Approximate Solutions of the Grad-Schlüter-Shafranov Equation

Gerson Otto Ludwig

Associated Plasma Laboratory, National Space Research Institute
12227-010, São José dos Campos, SP, Brazil
ludwig@plasma.inpe.br

Abstract. Approximate solutions of the Grad-Schlüter-Shafranov equation based on variational methods are developed. The power series solutions of the Euler-Lagrange equations for equilibrium are compared with direct variational results for a low aspect ratio tokamak equilibrium.

1. Introduction

The accurate solution of the nonlinear Grad-Schlüter-Shafranov (GSS) equation for plasma equilibria requires, in general, numerical iterative methods. Nevertheless, approximate analytical solutions are still very useful for calculating global plasma equilibrium parameters and for simulating the equilibrium evolution in tokamaks. The use of flux coordinates is particularly convenient in this respect. They allow the source term in the GSS equation to be easily described in terms of any convenient pair of input profiles, extending the range of possible solutions [1] . Moreover, using appropriate spectral representations for the transformation functions R(½,µ), Z(½,µ), from cylindrical to flux coordinates, the dependence upon the poloidal-angle µ can be exactly removed in the calculation of flux surface averaged quantities [2] . Then, the radial Fourier coefficients in the coordinates transformation can be evaluated by variational methods that render the internal energy $U(a)$ of the plasma stationary [3] [4] [5] . In this way, the problem of finding the flux coordinates is reduced to the solution of one-dimensional equations for the spectral coefficients in the radial coordinate ½.

Following the Lagrangian approach, the variational procedure leads to a set of coupled, nonlinear Euler-Lagrange (EL) equations for the radial Fourier coefficients [3] . This system of equations can be solved numerically in a straightforward manner according to the variational moment method [4] . Alternatively, in the direct variational or Rayleigh-Ritz method one strives for parametric forms of these coefficients that lead to a stationary value of $U(a)$ [2] [5] .

This paper compares power series solutions of the EL equations with direct variational solutions for the spectral coefficients. In Section 2 the variational principle, basic definitions and the EL equations for the spectral coefficients are briefly reviewed. In Section 3 the power series solutions of the EL equations are presented. In Section 4 approximate solutions obtained using the direct variational method are described. Finally, in Section 5 a comparison and a short discussion of the previous results are made.

2. Variational methods

The equilibrium solution of an axially symmetric plasma configuration is described by the GSS equation. The solution of this equation, inside a given constant flux surface, corresponds to the extremum of the internal energy of the plasma [3] [5]

$$U(a) = \iiint_{V(a)} \left(\frac{B_P^2}{2{}^{1}{}_0} + \frac{B_T^2 \text{ ¡ } B_0^2}{2{}^{1}{}_0} + p \right) d^3r; \qquad (1)$$

where B_P, B_T, B_0, and p denote the poloidal, toroidal and external magnetic field components, and the plasma pressure, respectively. The volume $V(a)$ extends to the boundary magnetic flux surface denoted by the minor radius a of the plasma. The flux surface averaging procedure leads to the one-dimensional form

$$U(a) = \int_0^a \left[\frac{I_T(\rho)}{2K(\rho)} \left(I_T(\rho) + \frac{2L(\rho)}{dL/d\rho}\frac{dI_T}{d\rho} \right) + \left(p(\rho) + \frac{L(\rho)}{dL/d\rho}\frac{dp}{d\rho} \right) \frac{dV}{d\rho} \right] d\rho, \qquad (2)$$

where $p(\rho)$ and $I_T(\rho)$ are the plasma pressure and toroidal current profiles, respectively. The geometry dependent quantities $V(\rho)$, $L(\rho)$ and $K(\rho)$ are given in terms of the metric coefficients in the transformation $(R, Z) \to (\rho, \theta)$ by [6]

$$V(\rho) = 4\pi^2 \int \langle \sqrt{g} \rangle_\theta d\rho, \qquad L(\rho) = \mu_0 \int \langle \sqrt{g}/h_3^2 \rangle_\theta d\rho \quad \text{and} \quad K(\rho) = \mu_0^{-1} \langle h_\theta^2/\sqrt{g} \rangle_\theta . \qquad (3)$$

