An Innovative Method for Ideal and Resistive MHD Stability Analysis of Tokamaks

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Abstract. An advanced asymptotic matching method of ideal and resistive MHD stability analysis in tokamak is reported. The report explains a solution method of two-dimensional Newcomb equation, dispersion relation for an unstable ideal MHD mode in tokamak, and a new scheme for solving resistive MHD inner layer equations as an initial-value problem.

1 Introduction

Since the classical work by Furth-Killeen-Rosenbluth [1], the theoretical framework for the resistive magnetohydrodynamic (MHD) stability analysis of tokamak has been well established in terms of the asymptotic matching method. In the present paper we report some innovations in the MHD stability analysis of tokamak by the asymptotic matching method. The main motivation of the present work is to get rid of obstacles inherent in a toroidal plasma to which we apply the asymptotic matching method. First, we present an eigenvalue method for solving the Newcomb equation [2] in a toroidal plasma. Second, by making good use of the eigenfunction, we derive the dispersion relation of an unstable ideal MHD mode in a toroidal plasma for the first time. Finally, we propose a new scheme for solving resistive MHD inner layer equations numerically in the form of an initial-value problem.

The eigenvalue method not only identifies ideal MHD stable states, but also computes the outer region matching data of a toroidal mode even close to marginal stability against ideal MHD perturbations. Furthermore, the eigenvalue method gives a new way to compute the growth rate of an unstable ideal MHD mode by applying the asymptotic matching method. The eigenfunction with the minimum eigenvalue serves as the outer ideal MHD solution. Matching this outer solution with solutions of ideal MHD inner layer equations yields the dispersion relation of the mode, an equation for the growth rate. It has been impossible to derive such a dispersion relation since the outer ideal MHD solution was not known so far.

As for the resistive MHD inner layer equations, any numerical methods have not been established which solve them as an initial-value problem. The lack of an effective method of the initial-value problem makes us solve the resistive MHD equations in the whole plasma region even for the simulation of the inner layer dynamics. The origin of difficulties in the inner layer problem is that it is not a well-posed initial-value problem for which numerical methods have been established. We can conquer these difficulties by transforming the original inner layer problem into a standard initial-value problem. To this end, we adapt the response formalism developed for the Newcomb equation [3]. This transformation can be shown to be applicable to the toroidal problem.

2 Eigenvalue method for the Newcomb equation

We have devised an eigenvalue problem [4, 5] associated with the Newcomb equation

$$\mathcal{N}\mathbf{Y} = -\lambda\mathcal{R}\mathbf{Y}. \qquad (1)$$

Here, $\mathbf{Y}(r)$ is a vector function made from the Fourier harmonics of the radial displacement, $\mathcal{N}$ the Newcomb operator, $\mathcal{R}$ the diagonal multiplication operator whose diagonal elements are defined such that $\propto (n/m - 1/q(r))^2$ (m, n: poloidal and toroidal mode numbers, $q(r)$: safety factor). This eigenvalue problem has only discrete eigenvalues without continuous spectra. Therefore, we can identify a stable state against ideal MHD perturbations as an eigenstate. Next, we express the $2N_R$ (N_R: Number of rational surfaces) linear independent global solutions of the Newcomb equation $\mathcal{N}\mathbf{Y} = 0$ as

$$\mathbf{Y}_{q,k}(r) = \mathbf{Y}_0(r) + \lambda_0[\vec{\eta}_{q,k}(r) + \Omega_{(q,k)}\hat{\mathbf{Z}}_{q,k}(r)], \tag{2}$$

$$\mathbf{Y}_{p,m}(r) = \vec{\eta}_{p,m}(r) + \Lambda_{(p,m)(q,k)}\hat{\mathbf{Z}}_{q,k}(r) + \hat{\mathbf{Z}}_{p,m}(r), \tag{3}$$

where $p, q = L, R,\ m = nq(r_m),\ k = nq(r_k)$ (k is fixed in the following), $\mathbf{Y}_0$ the eigenfunction for the minimum eigenvalue λ_0 , $\hat{\mathbf{Z}}_{q,k}(r)$ the local function made from the big solution around the rational surface r_k, and $\Omega_{(q,k)}, \Lambda_{(p,m)(q,k)}$ undetermined constants. The orthogonal conditions

$$(\mathcal{R}\mathbf{Y}_0, \vec{\eta}_{q,k}) = 0, \text{ and } (\mathcal{R}\mathbf{Y}_0, \vec{\eta}_{p,m}) = 0,$$

