Statistical Theory of L-H Transition and its Implication to Threshold Database

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Abstract

A statistical model for the bifurcation of the radial electric field E_r is analyzed in view of describing L-H transitions of toroidal plasmas. Noise in micro fluctuations is shown to lead to random changes of E_r, if a deterministic approach allows for more than one solution. The probability density function for and the ensemble average of E_r are obtained. The L-to-H and the H-to-L transition probabilities are calculated, and the effective phase limit is derived. Due to the suppression of turbulence by shear in E_r, the limit deviates from Maxwell's rule.

I. Introduction

The identification of the mechanism of L-H transition is fundamental to the reliable prediction. The tests of theories both with observation of event and with the statistical database are necessary. For the explanation of the rapid transition, strong nonlinearity has been taken in theories providing hysteresis of radial electric field E_r [1]. Comparison of theories with database has not yet given the conclusive transition mechanisms [2].

We present a statistical model of the E_r bifurcation underlying the L-H transition in toroidal plasmas. Nonlinearity of micro-fluctuations statistically induces random noise in the meso-scale E_r. Being kicked by this random noise, transitions between the L- and H-states occur in a probabilistic manner. A Langevin equation can then be formulated including the mechanism for hysteresis of E_r. The probability density function (PDF) for and the ensemble average of E_r are obtained. The flux of probability density and the transition rate between L-and H-states are calculated. The ensemble average of E_r does not show hysteresis in contrast to the deterministic model. The phase limit is different from the cusp boundaries and is given by the condition that the H- and L- states have equal probability. This article presents the statistical theory by which theoretical models could be compared with statistical threshold database.

II. Statistical equation

We consider a thin layer near the tokamak edge and analyze the dynamics of E_r (averaged over the magnetic surface) in the presence of micro fluctuations. The radial extent of E_r has the scale length ℓ which is assumed constant here for the simplicity. The random noise is induced by the convective nonlinearity in the vorticity equation $\tilde{V}\cdot\nabla\tilde{V}$ associated with micro fluctuations. The dynamical equation of E_r is given as a Langevin equation as [3]

$$\frac{\partial}{\partial \tau} X + \Lambda X = w(\tau)\, g, \qquad (1)$$

where normalization is introduced for the electric field and time as $X = e\rho_{\mathrm{p}} E_r / T$ and $\tau = t\, c_{\mathrm{s}}/2qR$ and $w(t)$ is a white-noise. (ρ_{p}: ion gyroradius at poloidal magnetic field, T : plasma temperature.) The damping term, $\Lambda X = \left(1 + 2q^2\right)^{-1}\left(qR/\rho_s e c_s n_i\right) \bar{J}_r$, is the normalized current. The term g denotes the noise current J_r^{n}. Explicit forms of average current $\bar{J}_r$ and g are discussed in ref.[3]. One chooses an example case as

$$\Lambda X = \mathrm{Im}\, Z\left(X + i\nu_*\right) \cdot \left(X + X_{\mathrm{NC}}\right) + \frac{\nu_b}{\left(\nu_b + \alpha X^4\right)^{1/2}} \exp\left(-\left(\nu_b + \alpha X^4\right)^{1/2}\right) - \gamma_{zonal} X \quad (2)$$

where $Z(X)$ is the plasma dispersion function, X_{NC} is the neoclassical drive and is of the order of $-\rho_p n_e^{-1} \mathrm{d}n_e/\mathrm{d}r$, $\nu_* = \nu_{ii} q R c_s^{-1}$ is the normalized ion collision frequency, $\nu_b = \varepsilon^{-3/2}\nu_*$, $\varepsilon = a/R$, α denotes the orbit squeezing [4,5] and γ_{zonal} is the zonal flow excitation rate combined with shear viscosity damping [6]. Zeros of Λ and relation with L-H transition have been discussed in literature [1]. When $\Lambda X = 0$ has one solution, the solution describes either the L-state or H-state. If multiple solutions exist, the bifurcation has a hysteresis and the hard transition is possible to occur.

