TH/P1-10

Relativistic Fluid Theories — Self Organization

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Abstract. Developments in two distinct but related subjects are reviewed: 1) Formulation and investigation of closed fluid theories which transcend the limitations of standard magnetohydrodynamics (MHD), in particular, theories which are valid in the long mean free path limit and in which pressure anisotropy, heat flow, and arbitrarily strong sheared flows are treated consistently, and 2) Exploitation of the two-fluid theories to derive new plasma configurations in which the flow-field is a co-determinant of the overall dynamics; some of these states belong to the category of self-organized relaxed states. Physical processes which may provide a route to self-organization and complexity are also explored.

1. Relativistic Magnetized Fluid Plasmas

The subject matter of this paper will be described under two separate headings. In this section we give a compressed summary of our recent efforts in deriving a relatively complete set of fluid equations describing the dynamics of a relativistic magnetized plasma with arbitrary temperature and flow-speed.

The fundamental generic hurdle to the construction of fluid theories is the problem of closure. The evolution of each successive moment depends on a higher order moment leading to an infinite set of equations which are of little use unless a truncation prescription can be devised i.e., we can express the nth order moment fully in terms of the the lower $n-1$ moments. The most successful and widely used fluid theory of magnetized plasmas, the magnetohydrodynamics (MHD), affects this closure by assuming a plasma stress tensor (energy-momentum tensor) dictated fully by a local thermodynamic equilibrium. We remind the reader that a complete knowledge of the stress tensor is sufficient to calculate the charge and current densities needed to complete the Plasma-Maxwell system.

Although MHD does capture several key features of a magnetized plasma (the electromagnetic nature of its flow ($E \times B$ drift), for example), its reliance on the thermodynamic closure fails to do justice to the dominant determinant of plasma dynamics, the electromagnetic force.

It turns out that the quest for a more consistent treatment (which, for example, allows a proper role for electromagnetism in determining the stress tensor) is greatly aided and abetted by the demands of special relativity; the form of the energy momentum tensor, the centerpiece of the theory, is dictated almost entirely by the constraints of Lorentz invariance. Fortunately the most general energy-momentum tensor, so found, is also physically reasonable (and warranted) and leads to a consistent and relatively clean fluid description.

Space limitations permit only a highly schematic review of the derivation. We begin with the following three exact (collisionless) conservation laws for each species of the plasma Ref. [1],

$$\frac{\partial \Gamma^{\nu}}{\partial x^{\nu}} = 0 \tag{1}$$

$$\frac{\partial T^{\mu\nu}}{\partial x^{\nu}} - eF^{\mu\nu}\Gamma_{\nu} = 0 \tag{2}$$

$$\frac{\partial M^{\mu\alpha\beta}}{\partial x^{\mu}} - e(F^{\alpha\nu}T_{\nu}^{\ \beta} + F^{\beta\nu}T_{\nu}^{\alpha}) = 0. \tag{3}$$

Here the flux density $\Gamma^{\mu} = n_R U^{\mu}$ (n_R is the rest frame density, and U^{μ} is the velocity four vector), the energy-momentum tensor $T^{\mu\nu}$, and the "stress-flow" tensor $M^{\mu\alpha\beta}$ are, respectively, the first second and third momentum moments of the distribution function which obeys

$$\frac{p^{\mu}}{m}\frac{\partial f}{\partial x^{\mu}} + g^{\mu}\frac{\partial f}{\partial p^{\mu}} = 0 \tag{4}$$

where $g^{\mu} = eF^{\mu\nu}p_{\nu}$ is the electromagnetic four-force experienced by the particle of charge e, $F^{\mu\nu}$ is the Faraday tensor and p^{ν} is the particle momentum.

The magnetized plasma assumption allows us to order the derivative terms in Eqs. (2) and (3) to be small compared to the terms proportional to the Faraday tensor. As a consequence we find the emergence of the MHD ($E \times B$) drift from (2), while the solution of (3) leads to a stress tensor of the form

$$T^{\mu\nu} = b^{\mu\nu}p_{||} + e^{\mu\nu}p_{\perp} + hU^{\mu}U^{\nu} + q_{||}(k^{\mu}U^{\nu} + U^{\mu}k^{\nu}) \tag{5}$$

In (5), $p_{||}$, $p_{\perp}$, h, and $q_{||}$ are Lorentz scalars corresponding respectively to parallel pressure, perpendicular pressure, enthalpy density, and parallel heat flow in the rest-frame, The four-vector k^{μ} is orthogonal to U^{μ}, and the tensors $b^{\mu\nu}$ and $e^{\mu\nu}$ are parallel and perpendicular quasi projectors constructed from the Faraday tensor Ref. [1].

For a complete knowledge of the stress tensor, then, we must have eight evolution equations for the eight unknown scalar functions occurring in (5): $n_R(x,t)$, $p_{||}(x,t)$, $p_{\perp}(x,t)$, $h(x,t)$, the three independent components of $U^{\mu}(x,t)$, and $q_{||}$. With (1), and ($E \times B$) drift providing three such relations, closure requires finding five additional equations for each plasma species. By an appropriate manipulation of Eqs. (2) and (3), and by invoking a representative generalized distribution function, this task was successfully accomplished in Ref. [1] and Ref. [3]. Nonrelativistic approximations of the final equations, relevant for fusion physics, are explicitly displayed in Ref. [3] and collisions are added to the general formalism in Ref. [2]. The newly constructed framework provides a much more general (than MHD, for example) system of closed fluid equations capable of handling physics (like pressure anisotropy, parallel heat flow) inaccessible to earlier theories.

