Nonlinear Evolution of Neoclassical Tearing Modes in the Presence of Sheared Flows

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Abstract. The center manifold reduction technique is employed to study the nonlinear evolution of an (m=2, n=1) neoclassical tearing mode and its first harmonic, in the presence of equilibrium sheared flows. A detailed bifurcation diagram of the reduced amplitude equations is presented delineating the parametric regimes for the occurrence of single saturated island states and oscillating island solutions.

1. Introduction

Neoclassical tearing modes have attracted a great deal of attention in recent years due to the constraint they impose on the attainment of the ideal MHD beta limit in high temperature long pulse tokamak discharges [1]. They arise in the collisionless regime where the growth rate of the classical tearing mode is significantly enhanced by the perturbation in the bootstrap current due to the local flattening of the pressure profile inside the magnetic island. The basic dynamical behaviour of this mode can be understood from an extension of the Rutherford theory [2] to include neoclassical effects. Several such analytic studies and some numerical modeling have considerably advanced our understanding of the evolution of these modes [3–5] but the effect of sheared equilibrium flows on their nonlinear evolution has not been studied. Sheared flows can occur in a tokamak discharge under a variety of conditions, e.g. due to unbalanced parallel injection of neutral beams leading to large scale toroidal rotation in the plasma or poloidal flows arising from turbulence and/or radial electric fields. They are known to have a strong influence on the onset and nonlinear evolution of resistive tearing modes and it is important to investigate their effect on the final nonlinear states of the neoclassical tearing modes. The conventional Rutherford type analysis is difficult to apply in the presence of equilibrium flows since the plasma current density and the plasma pressure are no longer flux functions. In this paper we have adopted the method of center manifold reduction to overcome this difficulty [6] and reduced the resistive MHD equations to a set of coupled nonlinear amplitude equations which are easier to analyze. In particular, we examine the time asymptotic states of these equations, which can describe the final nonlinear states of the neoclassical tearing mode, and delineate the domain of existence of these states and their stability properties through a detailed bifurcation analysis.

2. Model Equations

We start with an extended model of the resistive MHD equations which includes a bootstrap current contribution in the Ohm's law and which is evolved self-consistently through a pressure equation. This set of equations is related to the generalized reduced magnetohydrodynamic equations of Kruger *et al* [3] and in the absence of flows has been used by Yu and Gunter [5] to numerically study the nonlinear evolution of the neoclassical tearing mode. The equations are,

$$\frac{\partial}{\partial t}\nabla^2\phi_1 + (\vec{v_0}\cdot\vec{\nabla})\nabla^2\phi_1 + (\vec{v_1}\cdot\vec{\nabla})\nabla^2(\phi_0+\phi_1) = \hat{z}\cdot(\vec{\nabla}\psi_0\times\vec{\nabla}j_{z1}) + \hat{z}\cdot\vec{\nabla}\psi_1\times\vec{\nabla}(j_{z0}+j_{z1}) + \mu\nabla^4\phi_1 \quad (1)$$

$$\frac{\partial\psi_1}{\partial t} + (\vec{v_0}\cdot\vec{\nabla})\psi_1 + (\vec{v_1}\cdot\vec{\nabla})(\psi_0+\psi_1) = -\eta(j_{z1}-j_{b1}) \quad (2)$$

$$\frac{\partial p_1}{\partial t} + (\vec{v_0}\cdot\vec{\nabla})p_1 + (\vec{v_1}\cdot\vec{\nabla})(p_0+p_1) = \vec{\nabla}\cdot\chi_{||}\nabla_{||}p_1 + \vec{\nabla}\cdot\chi_{\perp}\nabla_{\perp}p_1 \quad (3)$$

where $j_{z1} = -\nabla^2\psi_1$, $j_{b1} = -g\frac{c\sqrt{\epsilon}}{B_\theta}\frac{\partial p}{\partial x}$ is the perturbed bootstrap current and other notations are standard. g is a smooth function of the minor radius which vanishes at the center and the plasma edge. We assume a simple slab geometry where x corresponds to the radial direction and all perturbations are assumed to be periodic in the y and z directions (corresponding to the poloidal and toroidal directions) All equilibrium quantities are assumed to be a function only of x. Equations (1-3) can be written in a more compact form as,

$$\frac{\partial}{\partial t}\tilde{R} = LR + N(\phi_1,\psi_1,p_1), \quad (4)$$

where R is the column vector $(\ \phi_1,\ \ \psi_1,\ \ p_1\ )^T$, $\tilde{R} = (\ \nabla_\perp^2\phi_1,\ \ \psi_1,\ \ p_1\ )^T$, L is a linear operator matrix,

$$\begin{pmatrix} \mu\nabla_\perp^4 - \phi_0'(x)\frac{\partial}{\partial y}\nabla_\perp^2 + \phi_0'''(x)\frac{\partial}{\partial y} & -\psi_0'(x)\frac{\partial}{\partial y}\nabla_\perp^2 + \psi_0'''(x)\frac{\partial}{\partial y} & 0 \\ \psi_0'(x)\frac{\partial}{\partial y} & \eta\nabla_\perp^2 - \phi_0'(x)\frac{\partial}{\partial y} & -\frac{\eta g c\sqrt{\epsilon}}{\psi_0'(x)}\frac{\partial}{\partial x} \\ p_0'(x)\frac{\partial}{\partial y} & 0 & -\phi_0'(x)\frac{\partial}{\partial y} + \chi_b\nabla_b^2 + \chi_\perp\nabla_\perp^2 \end{pmatrix}$$

and N the nonlinear vector, $(\ -\{\phi_1,\nabla_\perp^2\phi_1\} - \{\psi_1,\nabla_\perp^2\psi_1\},\quad -\{\phi_1,\psi_1\},\quad \{\phi_1,p_1\}\ )^T$. In the above, the superscript T stands for the transposed quantity, the primes denote differentiation with respect to x ($x = 0$ corresponds to the mode rational surface) and $\{a,b\}$ represents a Poisson bracket.

