Intermittency and Structures in Drift Wave Turbulence: Towards a Probabilistic Theory of Anomalous Transport

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Abstract. For forty years, plasma turbulence theory has been based on notions of ‘mixing length’, ‘diffusion’, and other ideas from mean field theory. However, there is now a plethora of evidence, from both simulations and experiments, that plasma turbulence is strongly intermittent and that transport is bursty, and composed of a multiplicity of "events" of differing scales. In this case, "predictive modelling via transport coefficients" is an oxymoron. Thus, a fundamentally probabalistic treatment is necessary, i.e. one should aim to calculate probability distribution function (pdf) of flux, rather than an anomalous diffusion coefficient. Of particular interest in this approach is the tail of the pdf, which describes large ‘events’ or avalanches. In this paper, we discuss recent progress in formulating a probabalistic theory of drift wave turbulence and transport. We also report on progress in understanding the more general question of structure formation, particularly in the context of sheared flow.

1. Introduction

There is now a plethora of evidence from simulation and experiment that plasma turbulence is highly intermittent and turbulent transport has a fundamentally "bursty'" character [1]. It is thus necessary to develop a probabilistic theory of plasma transport, focusing on calculating the probability distribution function (PDF) of flux, rather than anomalous transport coefficients. This follows from, say, the need to understand the frequency of large heat discharges from the plasma, which in turn affects the distribution of peak heat loads. Interestingly, intermittent transport often results from rare, large events which are accompanied by coherent structures such as zonal flows, streamers, blobs, and vortices. These structures are well known to play a crucial role in transport dynamics [2]. For instance, zonal flows (mainly poloidal flows that are radially localized) inhibit the radial transport by shearing eddies making up turbulence while streamers (radially elongated and poloidally localized flows) enhance it. Therefore, two of the most fundamentally important questions in the prediction of transport are (i) the formation of coherent structures and (ii) the effects of these structures on transport.

Most of the works dealing with the first issue (the formation of structures) has so far adopted a mean field theoretical view on the basis of quasi-linear closure and ray chaos. Although much insight has been gained in this approach, the formation of structure may be a strongly nonlinear phenomenon, whose description requires a non-perturbative method. Indeed, renormalized perturbation theory can easily make an exponentially large error in predicting PDFs in cases where structure formation is crucial. In particular, the formation of structure can be triggered by noise, in which case the PDF of the formation of structure itself is a quantity of ultimate interest. (For instance, an interesting issue is the prediction of the PDF of L→H transition. [3]) On the other hand, these coherent structures, once formed, can cause intermittent and bursty transport. Especially, intermittent transport leads to non-Gaussian PDF of flux. The deviation from Gaussian statistics manifests the failure of random phase approximation, underscoring the need for a non-perturbative method. Note that rare events contributing to PDF tails can play a major role in transport when PDF tails are significantly enhanced over Gaussian PDF.

In this paper, we focus on the effects of structures on transport. In Section 2, we study the effect of coherent structure on PDF of flux by employing a non-perturbative method, the so–called instantons [4]. Specifically, we present results of PDF tails for Reynolds stress in a simple drift wave turbulence [5] and for heat flux in an toroidal ion temperature gradient (TITG) mode. In Section 3, we discuss a simple model of intermittency in drift wave-zonal flow turbulence, based, in part, on analogies with continuum models of self-organized criticality [6]. The model is non-perturbative, as basic symmetry principles, rather than renormalized techniques presuming Gaussianity, are used in the derivations. The conclusion is provided in Section 4.

TABLE 1

Contrast	Formation of structure (conventionally)	Effect of structure on PDF flux
Method	Mean field theory (quasi-linear closure)	Instantons (non-perturbative)
Assumption	Ray chaos, random coupling, etc.	Spatial form of coherernt structure

2. Non-perturbative computation of PDF flux

Coherent structures often accompany bursty and intermittent transport, leading to non-Gaussian PDF of flux. In particular, this means that heat and particle loads may be concentrated in "large event", the frequency of which should be determined. The effect of coherent structure on PDF of flux is investigated in the following. A non-perturbative method that is utilized in our analysis is called instantons. Before proceeding with the computation of PDF by using this method, we provide some explanation for the physical meaning of instantons.

