Nonlinear Diffusion Regimes in Stochastic Magnetic Fields

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Abstract. The transport of collisional particles in stochastic magnetic fields is studied using the decorrelation trajectory method. The nonlinear effect of magnetic line trapping is considered together with particle collisions. The running diffusion coefficient is determined for arbitrary values of the statistical parameters of the stochastic magnetic field and of the collisional velocity. New diffusion regimes are found in the conditions for which the trapping of magnetic field lines is effective.

1. Introduction

The problem of anomalous transport induced by the presence of a stochastic magnetic field in a tokamak plasma is studied in the framework of the test particle approach. It is already known that particle collisions have a very strong influence on the effective diffusion [1]. Interacting with diffusion of magnetic field lines, the collisions can determine, depending on the specific conditions, either a much enhanced diffusion (Rochester-Rosenbluth regime) or a subdiffusive cross-field particle transport. Although there are some estimations and even an exact solution for a particular case, this complex process is not completely understood [2]. On the other hand, the diffusion of the magnetic field lines is influenced by the nonlinear process of magnetic line trapping. This trapping process was recently analyzed by means of the decorrelation trajectory method developed in [3]. It was shown that it leads to the decrease of the diffusion coefficient of the magnetic lines and to the change of its scaling law. A realistic description of particle diffusion that includes both particle collisions and magnetic line trapping was not performed until now. This process is studied in this paper using the decorrelation trajectory method [3], [4] which is extended to this much more complicated problem. We show that the interaction of these two nonlinear processes determines new anomalous diffusion regimes.

2. Solution by the decorrelation trajectory method

We consider a magnetic field $\mathbf{B} = B_0 (\mathbf{e}_z + \mathbf{b}(\mathbf{x}, z, t))$, where the confining magnetic field $\mathbf{B}_0 = B_0 \mathbf{e}_z$ is directed along z axis (slab geometry) and the small stochastic component $\mathbf{b}(\mathbf{x}, z, t)$ is in the plane $\mathbf{x} = (x, y)$ perpendicular to $\mathbf{B}_0$. Since the magnetic field is divergence-free, $\nabla \cdot \mathbf{b} = 0$, its two components can be determined from a scalar function $\phi(\mathbf{x}, z)$ as $\mathbf{b}(\mathbf{x}, z, t) = \nabla \times \phi(\mathbf{x}, z, t) \mathbf{e}_z$. The system of equations for guiding center motion is:

$$\frac{d\mathbf{x}}{dt} = \mathbf{b}(\mathbf{x}, z, t) \eta_{\parallel}(t) + \eta_{\perp}(t), \quad \frac{dz}{dt} = \eta_{\parallel}(t). \tag{1}$$

The three stochastic functions $\mathbf{b}(\mathbf{x}, z, t)$, $\eta_{\perp}(t)$ and $\eta_{\parallel}(t)$ are statistically independent: all cross correlations are zero. All these stochastic functions are assumed to be Gaussian, stationary and homogeneous, with zero averages. The Eulerian correlation (EC) of the stochastic potential $\phi(\mathbf{x},z,t)$ is modeled by:

$$A(\mathbf{x},z,t) \equiv \langle \phi(\mathbf{0},0,0)\,\phi(\mathbf{x},z,t)\rangle = \beta^2\lambda_\perp^2 \exp\left(-\frac{z^2}{2\lambda_\parallel^2} - \frac{x^2+y^2}{2\lambda_\perp^2}\right)\exp\left(-\frac{|t|}{\tau_c}\right) \tag{2}$$

where $\langle ...\rangle$ is the average over the realizations of the stochastic potential ϕ, β is the mean square value of the reduced magnetic field $\mathbf{b}$, $\lambda_\parallel$ is the correlation length of the potential ϕ along the main magnetic field $\mathbf{B}_0$, $\lambda_\perp$ is the correlation length in the plane perpendicular to $\mathbf{B}_0$ and τ_c is the correlation time of ϕ. The autocorrelation tensor of the reduced magnetic field components $B_{ij} \equiv \left\langle b_i(\mathbf{0},0,0) b_j(\mathbf{x},z,t)\right\rangle$, $i,j = x,y$, is determined from $A(\mathbf{x},z)$. The collisional velocities $\eta_\perp$, $\eta_\parallel$ are modeled by colored noises with the correlations