fix the undetermined constants, which are given by

$$\Omega_{(q,k)} = \frac{(\mathbf{Y}_0, \mathcal{R}\mathbf{Y}_0)}{(\mathbf{Y}_0, \mathcal{N}\hat{\mathbf{Z}}_{q,k})},\quad \Lambda_{(p,m)(q,k)} = -\frac{(\mathbf{Y}_0, \mathcal{N}\hat{\mathbf{Z}}_{p,m})}{(\mathbf{Y}_0, \mathcal{N}\hat{\mathbf{Z}}_{q,k})} = -\frac{\Omega_{(q,k)}}{\Omega_{(p,m)}}, \tag{4}$$

where (,) is the usual inner product of two functions. Then the equations for $\vec{\eta}_{q,k}(r), \vec{\eta}_{p,m}(r)$ can be solved as boundary value problems even for the marginal stability ($\lambda_0 = 0$). The coefficient for the small solution at each rational surface extracted from each global solution gives the matching data for a toroidal mode [5].

A code MARG2D has been developed which solves the 2-D Newcomb equation. This code allows detailed analysis of toroidal effects on the matching data (Δ') such as those of finite β and of non-circular cross section. It has been used in the Δ' calculation for the analysis of JT-60U reversed shear discharges [6] and of the neoclassical tearing mode in JT-60U high β_p H-mode discharges [7]. In the next section we inquire into the physical meanings of the constants $\Omega_{(q,k)}$, $\Lambda_{(p,m)(q,k)}$ and the eigenvalue λ_0. These meanings can be clarified by investigating the relation between λ_0 and the growth rate of an ideal MHD mode when the mode is unstable ($\lambda_0 < 0$).

3 Dispersion relation for ideal MHD modes in tokamak

The solutions given by Eqs.(2) and (3) serve as the outer solution even for the case of marginal stability. Therefore, we can apply the asymptotic matching method for computing the growth rate of an unstable ideal MHD mode. In this case the width of the layer around each rational surface is determined by plasma inertia instead of electrical resistivity in the case of resistive MHD stability analysis.

Since we are interested in the MHD stability of a toroidal plasma and we make the problem specific, we assume there exist two rational surfaces. We expand the independent variable around each rational surface r_j ($j = 1, 2$) as $r - r_j = \gamma\alpha_j z_j$, where γ is the growth rate of the mode normalized to the poloidal Alfvén frequency. The constants α_j ($j = 1, 2$) are positive and determined by the inner layer equation for the ideal MHD mode. Let $\mu_j(\nu_j = -1/2 + \mu_j)$ be the Mercier-Suydam index at each rational surface r_j. We consider the case that all of them are positive and the Mercier stability condition for a local interchange mode are satisfied. A general solution $\mathbf{Y}(r)$ of the Newcomb equation can be expressed as

$$\mathbf{Y}(r) = c_{L,1}\mathbf{Y}_{L,1}(r) + c_{R,1}\mathbf{Y}_{R,1}(r) + c_{L,2}\mathbf{Y}_{L,2}(r) + c_{R,2}\mathbf{Y}_{R,2}(r), \tag{5}$$

where $\mathbf{Y}_{R,2}(r)$ has the form of Eq.(2) and the others have the form of Eq.(3), and $c_{L,j}$ and $c_{R,j}$ are four arbitrary constants ($j = 1, 2$). Since $\gamma << 1$ ($|\lambda_0| << 1$), we can assume these coefficients satisfy

$$|c_{L,1}| \simeq |c_{R,1}| \simeq |c_{L,2}| << |c_{R,2}|. \tag{6}$$

We asymptotically connect $\mathbf{Y}(r)$ at each rational surface to the solution of the inner layer equation. Then we obtain linear equations for the coefficients, $c_{L,j}$ and $c_{R,j}$. The condition that the equations have non-trivial solutions yields an equation for γ, the dispersion relation for the ideal MHD mode. Let us define

$$\Delta_{(j)}^{'(0)} := \Delta_{R,j} + \Delta_{L,j}, \quad \Gamma_{(j)}^{'(0)} := \Delta_{R,j} - \Delta_{L,j}, \tag{7}$$

where $\Delta_{p,j}$ $(p = L, R)$ is the matching data of the eigenfunction $\mathbf{Y}_0(r)$ at each side of the rational surface r_j. Also let us introduce $A_j,\ B_j$ $(j = 1, 2)$ by

$$A_j := \frac{\alpha_j^{2\mu_j}}{2\Omega_{(R,j)}}[\Delta_{(j)}^{'(0)}\Delta_{\rm in,e}(\nu_j) + \Gamma_{(j)}^{'(0)}\Delta_{\rm in,o}(\nu_j)], \tag{8}$$