In a classical dynamical system, instantons give the transition probability amplitude between two states which have different nonlinear structures. This may be understood intuitively as follows. Associated with each coherent structure is a (nonlinear) solution which takes certain value of an ideal (topological) invariant. In the presence of dissipation and an external noise, the ideal invariant is broken, and thus there is a finite probability that a system evolves from one state to another with different solutions. Instantons capture the probability of the transition between two nonlinear solutions (or structures). Since this transition occurs rapidly in time, instantons are temporary localized (as its name indicates) and can thus naturally be related to the burstiness of events (see *FIG*. 1).

To exploit this idea in the prediction of PDF of flux in an analytically tractable manner, we take one structure to be the vacuum and the other to be a non-trivial entity. That is, we assume that a system is initially in a quiet state with no energy when an external random forcing is turned on. As the forcing injects energy into the system, there is a finite probability of the formation of coherent structures in the long time limit. Instantons capture the creation process of these structures. Once these structures are formed, they participate in transport, thereby contributing to the PDF tails of flux. As may be clear from this argument, the PDF tails of flux will then be determined once the transition amplitude to various structures is available. Unfortunately, the latter, however, requires the knowledge of a complete set of coherent structures in a system, which is almost impossible. Therefore, to utilize an instanton method, some insight is necessary as to what kind of structure is likely to be excited by a given forcing. One possible candidate for this nonlinear structure, which we are going to use, is obviously an exact nonlinear solution in the absence of dissipation and forcing. Once its spatial form is fixed, an instanton then give the transition probability to different amplitude of this solution. For instance, an instanton for a dynamical variable u takes the form of $u(\underline{x},t) = F(t)u_0(\underline{x})$ with $F(t \to -\infty) = 0$. Here, $u_0(\underline{x})$ denotes the spatial form of a coherent structure and *F(t)* is a temporally localized amplitude, representing its creation process (see *FIG.* 1). The distribution of *F(t)* determines the PDF of any flux which is a function of u in the long time limit (see *FIG. 2*).

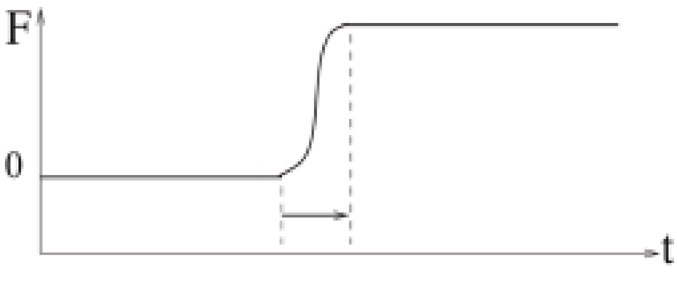

FIG. 1: Instanton is localized in time

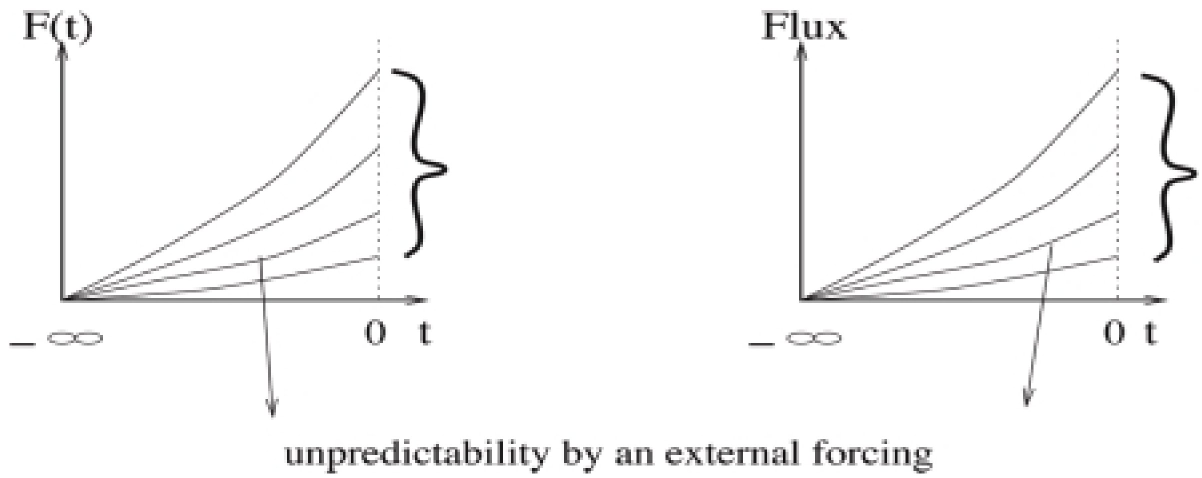

FIG. 2: Uncertainty in F determines PDF of flux

In the following, we consider the two examples. First, we show in detail how to compute PDF tail of momentum flux in a simple drift wave turbulence. We then extend this model to a toroidal ITG and study heat flux PDFs.