$$\left\langle \eta_\parallel(0)\eta_\parallel(t)\right\rangle_c = \chi_\parallel \nu R(\nu t), \quad \left\langle \eta_\perp^i(0)\eta_\perp^j(t)\right\rangle_c = \delta_{ij}\chi_\perp \nu R(\nu t) \tag{3}$$

where $\langle ...\rangle_c$ is the average over the collisional velocity realizations, ν is the collision frequency, $\chi_\parallel = \lambda_{mfp}^2\nu/2$ is the parallel collisional diffusivity, λ_{mfp} is the parallel mean free path, $\chi_\perp = \rho_L^2\nu/2$ is the perpendicular collisional diffusivity, ρ_L is the Larmor radius relative to the reference field and $R(\nu t) = \exp(-\nu|t|)$.

Four dimensionless parameters appear naturally in this problem:

$$\overline{\chi}_\perp \equiv \frac{\chi_\perp}{\lambda_\perp^2\nu}, \quad \overline{\chi}_\parallel \equiv \frac{\chi_\parallel}{\lambda_\parallel^2\nu}, \quad M = \frac{V}{\lambda_\perp\nu} = \frac{\beta\lambda_\parallel}{\lambda_\perp}\overline{\chi}_\parallel^{1/2}, \quad \overline{\tau}_c = \tau_c\nu. \tag{4}$$

They describe respectively the perpendicular and parallel diffusivities, the effect of the stochastic magnetic field and the dimensionless decorrelation time. The parameter which describes the evolution of the magnetic lines, the magnetic Kubo number $K_m = \beta\lambda_\parallel/\lambda_\perp$, appears here as a factor in M, which can be written as $M = K_m\sqrt{\overline{\chi}_\parallel}$.

We use the decorrelation trajectory method following the recent calculations for the influence of particle collisions on the diffusion in electrostatic turbulence [4]. The difference and the supplementary difficulty of the magnetic problem comes from the structure of the velocity $\mathbf{v} = \mathbf{b}(\mathbf{x},z,t)\eta_\parallel(t)$ which is the product of two stochastic processes. They are statistically independent but in the Lagrangian frame they are correlated through the trajectories, due to the space dependence of the magnetic field fluctuations. The latter makes this problem strongly nonlinear. The trajectories also depend on the collisional velocity $\eta_\perp$ and the velocity $\mathbf{v}$ is thus a triple stochastic process in the Lagrangian frame. The calculations evolve according to the following steps.

1) We make the change of variable $\mathbf{x}'(t) = \mathbf{x}(t) - \xi(t)$ in Eq.(1), which introduces the collisional displacements $\xi(t) = \int_0^t \eta_\perp(\tau)d\tau$ in the argument of the magnetic field fluctuations. The Eulerian correlation of $\widetilde{\phi}(\mathbf{x},z,t) \equiv \phi\left[\mathbf{x}+\xi(t),z,t\right]$ is calculated as in [4] and the average effect of the perpendicular collisional velocity $\eta_\perp(t)$ is determined. It consists of the modification of the EC of the magnetic potential (2) by introducing a supplementary time-dependence in addition to the one determined by the finite correlation time of the stochastic magnetic field. The effect of collisions consists in progressively smoothing out the EC of the magnetic potential and in eliminating asymptotically its $\mathbf{x}$-dependence.