and

$$B_j := \frac{\alpha_j^{2\mu_j}}{2\Omega_{(L,j)}}[\Delta_{(j)}^{'(0)}\Delta_{\rm in,e}(\nu_j) - \Gamma_{(j)}^{'(0)}\Delta_{\rm in,o}(\nu_j)], \tag{9}$$

where $\Delta_{\rm in,p}(\nu_j)$'s , which are independent on the growth rate, are the inner layer matching data of the solutions of the ideal MHD inner layer equations. The dispersion relation is expressed by an equation for the growth rate γ

$$\lambda_0 = \gamma^{2\mu_1}(A_1 + B_1) + \gamma^{2\mu_2}(A_2 + B_2). \tag{10}$$

The general dispersion relation where there exists M rational surfaces is easily obtained, which is given by

$$\lambda_0 = \sum_{j=1}^{M} F_j\gamma^{2\mu_j}, \quad F_j = A_j + B_j. \tag{11}$$

Equation (11) clarifies the relation between the eigenvalue λ_0 in Eq.(1) and the physical growth rate γ. Only $\mathbf{Y}_0(r)$ is necessary to compute the building blocks for the dispersion relation. The dispersion relation enables precise stability analysis of an ideal MHD mode around the marginal stability. Ideal MHD spectral codes such as ERATO can hardly deal with such a problem.

4 Initial value problem for resistive layer equations

In this section we present a new scheme that numerically solves inner layer equations in the resistive MHD stability analysis. The new scheme solves the equations as an initial-value problem. Let us consider the simple inner layer equation in the form of evolution equation

$$\partial_t \frac{d^2\Phi}{dz^2} = z\Xi, \quad \partial_t\Xi = -\frac{d^2}{dz^2}(z\Phi) + \frac{d^2\Xi}{dz^2}, \tag{12}$$

where z is the radial coordinate stretched around a rational surface, Φ the electrostatic potential, and Ξ the electric field parallel to the equilibrium magnetic field. We impose on Ψ, Ξ the tearing parity condition. The asymptotic conditions at $z = \infty$ are

$$\Phi(z, t) \sim \phi_\infty(t)(1 + c/z), \quad \Xi(z, t) \sim 0. \tag{13}$$

Here $c = 1/\Delta_o$ is given by the outer region matching data. Equation (13) states that the solution should be matched at infinity to the ideal MHD solution for which the parallel electric field vanishes. The inner layer problem is to obtain $\phi_\infty(t)$ when c is given.

An unknown constant $\phi_\infty(t)$ appears in the asymptotic condition, Eq.(13). We have to determine this unknown constant at each time so that $\Phi(z,t)$ and $\Xi(z,t)$ should satisfy Eq.(13), which is the origin we meet in the inner layer problem. We apply the response formalism developed for the Newcomb equation [3]. In applying the response formalism to the inner layer problem, we employ an implicit finite difference approximation to time in Eq.(12); the special derivatives are left intact. Thus we obtain

$$\gamma_t \left(\frac{d^2\Phi^{n+1}}{dz^2} - \frac{d^2\Phi^n}{dz^2} \right) = z\Xi^{n+1}, \; \gamma_t \left(\Xi^{n+1} - \Xi^n \right) = -\frac{d^2}{dz^2}(z\Phi^{n+1}) + \frac{d^2}{dz^2}\Xi^{n+1}, \tag{14}$$

where $\gamma_t = 1/\Delta t$ (Δt: time step). Equation (14) can be rewritten in a vector form

$$\mathcal{L}\hat{\mathbf{Z}}^{n+1} = (1/\phi_\infty^{n+1})\mathbf{H}(\mathbf{Z}^n), \;\; \hat{\mathbf{Z}}^{n+1} = (\hat{\Phi}^{n+1}, \hat{\Xi}^{n+1})^t, \tag{15}$$

where $\mathcal{L}$ is the operator corresponding to Eq.(14),

$$\mathbf{Z}^{n+1} = \phi_\infty^{n+1}\hat{\mathbf{Z}}^{n+1} = (\Phi^{n+1}, \; \Xi^{n+1})^t,$$

and

$$\mathbf{H}(\mathbf{Z}^n) = \gamma_t(-d^2\Phi^n/dz^2, \; \Xi^n)^t.$$

We can now apply the response formalism to Eq.(14). Let $\mathbf{Y} = (\Phi_2, \Xi_2)^t$ be a vector function satisfying the parity condition and