2.1 PDF for momentum flux in drift wave turbulence

We adopt a forced Hasegawa-Mima turbulence [7] as the simplest paradigm of drift wave turbulence and study the PDF tail of local Reynolds stress *R* (momentum flux) [5]. The heat transport will be discussed in Section 2.2. Momentum transport is responsible for the generation of a shear flow, and thus a 'momentum transport event' may be thought of as a localized burst of shear generation such as that which occurs when the L $\rightarrow$ H transition is triggered.

A forced Hasegawa-Mima equation [7] takes the following form:

$$\left(1-\nabla^2\right)\partial_t\phi + v_*\partial_y\phi - v\cdot\nabla\nabla^2\phi = f\,. \qquad (1)$$

Here, the notation is standard; *x* and *y* denote local radial and poloidal directions, respectively; $v_* = \rho_s^2\Omega_i/L_n$ is the drift velocity due to radial density gradient; $L_n = -\left(\partial_X n_0/n_0\right)^{-1}$ is the (background) density length scale; ϕ, $v = -\nabla\times\phi\hat{z}$, and *f* are electric potential, $\boldsymbol{E}\times\boldsymbol{B}$ advection velocity, and external forcing. Note that Eq. (1) is non-dimensionalized by measuring the length, velocity, and ϕ in units of ρ_s, c_S, and T_e/e.

The external forcing *f* introduces a random noise in the system and thus leads to the formation of a coherent structure (instanton). To simplify analysis, we assume the Gaussian statistics for the forcing with white noise in time as follows:

$$\left\langle f(\underline{x},t)\,f(\underline{x}',t')\right\rangle = \delta(t-t')\kappa(\underline{x}-\underline{x}'), \qquad (2)$$

and $\langle f\rangle = 0$. Furthermore, we assume κ in Eq. (2) is roughly parabolic for $|\underline{x}-\underline{x}'| < L$.

By exploiting the Gaussian statistics of the forcing [8], we can formally express the PDF for the local Reynolds stress $v_x v_y(\underline{x}_0) = -\partial_x\phi\,\partial_y\phi(\underline{x}_0)$ in terms of a path integral [9,8,4]:

$$P(R;\underline{x}_0) = \left\langle \delta\left(v_x v_y\Big|_{\underline{x}_0} - R \right) \right\rangle = \int d\lambda\, e^{i\lambda R} I_\lambda, \tag{3}$$

where $I_\lambda = \int D\phi D\bar{\phi}\, e^{-S_\lambda}$. In Eq.(3), the angular brackets denote the average over the random forcing *f*, and S_λ is the effective action given by

$$S_\lambda = -i\int d^2xdt\bar{\phi}\left[(1-\nabla^2)\partial_t\phi + v_*\partial_y\phi - v\cdot\nabla\nabla^2\phi\right] + \frac{1}{2}\int d^2x\, d^2x' dt \bar{\phi}(\underline{x})\kappa(\underline{x}-\underline{x}')\bar{\phi}(\underline{x}')$$
$$+ i\lambda\int d^2xdt\left(-\partial_x\phi\,\partial_y\phi\right)\delta(t)\delta(\underline{x}-\underline{x}_0). \tag{4}$$

Here, $\bar{\phi}$ is a conjugate variable, playing the role of Lagrange multiplier. Since we are interested in PDF tails, we can calculate the path integral in Eq.(3) by a saddle–point method for large *R* (and λ). Saddle–point solutions for ϕ and $\bar{\phi}$, which minimizes the effective action with the initial condition $\phi(t\to-\infty) = \bar{\phi}(t\to-\infty) = 0$, constitute instantons. Once instanton solutions are found, the effective action, and consequently PDF tails, can be straightforwardly computed to leading order.