2) We define a set of subensembles S of the realizations of the stochastic functions that have given values of the potential $\widetilde{\phi}$, of the magnetic field $\widetilde{\mathbf{b}}$ and of the parallel velocity $\eta_\parallel$ in the point $\mathbf{x} = \mathbf{0}$, $z = 0$ at time $t = 0$: $\widetilde{\phi}(\mathbf{0},0,0) = \phi^0$, $\widetilde{\mathbf{b}}(\mathbf{0},0,0) = \mathbf{b}^0$, $\eta_\parallel(0) = \eta^0$. The correlation of

the Lagrangian velocity can be represented by a sum over the subensembles of the correlations appearing in each subensemble, weighted by the known probability $P\left(\mathbf{b}^0, \phi^0, \eta^0\right)$ of having $\mathbf{b}^0, \phi^0, \eta^0$ at $\mathbf{x} = \mathbf{0}$, $z = 0$ and $t = 0$. The subensemble average Eulerian velocity $\mathbf{V}^S(\mathbf{x},t) \equiv \langle \mathbf{v}\left[\mathbf{x}, z(t), t\right]\rangle_S = \left\langle \mathbf{b}\left[\mathbf{x}, z(t), t\right] \eta_{\|}(t)\right\rangle_S$ is determined.

3) The next step in the decorrelation trajectory method consists to find a deterministic trajectory $\mathbf{X}^S(t)$ in each subensemble S as the solution of the equation

$$\frac{d\mathbf{X}^S}{dt} = M\mathbf{V}^S(\mathbf{X}^S, t) \tag{5}$$

with $\mathbf{X}^S(0) = \mathbf{0}$. The average Lagrangian velocity is estimated as in [4] by the average Eulerian velocity along this decorrelation trajectory $\langle \mathbf{v}\left[\mathbf{x}(t), t\right]\rangle_S \cong \mathbf{V}^S\left[\mathbf{X}^S(t), t\right]$.

We finally obtain the running diffusion coefficient for arbitrary values of the four dimensionless parameters (4) and for given Eulerian correlations of the three stochastic processes that combine in the equations of motion (1). It is the sum of two terms: a direct contribution of the collisional velocity $\eta_\perp$ and the contribution of the stochastic magnetic field:

$$D(t; M, \overline{\chi}_{\|}, \overline{\chi}_\perp, \overline{\tau}_c) = \chi_\perp \left(1 - \exp(-\nu t)\right) + (\nu\lambda_\perp^2) D_{int}(t; M, \overline{\chi}_{\|}, \overline{\chi}_\perp, \overline{\tau}_c). \tag{6}$$

$$D_{int}(t; M, \overline{\chi}_{\|}, \overline{\chi}_\perp, \overline{\tau}_c) = \frac{M}{2\pi}\int_0^\infty dp \int_0^\infty db\, b^3 \exp\left(-\frac{b^2}{2}(p^2+1)\right) \int_{-\infty}^\infty d\eta^0 \eta^0 \exp\left(-\frac{\eta^{02}}{2}\right) X^S(t) \tag{7}$$

where $X^S(t)$ is the component along x axis of the solution of Eq.(5). It depends on the parameters M, $\overline{\chi}_{\|}$, $\overline{\chi}_\perp$ and $\overline{\tau}_c$ as well as on the shape of the Eulerian correlations. This contribution (7) results from the nonlinear interaction of the three stochastic processes. The asymptotic diffusion coefficient is

$$D(M, \overline{\chi}_{\|}, \overline{\chi}_\perp, \overline{\tau}_c) = (\nu\lambda_\perp^2)\left[\overline{\chi}_\perp + D_{int}(M, \overline{\chi}_{\|}, \overline{\chi}_\perp, \overline{\tau}_c)\right] \tag{8}$$

where $D_{int}(M, \overline{\chi}_{\|}, \overline{\chi}_\perp, \overline{\tau}_c)$ is the limit for $t \to \infty$ of $D_{int}(t; M, \overline{\chi}_{\|}, \overline{\chi}_\perp, \overline{\tau}_c)$.

A computer code that calculates the running diffusion coefficient starting from the analytical expression (7) has been developed. It determines the decorrelation trajectories (5) for a large enough number of subensembles and performs the integrals in Eq.(7). The code was tested and the parameters in the numerical calculation were established using the known analytical results [2] concerning the subdiffusive transport.