Now, if $C(\rho)$ denotes one of the Fourier coefficients in the spectral representation for $R(\rho,\theta)$ and $Z(\rho,\theta)$ it follows that $V = V(\rho; C)$, $L = L(\rho; C)$ and $K = K(\rho; C; C')$ [3] . The EL equation for $C(\rho)$ becomes

$$\frac{d}{d\rho}\left(\frac{I_T^2(\rho)}{2K^2(\rho)} \frac{\partial K}{\partial C'} \right) - \frac{I_T^2(\rho)}{2K^2(\rho)} \frac{\partial K}{\partial C} - \frac{\partial V}{\partial C}\frac{dp}{d\rho} + \frac{\partial L/\partial C}{dL/d\rho} \left(\frac{I_T(\rho)}{K(\rho)}\frac{dI_T}{d\rho} + \frac{dV}{d\rho}\frac{dp}{d\rho} \right) = 0. \qquad (4)$$

3. Variational power series solutions

The flux surfaces of a tokamak plasma are well represented by the truncated Fourier series

$$R(\rho; \theta) = R_0(\rho) + \rho \cos(\theta) - \tfrac{1}{2}\rho T(\rho) \sin^2(\theta); \qquad Z(\rho; \theta) = \rho E(\rho) \left(\sin(\theta) - \tfrac{1}{4} T(\rho) \sin(2\theta) \right); \qquad (5)$$

which include displacement, elongation and triangularity effects. Quadrangularity and higher order effects can also be included in a straightforward manner. Furthermore, the radial and poloidal integrals defining the geometric coefficients in (3) can be performed analytically for an arbitrary number of terms in the spectral representation [2] .

For each one of the coefficients $R_0(\rho)$, $E(\rho)$ and $T(\rho)$ in the representation defined by (5) one can write an EL equation in the form of (4). Taylor series expansions of these equations were performed to generate second, fourth and sixth order power series expansions of the Fourier coefficients in the radial coordinate ρ. The second order approximations are

$$R_0(\rho) \cong R_m + \tfrac{1}{2} R_0''(0)\rho^2; \qquad E(\rho) \cong E_m + \tfrac{1}{2} E''(0)\rho^2; \quad \text{and} \quad T(\rho) \cong T'(0)\rho; \qquad (6)$$

where R_m and E_m correspond to the major radius and elongation at the magnetic axis, respectively. The poloidal flux function, in this same approximation, is

$$\Psi_P(\rho) = \frac{E_m R_m \mu_0 I_T''(0) \rho^2}{2(E_m^2 + 1)} + O[\rho]^4. \qquad (7)$$

Now, to this order the EL equation for $R_0(\rho)$ gives the following relation between coefficients

$$8\pi^2 \left(E_m^2 + 1\right)^2 p''(0) - \left[E_m^2 + \left(3E_m^2 + 1\right) R_m R_0''(0) - 2R_m T'(0)\right] \mu_0 I_T''(0)^2 = 0; \qquad (8)$$

while the equation for $E(\rho)$ gives

$$\begin{aligned}
&-96\pi^2 E_m (E_m^2 + 1)^3 (1 - 2R_m(2R_0''(0) - T'(0)) p''(0) \\
&+4E_m(E_m^4 - 1)R_m^2 \mu_0 I_T^{(4)}(0) I_T''(0) \\
&+ [3E_m(4(E_m^4 + 6E_m^2 + 1) + 4(3E_m^4 + 2E_m^2 - 5)R_m T'(0) \\
&- (19E_m^4 + 62E_m^2 + 23)R_m^2 T'(0)^2) \\
&+12E_m(1 + 12R_m T'(0) + E_m^2(3E_m^2 + 8)(1 + 4R_m T'(0))R_m R_0''(0) \\
&- 12E_m(3E_m^4 + 2E_m^2 + 3)R_m^2 R_0''(0)^2 \\
&+12(E_m^4 + 18E_m^2 + 5)R_m^2 E''(0)] \mu_0 I_T''(0)^2 \qquad = 0.
\end{aligned} \qquad (9)$$