As discussed previously, the spatial form of instantons can be given by coherent structure which is an exact nonlinear solution of a dynamical system in the absence of dissipation and external forcing. In the case of Hasegawa-Mima equation, there is such an exact nonlinear solution, known as a modon [10]. This is a bipolar vortex soliton, propagating perpendicular to both the density gradient and toroidal magnetic field. Thus, we substitute the ansatz in Eq.(4) that an instanton has a spatial form given by a modon with unknown amplitude depending on time. That is,

$$\phi(\underline{x},t) = \psi(\underline{x},t)F(t), \tag{5}$$

where $\psi(\underline{x},t) = \psi(x, y-Ut)$ is a modon solution [10]. The time variation of ϕ, i.e., *F(t)* in Eq.(5), representing the excitation of a modon by an external forcing, can be associated with the degree of burstiness of an event. This temporal evolution *F(t)* of the instanton is found to be

$$F(t) = \frac{1}{1 - \dfrac{F_0 - 1}{F_0}\exp\left\{-\sqrt{\gamma}t\right\}}, \tag{6}$$

where $F_0 = 1 - \left(R/\xi_0\right)^{1/2}$. Here, γ is a constant depending on the parameters values for forcing and modon, and the boundary conditions $F(t=0) = F_0$ and $F(t\to-\infty) = 0$ have been used in obtaining *F(t)*; $\xi_0 = -\partial_X\psi\,\partial_y\psi(\underline{x}_0)$ is local Reynolds stress associated with the modon solution.

As expected, the instanton is localized within a time interval proportional to $1/\sqrt{\gamma}$. Note that a non-vanishing projection of the forcing onto the modon is necessary for the existence of a non-trivial solution for *F*. Thus, the (spatial) 'overlap' between the forcing and modon is critical for the generation of the modon. This projection is likely to be maximized by choosing the characteristic scale of the forcing to be comparable to that of a modon. In more general terms, which coherent structure is likely to be generated is determined by the nature of the forcing, with different forcings giving rise to different manifestations of intermittency. Finally, note that the use of the special modon solution here is motivated mainly by convenience. In general, the instanton procedure could be implemented for any emperical eigenfunction.

By using instanton solution Eq. (6), the PDF tail for local Reynolds stress R can easily be obtained, to leading order, as

$$P(R;\underline{x}_0) \sim \exp\left\{-\frac{2}{3q}\left(\frac{R}{\xi_0}\right)^{3/2}\right\}, \tag{7}$$

for $R/\xi_0 > 0$. Here, $q = \left|4\kappa_0/\left(\sqrt{\gamma}Q\right)\right|$; Q is a parameter depending on the forcing and modon. Eq. (7).is the probability of finding a local Reynolds stress R, normalized by ξ_0, at $\underline{x} = \underline{x}_0$, which can be viewed as a transition amplitude from an initial state, with no fluid motion, to final states with different values of R due to a modon in the long time limit. Interestingly, it is a stretched exponential, exhibiting non-Gaussian statistics and intermittency. This stretched exponential PDF tail implies that the probability of the generation of (large-scale) shear flow with a large amplitude is likely to be enhanced over Gaussian prediction. Note that in the absence of forcing (i.e., $\kappa_0 \to 0$), $P \to 0$, simply because the instanton cannot form without the forcing.

2.2 PDF for heat flux in a TITG

We now consider PDF for heat flux in the two dimensional TITG in a slab geometry [11]. The governing for electric potential ϕ and pressure perturbation p are are given by

$$\begin{aligned}&\partial_t(1-\nabla_\perp^2)\phi - [\phi + \tau p, \nabla_\perp^2\phi] + \tau[\partial_i\phi, \partial_i p] \\ &\quad + v_*[1 - 2\varepsilon_n + \tau(1+\eta_i)\nabla^2]\,\partial_y\phi - 2\varepsilon_n v_* \tau\,\partial_y p = f, \qquad (8)\\ &\quad \partial_t p + [\phi, p] + v_*(1+\eta_i)\partial_y\phi = 0, \qquad (9)\end{aligned}$$

where $\tau = T_{i0}/T_{e0}$, $\varepsilon_n = L_n/R$, $\eta_i = L_n/L_T$, $L_n = -\left(\partial_x \ell n n_0\right)^{-1}$, $R = \left(\partial_x \ell n B_0\right)^{-1}$, $L_T = -\left(\partial_x \ell n T_{i0}\right)^{-1}$, and $v_* = \rho_s/L_n$. f is an external random forcing.