The general solution (6)-(8) shows that collisional particle diffusion in stochastic magnetic fields is characterized by two kinds of trajectory trappings and contains two decorrelation mechanisms. The latter are produced by the collisional cross field diffusion $\overline{\chi}_\perp$ and by the time variation of the stochastic magnetic field. One of the trapping processes concerns the parallel motion and is determined by collisions which constrain the particles to return in the already visited places with probability one. This parallel trapping leads to a subdiffusive transport in the absence of a decorrelation mechanism. The second kind of trapping concerns the magnetic lines which at $K_m > 1$ wind around the extrema of the vector potential.

3. Subdiffusive transport

In the case of a static stochastic magnetic field ($\tau_c \to \infty$) and in the zero Larmor radius limit corresponding to negligible cross field collisional diffusion, $\chi_\perp = 0$, a subdiffusive transport was obtained. The magnetic line trapping that appears when $K_m > 1$ does not affect the asymptotic time-dependence of the running diffusion coefficient. There is however a significant effect of the nonlinear process of magnetic line trapping but it appears to be localized in time. It determines

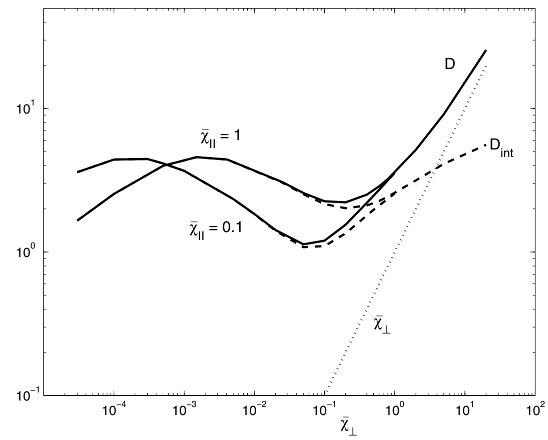

Figure 1: The asymptotic diffusion coefficient as a function of $\overline{\chi}_{\perp}$ (continuous line) compared with the two terms in Eq.(8). $M = 10$, $\overline{\tau}_c = \infty$.

a transient decay of the running diffusion coefficient $D(t)$. At later times, after the characteristic return time of the parallel motion, a nonlinear build-up of Lagrangian velocity correlation is obtained. The parallel motion eventually washes out the effect of the magnetic line trapping. Consequently, the asymptotic behavior of the running diffusion coefficient in the collisionless case is exactly the same as in the quasilinear conditions when the stochastic magnetic field does not generate magnetic line trapping. However, the rather nontrivial evolution of the running diffusion coefficient leads to anomalous diffusion regimes when a decorrelation mechanism is present.

4. Diffusive transport induced by collisional decorrelation

The stochastic collisional velocity $\eta_{\perp}(t)$ in Eq.(1) moves the particles away from the magnetic lines and consequently it has a decorrelation effect leading to diffusive transport. The asymptotic diffusion coefficient is determined from Eqs.(6)-(8) using the numerical code we have developed. Some results are presented in Figure 1 where the asymptotic diffusion coefficient Eq.(8) is represented as a function of $\overline{\chi}_{\perp}$. The two components D_{int} and $\overline{\chi}_{\perp}$ are also represented. One can see that at small collisional diffusion $\overline{\chi}_{\perp} \ll 1$, the non-linear interaction term largely dominates the collisional term while at large collisional diffusion $\overline{\chi}_{\perp} \gtrsim 1$, the non-linear term is only a correction to $\overline{\chi}_{\perp}$. Thus, the subdiffusive transport appearing at $\overline{\chi}_{\perp} = 0$ is transformed by a small collisional cross field diffusion into a diffusive transport with a diffusion coefficient that can be several orders of magnitude larger than $\overline{\chi}_{\perp}$. The dependence of the diffusion coefficient on $\overline{\chi}_{\perp}$ is rather nontrivial. There is at very small $\overline{\chi}_{\perp}$ an increase of D up to a maximum. Then, at larger $\overline{\chi}_{\perp}$, the nonlinear interaction of the parallel and perpendicular trapping with the collisional decorrelation generates an unusual transport regime, in which the effective diffusion coefficient decreases as the collisional diffusion $\overline{\chi}_{\perp}$ increases. A minimum of D is obtained when $\overline{\chi}_{\perp}$ determines a decorrelation time of the order of the return time of the parallel motion. At larger $\overline{\chi}_{\perp}$ the nonlinear contribution D_{int} increases again with the increase of $\overline{\chi}_{\perp}$ but this contribution begins to be comparable and eventually negligible compared to the collisional diffusion coefficient $\overline{\chi}_{\perp}$.