2. Self Organization-Complexity

Recent theoretical investigations [4] have demonstrated the possible existence of new magnetofluid states that differ qualitatively from those accessible to either neutral fluids or to conventional MHD plasmas. These states originate from a strong interaction of the plasma flow-field with the magnetic field, and are predicted to appear if plasmas with strong velocity shear flows (with large initial values of both magnetic and magneto-fluid helicity) are created and allowed to relax. The relaxation process is supposed to take place under the dual constraints of the constancy of the standard magnetic helicity (electron helicity when electron inertia is neglected), $\int_\Omega \boldsymbol{A}\cdot\boldsymbol{B}\; dx$, and the generalized (ion) helicity, $\int_\Omega (\boldsymbol{A}+\boldsymbol{V})\cdot(\boldsymbol{B}+\nabla\times\boldsymbol{V})\; dx$. The dynamic invariance of these two helicities is expected to force the plasma to self-organize and relax to a possible long-lived quasi equilibrium state away from thermal equilibrium.

The investigation of these states bears critically upon basic plasma confinement and heating issues in both natural and laboratory plasmas because: a) unlike conventional confinement, based on the existence of closed nested surfaces, this mechanism pertains even in the absence of symmetry (relevant to most astrophysical plasmas), and b) unlike conventional relaxed states, the extra invariant enables additional non-thermal features such as pressure gradients to be sustained to reach large beta values.

Conventional route to self organized relaxation is through the construction of variational principles; one sets up a suitable target functional whose constrained minimization leads to the defining equations of the relaxed state. The well-known Taylor state, for instance, is obtained by minimizing the magnetic energy $\int_\Omega |\boldsymbol{B}|^2\; dx$ for a fixed magnetic helicity. Because of the fundamental importance of relaxed states it is imperative to seek answers to the following questions: 1) What constitutes a relaxed state, and 2) What may be the valid routes to its formation.

We have begun a critical examination of these and related issues. Preliminary investigations reveal that self-organized relaxation occurs only under rather restrictive conditions and is not a general tendency of plasmas. In particular not all variational principles are well-posed and not all solutions to seemingly standard variational principles qualify as relaxed states which must, at least, be stable equilibria.

Much of our exploration of the relaxation phenomena is carried out for an electron-ion plasma described by the two-fluid model known as Hall MHD. The idea is to delineate a mathematically sound and physically plausible pathway to the creation of new and interesting self-organized states. We must remind the reader that in all these considerations although one formally deals with an ideal system, dissipation is supposed to play a fundamental background role in bringing about relaxation. Thus of all the ideal functionals that are employed in the construction of a variational principle, the target functionals must be relatively more fragile (change under dissipation) or less rugged as compared to the functional which are kept invariant. Otherwise the variational principle turns out to be ill-posed leading to trivial solutions. For the Hall MHD system we show that the conventional method of taking the total energy (fluid plus magnetic) as the target functional to be minimized under the two helicity constraints is ill-posed due to the discontinuity of the ion helicity in the topology of the Hilbert space endowed with the energy norm [4].

Thus energy is not the principal variable that characterizes self-organization in the Hall vortex dynamics.

We have proposed the ion enstrophy $\int_\Omega |\boldsymbol{B} + \nabla \times \boldsymbol{V}|^2 \, dx$ as an appropriate target functional for a well-posed variational principle. Then aided by a process of adjustment of the most fragile of the three ideal constants of the system (two helicities and the energy), minimization of this measure of dissipation and turbulence can be shown to lead to the equilibrium Double Beltrami states [4]. A continued examination of this system has also led us to propose a reasonable definition of complexity and a possible theoretical route to it. Complexity may be defined as the simultaneous existence, and interaction of structures on disparate scales and the pathway to complexity may be the singular perturbations on nonlinear systems which constitute a robust mechanism for creating patterns at new characteristic scales.

3. Applications-Results

We are beginning to apply both these formulations towards solving concrete problems in laboratory as well as astrophysical plasmas. The new magnetized fluid theory takes the original magnetohydrodynamics (MHD) program to its logical limit; It has now becomes possiible to investigate in a systematic manner the exciting new physics associated with plasmas with strong flows, pressure anisotropies, relativistic regimes, and long mean free paths. Initial analytic studies in the non relativistic limit (appropriate, for example, for the toroidal confinement systems) reveal that the new theory's predictions, as compared to MHD, are nearer the predictions of the drift kinetic equation. In particular the prediction of the ion sound speed is closer to the exact kinetic value. The inclusion of parallel heat flow could significantly change island growth rates. The validity of this theory in the long mean free path limit makes it potentially valuable in the tokamak edge regions. The theory also provides, among other things, an accurate physical description of the decay of poloidal rotation in a tokamak. In summary, Some immediate applications include divertor flow analysis, H-mode studies, and the Maryland Centrifugal Experiment (MCX).

The ideas of treating plasma flows at par with the magnetic field as fundamental determinants of the plasma dynamics has led to a new and profitable approach to a host of unsolved problems in plasma astrophysics. The emergence of Double Beltrami states, which reflect a strong coupling and interaction between the flow-field and the magnetic filed, has been one of the most important consequences of this investigation. Double Beltrami states have been invoked in solving a variety of astrophysical problems: 1) Formation and heating of the solar corona [6], 2) Generation of flows in the subcoronal region by a magneto-Bernoulli mechanism [7], 3) Providing a mechanism for explosive events like coronal mass ejections (in which, for example, the magnetic energy is converted to mass flow energy) in the solar corona [8], and 4) Extension of the extant dynamo theories to Hall dynamos [9] which may be considerably more efficient in certain ranges of parameters. This approach seems to lend a physics "unification" to such different natural phenomena.

An experiment to create the double Beltrami equilibria in the laboratory has just begun at the university of Texas at austin. Creation of flows is one of the principal initial goals.

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