In the absence of an external forcing and dissipation, the coupled equations (8) and (9) have exact nonlinear solutions $p \sim \phi$, $\phi(\underline{x},t) = F(t)\psi(\underline{x},t)$ and where ψ is a modon [2]. Thus, we again assume that a coherent structure, causing bursty heat transport, is a modon so that instanton solutions take the following form $p \sim \phi$, $\phi(\underline{x},t) = F(t)\psi(\underline{x},t)$. A similar analysis then gives us the PDF tail for local heat flux $H = p v_x$ as

$$P(H;\underline{x}_0,t_0) \sim \exp(-\beta H^{3/2}), \tag{10}$$

where β is a constant depending on the property of the forcing and spatial structure of the modon. This simple example thus demonstrates a non-trivial intermittency in the heat flux. Note that Eq.(10) should be thought of as the PDF of local heat flux in the saturated state, as the physics of linear growth and structure formation are not addressed in its derivation.

3. SOC, zonal flows

There has been a great deal of recent theoretical activity driven by the realization that a principal self-regulatory mechanism for drift wave turbulence is shearing by self-generated zonal flows. Zonal flow generation is an intrinsically non-local interaction process in $\underline{k}$ space, so that larger scale flows exert a strain on smaller scale drift waves, and thus leave a 'footprint' on the small scale modes [13,14]. Such a 'footprint' is a prima-facie signature of intermittency, in that it necessitates corrections to simple scaling analyes. Thus, one *must* confront intermittency when constructing a model of drift wave-zonal flow turbulence, and its impace on transport.

A simple model, based on the analogy between sand flow in the pile and the flux of drift wave action in k_r, has been constructed [6]. Note that zonal flow generation is directly tied to the k_r-flux of the drift wave population density N, since shearing tilts vortices and generates high k_r while re-enforcing (and thus amplifying, via modulational instability) a seed shear. In this vein, transfer to large k_r is analogous to displacement down the pile, $N(k_r)$ to local pile height $h(x)$, and straining-induced Γ_{k_r} to the pile flux. Thus, the deviation from the SOC profile δN saisfies:

$$\frac{\partial \delta N}{\partial t} + v_g \frac{\partial \delta N}{\partial x} + \frac{\partial}{\partial k_r} \Gamma_{k_r}(\delta N) = \tilde{C}(N). \qquad (11)$$

Using arguments of joint reflection [15] symmetry the δN evolution equation can be approximated by

$$\frac{\partial}{\partial t}\delta N + v_g \frac{\partial}{\partial x}\delta N + \alpha \delta N \frac{\partial}{\partial k_r}\delta N - \beta \frac{\partial^2 \delta N}{\partial k_r^2} = -\gamma \tilde{N} + \tilde{f}, \qquad (12)$$

where α and β are model-dependent coefficients and $\tilde{f}$ is random forcing. Note that the straining term is replaced by a non-linear self-refraction term, since $V_E \sim N$ via Reynolds stress.

Equation (12) is manifestly a noisy Burgers equation in k_r-space, modulo the group-propagation and damping term. The straightforwardly predictable shock solutions may be interpreted as 'shearing events', i.e. bursts of quasi-coherent shearing during which $|k_r| \sim t$. Thus, the turbulence of wave-packets undergoing shearing by zonal flows can be thought of as a gas of k_r-space shocks.

A central goal of any theory of drift wave-zonal flow turbulence is to predict the pdf of drift wave intensity fluctuations. Here, this amounts to predictingthe probability distribution of packets $(\delta N > 0)$ and cavitons $(\delta N < 0)$. Using results available from studies of Burgers turbulence with forcing spectrum $\left\langle \tilde{f}^2 \right\rangle_k \sim 1/k$, we can predict $P(\delta N) \sim (\delta N)^{-4}$ for $\delta N < 0$, and that $P(\delta N)$ decays faster than Gaussian for $\delta N > 0$ [4,16]. Thus $P(\delta N)$ is *strongly asymmetric* - a consequence of the contrast between 'shock' and 'ramp' evolution. The power law tail of the caviton distribution is consistent with expectations of strnog intermittency.

4. Conclusions

We have examined two simple, non-perturbative models of intermittency in drift wave turbulence. Research is ongoing and will be presented in future publications.

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