5. Diffusive transport in time-dependent stochastic magnetic fields

In a time-dependent stochastic magnetic field with finite τ_c the configuration of the stochastic field $\mathbf{b}(\mathbf{x}, z, t)$ changes, the magnetic lines move and consequently the perpendicular velocity of the particles is decorrelated leading to diffusive transport. We determine here the diffusion

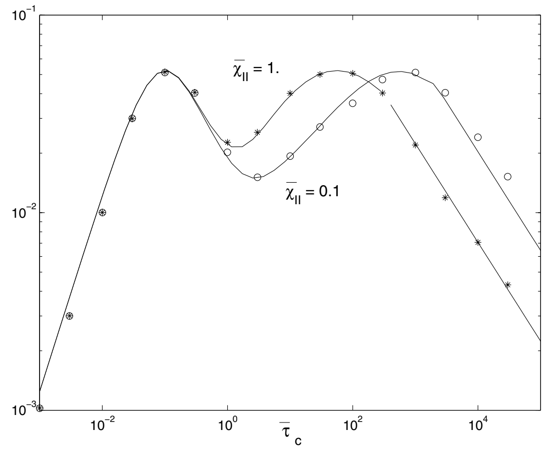

Figure 2: The asymptotic diffusion coefficient normalized with $(\lambda_\perp^2 \nu)M^2$ as a function of $\overline{\tau}_c$. The continuous lines represent the running diffusion coefficient as function of t for the subdiffusive transport obtained at $\overline{\tau}_c = \infty$. $M = 10$, $\overline{\chi}_\perp = 0$.

coefficient in such time-dependent fields in the limit of zero Larmor radius ($\overline{\chi}_\perp = 0$), starting from the general solution (6). The following diffusion regimes can be observed in Figure 2, in the nonlinear conditions when the trapping of the magnetic lines is effective ($K_m > 1$). The quasilinear regime at small correlation times with $D_0 \approx M^2\overline{\tau}_c$ is characterized by a fast time-variation which prevents trajectory trapping. At larger correlation times the magnetic lines can be trapped before the stochastic magnetic field changes and the parallel motion is ballistic. In these conditions the diffusion regime is similar to that described in [4] for the electrostatic turbulence: the diffusion coefficient decreases with the increase of $\overline{\tau}_c$. A minimum of the diffusion coefficient appears at $\overline{\tau}_c$ of the order of the return time of the parallel motion. This is followed at larger $\overline{\tau}_c$ by an anomalous increase determined by the interaction of the parallel trapping with the magnetic line trapping which generates correlation of the Lagrangian velocities. At very large correlation times the diffusion coefficient decreases as $D \approx K_m^2 \overline{\tau}_c^{-1/2} \overline{\chi}_{||}^{1/2}$.

6. Conclusions

We have studied here the transport of collisional particles in stochastic magnetic fields using the decorrelation trajectory method. A rather complex dependence of the diffusion coefficients on the plasma parameters was obtained. This is determined by the nonlinear process of magnetic line trapping interacting with the collisional velocity. We have shown that even without changing the characteristics of the stochastic magnetic field, the diffusion coefficient can be strongly influenced by the parameters which describe particle collisions. A minimum of the diffusion coefficient was obtained for decorrelation times of the order of the average return time for the parallel motion.

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