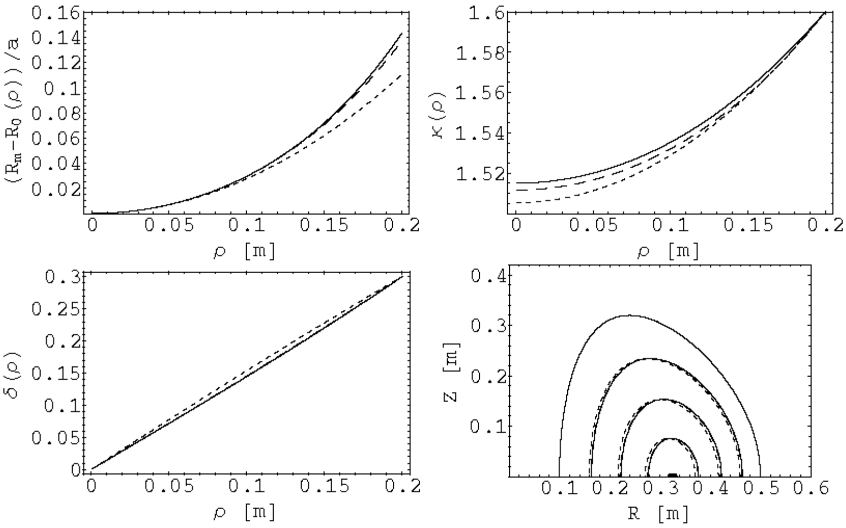

Fig. 1. Shafranov shift, geometric elongation and triangularity profiles, and flux surfaces contours for an ETE equilibrium, comparing results obtained with power series solutions of increasing order. The dotted line in this figure corresponds to the second order expansion, the dashed line to the fourth and the continuous line to the sixth order solution.

The EL equation for $T(\rho)$, to this order, gives the same equation (8) since the triangularity vanishes at the magnetic axis. The pair of equations (8) and (9) relate $\{R_m, E_m\}$ to the values of the derivatives $\{R_0''(0), E''(0), T'(0)\}$ at the magnetic axis. Taking the expansions of the Fourier coefficients up to fourth order in ρ, the EL equations generate three new relations $\{R_0''(0), E''(0), T'(0)\} \rightarrow \{R_0^{(4)}(0), E^{(4)}(0), T^{(3)}(0)\}$, which are linear in the higher order derivatives. Finally, taking expansions to sixth order in ρ one obtains three more relations $\{R_0^{(4)}(0), E^{(4)}(0), T^{(3)}(0)\} \rightarrow \{R_0^{(6)}(0), E^{(6)}(0), T^{(5)}(0)\}$, which are again linear in the higher order derivatives. These relations become increasingly cumbersome, having been derived using *Mathematica*. In order to close the solution the set of values $\{R_0^{(6)}(0), E^{(6)}(0), T^{(5)}(0)\}$ is calculated so that the expansions for $\{R_0(\rho), E(\rho), T(\rho)\}$ satisfy the given fixed boundary conditions $\{R_0(a), E(a), T(a)\}$. Then, the EL relations are used to calculate all the coefficients backwards to the magnetic axis $\{R_m, E_m\}$. Basically, this is the same procedure that would be used in the numerical solution of the Euler equations. The lowest order approximation is obtained simply by taking $\{R_0''(0), E''(0), T'(0)\} = \{2(R_0(a) - R_m)/a^2, 2(E(a) - E_m)/a^2, T(a)/a\}$.

This method gives very fast, convergent results even for a low aspect ratio tokamak as shown in figure 1, which illustrates an application to the ETE Spherical Tokamak Experiment ($R_0(a) = 0.30\,\mathrm{m}$, $a = 0.20\,\mathrm{m}$, $\kappa(a) = 1.6$, $\delta(a) = 0.3$, $I_T(a) = 0.30\,\mathrm{MA}$, $p(0) = 10\,\mathrm{kPa}$, $B_0 \approx 0.21\,\mathrm{T}$). A quasi-uniform current distribution was used in these calculations[7] .