When the correlation time of fluctuations τ_{ac} is much shorter than the response time of E_r , the statistical average of micro-fluctuations is calculated by treating E_r as a constant parameter. In this dc-limit, fluctuation level has been given as $\left|\tilde{\phi}\right|^2 = \left(1 + \omega_E^2 \tau_{\mathrm{ac}}^2\right)^{-1} \left|\tilde{\phi}\right|_{\mathrm{L}}^2$, where $\left|\tilde{\phi}\right|_{\mathrm{L}}^2$ is the fluctuation level in the L-mode state, $\omega_E = B^{-1}\, \mathrm{d}E_r/\mathrm{d}r$ is the $E \times B$ shearing rate [7, 8]. Using an evaluation $\mathrm{d}E_r/\mathrm{d}r \simeq E_r/\ell$, one has $\omega_E^2 \tau_{\mathrm{ac}}^2 = \tau_{\mathrm{ac}}^2 B^{-2} \ell^{-2} E_r^2$. In the following, $\left|\tilde{\phi}\right|_L^2$ and global plasma parameters (like temperature) are treated as control parameters. The amplitude of the noise is a nonlinear function of X , and is explicitly given as

$$g = \sqrt{\hat{\tau}_{\mathrm{ac}}}\, \frac{R^2 k_0^2 \rho_i^2 \hat{\phi}^2}{a\sqrt{\ell\, \ell_z}}\, \frac{1}{1 + U X^2} , \qquad (3)$$

where $\hat{\phi} = e\left|\tilde{\phi}\right|_{\mathrm{L}}/T$, $\hat{\tau}_{\mathrm{ac}} = \tau_{\mathrm{ac}} c_s / 2qR$, and $U = \left(\hat{\tau}_{\mathrm{ac}} a/\ell\right)^2$.

III. Statistical properties

The probability density function (PDF) of X, $P(X)$, ensemble average, and transition rate between the L-and H-modes are studied.

3.1 Probability density function

The Fokker-Planck equation of $P(X)$ is deduced from the Langevin equation, and the stationary solution of PDF $P_{\mathrm{eq}}(X)$ is expressed as $P_{\mathrm{eq}}(X) \propto g^{-1} \exp\left(-S(X)\right)$ by use of the nonlinear potential [9]

$$S(X) = \int^X 4\Lambda(X') g(X')^{-2} X'\, \mathrm{d}X' . \qquad (4)$$

The minimum of $S(X)$ (apart from a correction $\ln g$), i.e., zero of Λ, predicts the most probable state of X. Figure 1(a) illustrates PDF $P_{\mathrm{eq}}(X)$ for various values of parameter X_{NC}. The PDF has two peaks, representing the hysteresis. However, the state $X = X_{\mathrm{L}}$ is dominant if $X_{\mathrm{NC}} < X_{\mathrm{NC}}^{\mathrm{c}}$ holds ($X_{\mathrm{NC}}^{\mathrm{c}} \simeq 0.4$ for the parameters or Fig.1(a)), and $X = X_{\mathrm{H}}$ is dominant if $X_{\mathrm{NC}} > X_{\mathrm{NC}}^{\mathrm{c}}$.

3.2 Ensemble average

When one solution of bistable branches is chosen as an initial condition, many transitions in between X_{H} and X_{L} branches occur in a long time, and $P(X)$ reaches to $P_{\mathrm{eq}}(X)$.

