{B})$, with $\mathbf{j}_\perp = \mathbf{B} \times \nabla p / B^2$ and $\mathbf{B} \cdot \nabla (j_\parallel / \mathrm{B}) = -\nabla \cdot \mathbf{j}_\perp$. This form of the equations allows us to decouple the perpendicular and parallel force balance equations. This form also has the virtue that the stochasticity enters entirely through the solution of the magnetic differential equation along the chaotic field line trajectories.

To solve the magnetic differential equation, we cast it in the same form as some standard equations of turbulence theory. This suggests the application of resonance broadening theory. In applying resonance broadening theory there is an issue that the theory assumes causality, and we handle that by working in terms of causal and anti-causal Green's functions. There remains a periodicity issue, and we handle that by using the ballooning representation. We can obtain an analytic solution to the magnetic differential equation in the limit that the chaotic field line trajectories diffuse only weakly in the radial direction.

This model has been incorporated in the PIES 3D equilibrium code, and the code has been applied to calculate reconstructed W7AS equilibria.[The code finds that there is a threshold in β above which a stochastic region appears at the plasma edge, and that the width of the stochastic region increases with further increase in β. The calculated differences in the size of the stochastic region and in the field line diffusion coefficient provide a plausible explanation for the previously puzzling observations on the W7AS stellarator that the current in the divertor control coils has a large effect on the achievable β. Although the presence of such a stochastic region in the experiment could not be verified directly, the calculations were consistent with the available experimental data. The Rechester-Rosenbluth estimate for the contribution of the field-line stochasticity to the energy transport is consistent with the observations. (This estimate is sensitive to the local temperature, so the error bars here are large.) The difference in the pressure profiles observed in the presence of the divertor control coil current in the region predicted to be stochastic is also consistent with this picture.

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References

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