4. Direct variational solutions

The direct variational method provides a global solution of the problem. The method relies on the construction of trial functions for the spectral coefficients, which depend on a few parameters to define a stationary point of the internal energy $U(a)$. It is possible to develop a scheme of trial functions based on polynomials, by increasing the order of the power series solutions of Section 3 and making use of the full set of EL relations. The parameters that are more conveniently varied are the derivatives of the radial Fourier coefficients at the plasma edge, less one derivative that is calculated from the set of EL equations. In fact, using polynomials it

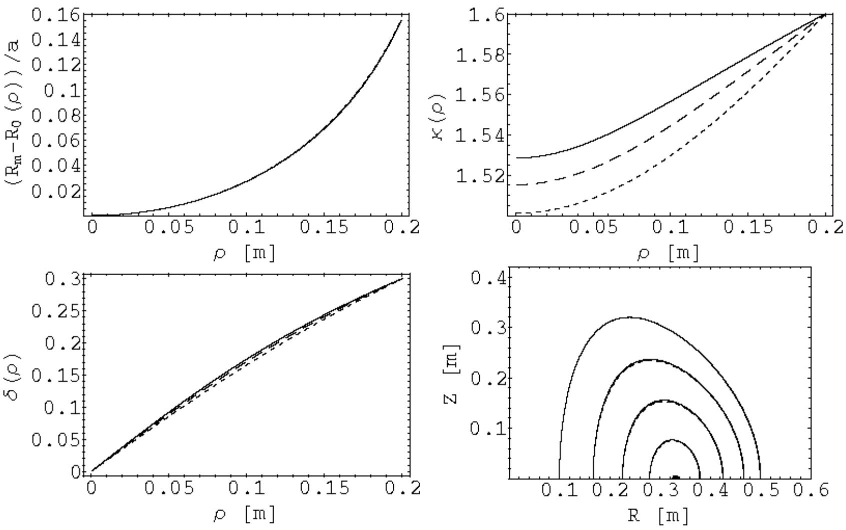

Fig. 2. Shafranov shift, geometric elongation and triangularity profiles, and flux surfaces contours for an ETE equilibrium, comparing results obtained with different trial functions. The dotted line in this figure corresponds to $c = 1$*, the dashed line to* $c = 2$ *and the continuous line to* $c = 2.65$*, close to the minimum value of* $U(a)$ *that can be attained considering the binomial family of trial functions for the equilibrium considered.*

is not possible to find a stationary point varying all the derivatives at the same time. This follows because the extremum of the energy is reached asymptotically, and an essentially infinite order polynomial or power series is required to arrive at the exact solution of the problem. The edge derivatives of the Fourier coefficients correspond to the Neumann conditions at the boundary, providing a link with the external magnetic field for future applications.

In this paper a set of relatively simple trial functions based on binomial functions is presented, which can determine stationary values of $U(a)$ through the variation of one parameter for each Fourier coefficient. The binomial trial functions are

$$
\begin{aligned}
R_0(\rho) &= R_m \left(1 - \left(1 - (R_0(a)/R_m)^{1/\alpha_R}\right) \rho^2/a^2\right)^{\alpha_R}; \\
E(\rho) &= E_m \left(1 - \left(1 - (E(a)/E_m)^{1/\alpha_E}\right) \rho^2/a^2\right)^{\alpha_E}; \\
T(\rho) &= T(a) 2^{\gamma_T} (1 + \rho^2/a^2)^{-\gamma_T} \rho/a;
\end{aligned}
\tag{10}
$$

with the exponents α_R and α_E defined by

$$
\alpha_R = \frac{1}{\gamma_R} \left(\frac{R_0(a)}{R_m}\right)^{\gamma_R} \left(1 - \frac{R_0(a)}{R_m}\right); \qquad \alpha_E = \gamma_E^c \left(\frac{E_m}{E(a)}\right)^{c\gamma_E} \left(1 - \frac{1}{E(a)}\right)^{c\gamma_E} \left(1 - \frac{1}{E_m}\right). \tag{11}
$$

These trial functions satisfy the symmetry and boundary conditions. The exponents γ_R, γ_E and γ_T are the parameters to be varied, while R_m and E_m are determined in consistence with (8) and (9). The forms adopted for α_R and α_E are such that, at the stationary values of γ_R and γ_E (of order unity), the solution is also stationary with respect to R_m and E_m, close to the extremum of $U(a)$. The parameter $c \geq 2$ may be adjusted up to the limit where real stationary points can still be determined. This limit corresponds to the minimum value that can be reached by the internal energy of the plasma with the binomial trial functions meaning, in principle, the "best" solution to the equilibrium problem. Again, it is not possible to vary all the parameters γ_R, γ_E, γ_T, R_m and E_m at the same time, since a simultaneous stationary result can only be attained with the exact solution and minimum absolute value of $U(a)$.