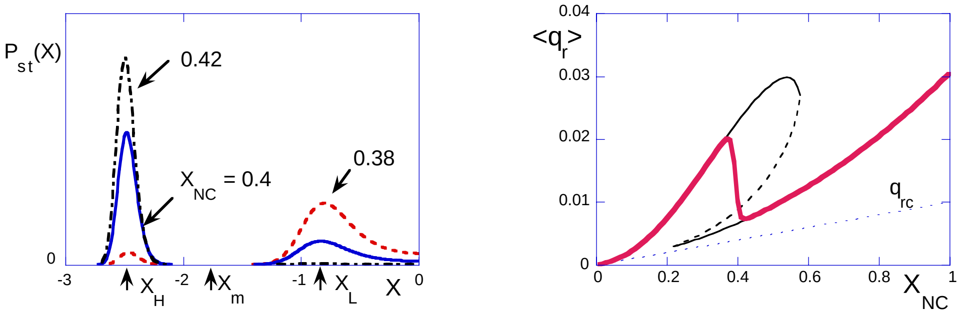

Fig.1 (a) PDF of X in a stationary state (for fixed collision frequency $\nu_* = 0.1$). Solid line is for $X_{\mathrm{NC}} = 0.4 \simeq X_{\mathrm{NC}}^{c}$, dotted line (L-mode is dominant), and broken line (H-mode is dominant) . (b) Heat flux $\langle q_r \rangle$ as a function of global gradient X_{NC} . (q_r is in a unit of $c_s \ell^2 p_0 / 2 R q \rho_p$.) Deterministic model shows the cusp catastrophe (thin line). Ensemble average is shown by the thick solid line. (For parameters, see [3].)

The ensemble average $\langle X \rangle = \int X \, P_{\mathrm{eq}}(X) \, \mathrm{d}X$ changes smoothly as the global control parameter varies. The heat flux is given by the relation

$$q_r = -\left(\chi_c + \chi_{turb}\right) \nabla \, p_0 \, . \tag{5}$$

The turbulent transport coefficient has the form $\chi_{turb} = \chi_{N0} X_{NC}^{1.5} \left(1 + U X^2\right)^{-1}$ including the effect of the electric field shear stabilization [1]. The ensemble average of the heat flux $\langle q_r \rangle$ is illustrated by a thick curve in Fig.1(b). Even though the deterministic theory gives a hysteresis, the ensemble average does not show the hysteresis.

3.3 Transition rates

The transition probability is obtained by calculating a flux of probability density from the Fokker-Planck equation, and is expressed by use of the potential $S(X)$.[10, 11] The rates (frequencies) of the L-to-H transition and back-transition are given as

$$r_{\mathrm{L} \to \mathrm{H}} = \frac{\sqrt{\Lambda_{\mathrm{L}} \Lambda_{\mathrm{m}}}}{2\pi} \exp\left(S(X_{\mathrm{L}}) - S(X_{\mathrm{m}})\right) , \tag{6a}$$

$$r_{\mathrm{H} \to \mathrm{L}} = \frac{\sqrt{\Lambda_{\mathrm{H}} \Lambda_{\mathrm{m}}}}{2\pi} \exp\left(S(X_{\mathrm{H}}) - S(X_{\mathrm{m}})\right) , \tag{6b}$$

respectively, where the time rates $\Lambda_{\mathrm{L,\,m,\,H}}$ are given as $\Lambda_{\mathrm{L,\,m,\,H}} = 2X \left| \partial \Lambda / \partial X \right|$ at $X = X_{\mathrm{L,\,m,\,H}}$. Note that $\Lambda_{\mathrm{L,\,m,\,H}}$ are normalized, being of the order unity. The transition rates is explicitly evaluated by use of the integrals

$$S(X_{\mathrm{L}}) - S(X_{\mathrm{m}}) = -\Gamma \, I_{\mathrm{L}} \equiv -\Gamma \int_{X_{\mathrm{m}}}^{X_{\mathrm{L}}} \Lambda \, X \left(1 + U X^2\right)^2 \mathrm{d}X \ , \tag{7a}$$

$$S(X_{\mathrm{H}}) - S(X_{\mathrm{m}}) = -\Gamma\, I_{\mathrm{H}} \equiv -\Gamma \int_{X_{\mathrm{H}}}^{X_{\mathrm{m}}} \Lambda\, X \left(1 + UX^2\right)^2 \mathrm{d}X \qquad (7b)$$

with the coefficient $\Gamma = 2\, \hat{\tau}_{ac}^{-1} a^2\, \ell\, \ell_z R^{-4} k_0^{-4} \rho_i^{-4}\, \hat{\phi}^{-4}$. Integrals I_{H} and I_{L} are calculated and are of the order unity. The transition and back-transition rates are $r_{\mathrm{L}\to\mathrm{H}} = \sqrt{\Lambda_{\mathrm{L}}\Lambda_{\mathrm{m}}}\, (2\pi)^{-1} \exp\left(-\Gamma\, I_{\mathrm{L}}\right)$ and $r_{\mathrm{H}\to\mathrm{L}} = \sqrt{\Lambda_{\mathrm{H}}\Lambda_{\mathrm{m}}}\, (2\pi)^{-1} \exp\left(-\Gamma\, I_{\mathrm{H}}\right)$, respectively. See ref.[12] for details.