$$\Phi_2(z) \equiv 1, \;\; \Xi_2(z) \equiv 0, \; z \to \infty.$$

By expressing as

$$\hat{\mathbf{Z}}^{n+1} = \mathbf{X}^{n+1} + \mathbf{Y}, \;\; \mathbf{X}^{n+1} = (\Phi_1^{n+1}, \Xi_1^{n+1})^t,$$

we have

$$\mathcal{L}\mathbf{X}^{n+1} = -\mathcal{L}\mathbf{Y} + (1/\phi_\infty^{n+1})\mathbf{H}, \tag{16}$$

and the asymptotic conditions

$$\Phi_1^{n+1}(z) \sim c/z, \; \Xi_1^{n+1}(z) \to 0, \; z \to \infty. \tag{17}$$

Next, let $\mathbf{X}$, $\mathbf{V}$ be, respectively, the solutions of

$$\mathcal{L}\mathbf{X} = -\mathcal{L}\mathbf{Y}, \;\; \mathcal{L}\mathbf{V} = \mathbf{H},$$

with the condition

$$\mathbf{X} \sim c_0(1/z, 0)^t, \;\; \mathbf{V} \sim c_h(1/z, 0)^t, \;\; (c_0, c_h\text{: unknown constants}).$$

By changing the asymptotic condition for $\mathbf{V}$ into the equivalent boundary condition

$$d\Phi_{\mathrm{V}}/dz = -\Phi_{\mathrm{V}}/z, \;\; \Xi_{\mathrm{V}} = 0,$$

we can solve the equation for $\mathbf{V}$ and obtain the constant c_h. Similarly, we obtain c_0. Finally, from $\mathbf{X}^{n+1} = \mathbf{X} + \mathbf{V}/\phi_\infty^{n+1}$ we have the relation,

$$\phi_\infty^{n+1} = c_h/(c - c_0), \tag{18}$$

to be obtained.

Figure 1 shows (a) $\phi_\infty(t)$ and (b) $\Phi(z)$ computed by the present method ($\Delta t = 0.04$), where Δ_o is set to be $\Delta_o = 0.2465$. The solution evolves with the linear growth rate $\gamma = 0.8$, which coincides with that given by the well known analytic dispersion relation, and shows the typical structure of the resistive kink mode. These results confirm the validity of the present method.

5 Conclusions

Some innovations in the MHD stability analysis have been reported in the present paper. The eigenvalue method enables detailed analysis of toroidal effects on the matching data. The dispersion relation of an unstable ideal MHD mode allows precise analysis of the ideal MHD stability of a toroidal plasma around the marginal stability. The new scheme for solving the resistive MHD inner layer equations as an initial-value problem can replace the methods based on the transcendental dispersion relation of a resistive MHD mode. These innovations resolve the difficulties inherent in a toroidal mode, and greatly facilitates getting deep insights into MHD stability of tokamaks.

Acknowledgments

The author thanks Ms. T. Ishizawa at JAERI (Japan Atomic Energy Research Institute) for her coding programs based on the proposed schemes and executing test runs. The author is also grateful to Prof. T. Kako at the University of Electro-communications, Dr. H. Naitou at Yamaguchi University, Dr. A. Ishizawa at Kyoto University, and Drs. Y. Kishimoto, T. Ozeki and S. Takeji at JAERI for their useful comments on the present work.

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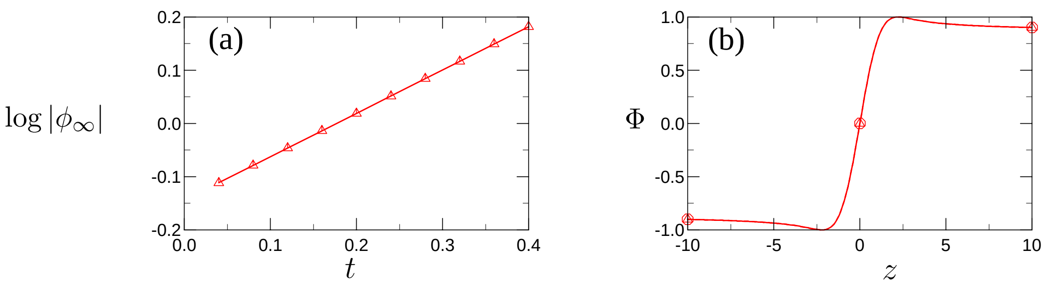

Fig. 1 (a) Time dependence of $\phi_\infty(t)$ and (b) solution $\Phi(z)$ at $t = 0.4$ for $\Delta_o = 0.2465,\ \Delta t = 0.04$. The computed growth rate of ϕ_∞ is $\gamma = 0.8$, which coincides with that predicted by the dispersion relation. The solution shows a typical structure of the resistive kink mode.