	2nd	4th	6th	c=1	c=2	c=2.65
R_m (m)	0.3220	0.3273	**0.3285**	0.3312	0.3310	**0.3309**
E_m	1.5053	1.5116	**1.5150**	1.5011	1.5153	**1.5285**
ℓ_i	0.3648	0.3570	**0.3541**	0.3536	0.3506	**0.3483**
μ_I	0.7116	0.4357	**0.4295**	0.4285	0.4270	**0.4259**
β_I	0.1775	0.1911	**0.1952**	0.1968	0.1977	**0.1983**
$\widehat{U}(a)$	0.5046	0.4024	**0.4019**	0.4053	0.4035	**0.4023**

Table 1. Comparison of results obtained for the ETE equilibrium using the variational power series of second, fourth and sixth order, and direct methods.

Figure 2 shows the results of the direct method for different values of c and the same equilibrium presented previously in figure 1. The power series and direct methods were also applied to high aspect ratio tokamak configurations. In this case even the lowest order power series solution gives reasonable results as can be inferred examining figure 1.

5. Discussion

The results of the calculations presented in this paper are summarized in table 1. The accuracy of the solutions can be measured by the last row, which gives the normalized internal energy $\widehat{U}(a) = U(a)/(\mu_0 a I_T^2(a))$. The table lists also the values of the internal inductance, the current diamagnetism and the current beta parameters, defined by, respectively,

$$\ell_i = \frac{\int B_P^2/(2\mu_0)dV}{\mu_0 R_m I_T^2(a)/4}; \qquad \mu_I = \frac{\int (B_T^2 - B_0^2)/(2\mu_0)dV}{\mu_0 R_m I_T^2(a)/4} \quad \text{and} \quad \beta_I = \frac{\int p\,dV}{\mu_0 R_m I_T^2(a)/4}.$$

The "best" solutions in each case are indicated in boldface. Convergence of the power series solution is clear, with a maximum deviation between the last two successive approximations of 2% for β_I, and less for all other parameters. The deviation is even smaller for the direct approximations. However, the decrease in energy between the last two cases in the direct method relates essentially to the increase in E_m, resulting from a stiffer profile of the elongation. This is a limitation due to the very simple dependence of the trial functions on the radial coordinate ρ. Figure 2 shows that the main difficulty with the direct method is to reproduce correctly the elongation profile. Otherwise, application of the direct method has no restrictions, in aspect ratio for example.

The time required by *Mathematica* for computing the stationary point, using the power series method, is $\approx$1 s in a 600 MHz PC, making this method very attractive for future studies concerning the plasma evolution. The direct method is slower, but has a simpler formulation. Possibly the greatest advantage of both methods is the simple analytic expressions obtained, which can be used in the calculation of all plasma equilibrium profiles.

References

[1] L.E. Zakharov and A. Pletzer, Phys.Plasmas **6**, 4693 (1999).

[2] G.O. Ludwig, Plasma Phys. Control. Fusion **39**, 2021 (1997).

[3] V.D. Khait, Sov. J. Plasma Phys. **6**, 476 (1980).

[4] L.L. Lao, S.P. Hirshman and R.M. Wieland, Phys. Fluids **24**, 1431 (1981).

[5] G.O. Ludwig, Plasma Phys. Control. Fusion **37**, 633 (1995).

[6] L.E. Zakharov and V.D. Shafranov, *Reviews of Plasma Physics* (Consultants Bureau, New York, 1986), Vol. 11, p. 244.

[7] L.S. Solov'ev, Sov. Phys. JETP **26**, 400 (1968).