3.4 Phase limit

The phase limit between the L-mode and H-mode (e.g., $X_{\mathrm{NC}}^{\mathrm{c}}$) is defined by the condition that both have equal probability. The probability that the state is found in the L-state is given as $P_{\mathrm{L}} = r_{\mathrm{H}\to\mathrm{L}} / \left(r_{\mathrm{L}\to\mathrm{H}} + r_{\mathrm{H}\to\mathrm{L}}\right)$. That for the H-state is $P_{\mathrm{H}} = r_{\mathrm{L}\to\mathrm{H}} / \left(r_{\mathrm{L}\to\mathrm{H}} + r_{\mathrm{H}\to\mathrm{L}}\right)$. The condition $P_{\mathrm{H}} = P_{\mathrm{L}}$, i.e., $r_{\mathrm{L}\to\mathrm{H}} = r_{\mathrm{H}\to\mathrm{L}}$, is given from Eq.(6) as

$$S(X_{\mathrm{H}}) = S(X_{\mathrm{L}}) + \frac{1}{2} \ln\left(\Lambda_{\mathrm{L}}/\Lambda_{\mathrm{H}}\right) . \qquad (8)$$

Apart from a weak logarithmic term, it is approximated as $S(X_{\mathrm{H}}) = S(X_{\mathrm{L}})$, i.e., $\int_{X_{\mathrm{H}}}^{X_{\mathrm{L}}} \Lambda\, X \left(1 + UX^2\right)^2 \mathrm{d}X = 0$. This result is an extension of Maxwell's rule. When the noise is independent of X, this relation reduces to the condition $\int_{X_{\mathrm{H}}}^{X_{\mathrm{L}}} \Lambda\, X\, \mathrm{d}X = 0$. The phase limit is different from the cusp boundaries. A phase diagram in a control parameter space $\left(\nu_b, X_{\mathrm{NC}}\right)$ is obtained explicitly, and is transformed onto the (n, T) plane (Fig.2).

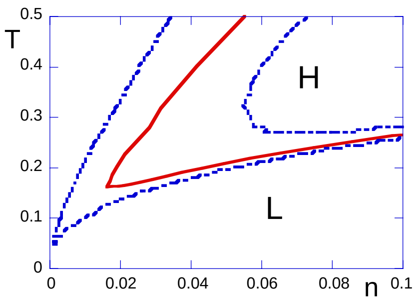

Fig.2 Domain of the L-mode and H-mode on the (n, T) plane. (n, T : normalized.) Solid line shows the ensemble average, while dotted lines indicate the ridges of the cusp.

IV. Summary and implication to experiments

A statistical model for the bifurcation of the radial electric field E_r is analyzed in view of describing L-H transitions of toroidal plasmas. The probability density function for and the ensemble average of E_r are obtained. The L-to-H and the H-to-L transition probabilities are calculated, and the effective phase limit is derived.

Experiments	Theories
Threshold Database	
Most probable transition boundary	Statistical theory (Ensemble average)
Range of data	(width of PDF)
Boundaries of possible transition points	Deterministic part of theory (ridges of cusp)
Study of an event	Deterministic part of theory
Rate of E_r-change at transition	(Model of nonlinearity)

Table 1: Approaches in comparison study of experiments and theories. Appropriate theoretical method must be employed to relevant experimental approaches.

Implications to experiments are as follows: First, the cusp-boundaries of H-mode and the ensemble average of the transition condition in plasma parameters are different. They may show the different parameter dependencies. They must be judged by both the ensemble averages of statistical models which have a noise source, and by a value of deterministic model. Due to the noise, each transition occurs being scattered around the ensemble average. This must be noticed in the future comparison of experimental database with many theories. Second, the ensemble averages of $\langle X \rangle$ and $\langle q_r \rangle$ do not show a hysteresis against global parameters X_{NC}, in contrast to the deterministic model. Third, the observation of hysteresis in experiments critically depends on the speed of global parameter change. Relevant comparison between the theory and experimental observations is summarized in Table 1. Analyses for more realistic cases are reported.

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