3. Center Manifold Reduction

We consider an equilibrium situation where the parameters of the magnetic field and flow are such that an $m = 2, n = 1$ and its first harmonic are simultaneously marginally stable at the $q = 2$ surface (m and n are the poloidal and toroidal mode numbers). Such a situation is possible for a variety of model equilibria as has been discussed in the literature [6–9]. We will examine the nonlinear interaction of these modes by first reducing Eq.(4), using the center manifold technique, to a set of amplitude equations of the form,

$$\dot{Z}_r = f_r(Z_1,\bar{Z}_1,Z_2,\bar{Z}_2,Z_0) \quad (5)$$

where $Z_{1,2}$ are the complex amplitudes of the two perturbed modes, bar denotes a complex conjugate quantity, overdot denotes time derivative and Z_0 denotes the distance of the system parameters from their critical values at marginal equilibrium. Note that $\dot{Z}_0 = 0$. The physical quantities are expanded as,

$$R(x,y,z,t) = \sum_{r=1,2} Z_r(t)R_r(x)e^{ir\beta\zeta} + c.c. + \sum_{r,s=0,1,2,r\le s} Z_r(t)Z_s(t)R_{rs}(x) + c.c. + ... \quad (6)$$

where the R_r are the linear eigenmodes of the eq.(4) and the functions R_{rs} and further higher order ones are chosen orthogonal to R_r. $\beta\zeta = (k_y y + k_z z)$ where ζ is the helical coordinate and β is the helical mode number corresponding to an $m = 2, n = 1$ mode. Close to the marginal state the functions f_r can be Taylor expanded in a power series of the amplitudes,

$$f_r = \sum_{s=0,1,2} A_r^s Z_s + c.c. + \sum_{s,p=0,1,2,s\leq p} A_r^{sp} Z_s Z_p + c.c. + ... \tag{7}$$

Substituting (6) and (7) in Eq.(4) and matching terms order by order in the amplitudes Z_r up to say third order terms, one can obtain expressions for the various coefficients A_r^s, A_r^{sp} etc. In general there are a large number of coefficients even with a truncation to third order and their evaluation is a laborious task. However the constraint imposed by the symmetry of the system can make many of these coefficients vanish. Our model equations are invariant to translation in y (actually to the helical coordinate in a toroidal system) so that as discussed in [6, 10] Eq.(5) can be reduced to the following generic form,

$$\dot{Z}_1 = (\lambda_1 + i\omega_1)Z_1 + a_1\bar{Z}_1 Z_2 + b_1 Z_1 \mid Z_1 \mid^2 + c_1 Z_1 \mid Z_2 \mid^2 \tag{8}$$

$$\dot{Z}_2 = (\lambda_2 + i\omega_2)Z_2 + a_2 Z_1^2 + b_2 Z_2 \mid Z_1 \mid^2 + c_2 Z_2 \mid Z_2 \mid^2 \tag{9}$$

The method of deriving the expressions for the coefficients is quite standard (see [6] for instance) and we omit the details. The final expressions are listed in the Appendix. These coefficients are complex and their imaginary contributions arise solely due to the presence of flow. The mathematical origin of this can be traced to a symmetry breaking in the system of equations - in this case the breaking of reflection symmetry by the flow terms. The real parts of λ_j provide a measure of the distance of the system parameters from the marginal state, while the imaginary contributions arise from the Doppler shift contribution to the natural mode frequencies due to the flow. The frequencies are further modified by amplitude dependent contributions from the terms proportional to b_1 and c_2 while the terms proportional to a_1, c_1, a_2 and b_2 provide cross coupling between the modes. Eqs.(8-9) are still difficult to solve analytically. We have therefore examined their equilibrium states and studied the stability of these states by a detailed numerical local bifurcation analysis. Our results are presented and discussed in the next section.

4. Bifurcation Analysis and Discussion

Setting $Z_i = r_i e^{\theta_i}$ eqs. (8 - 9) can be reduced to three equations for the variables r_1, r_2, the amplitudes of the modes and $\phi = 2\theta_1 - \theta_2$, the relative phase between them. These equations admit three different equilibrium states, namely, the origin $r_j = 0$ (often called the "death" state), $r_1 = 0, r_2 \neq 0$ (a single island state or "semi-death" state) and $r_1, r_2 \neq 0$ (a mixed mode state). We have studied the stability of these states in the functional space of the various coefficients of Eqs.(8- 9) and our results are displayed in the form of phase diagrams in selected parameter regions. The linear stability of the origin is determined by the eigenvalues λ_j. In Fig.1(a) we have shown a phase diagram in the space of Real(λ_1) and Imag(λ_1) keeping other coefficients constant. Physically this corresponds to the space of the parametric distance from the marginal state and the amount of flow induced Doppler frequency shift. The origin loses its stability by a Hopf bifurcation to yield a single island state (an $m = 4, n = 2$ island) which for higher values of Real(λ_1) goes over to a mixed state. Note that this mixed state is characterized by

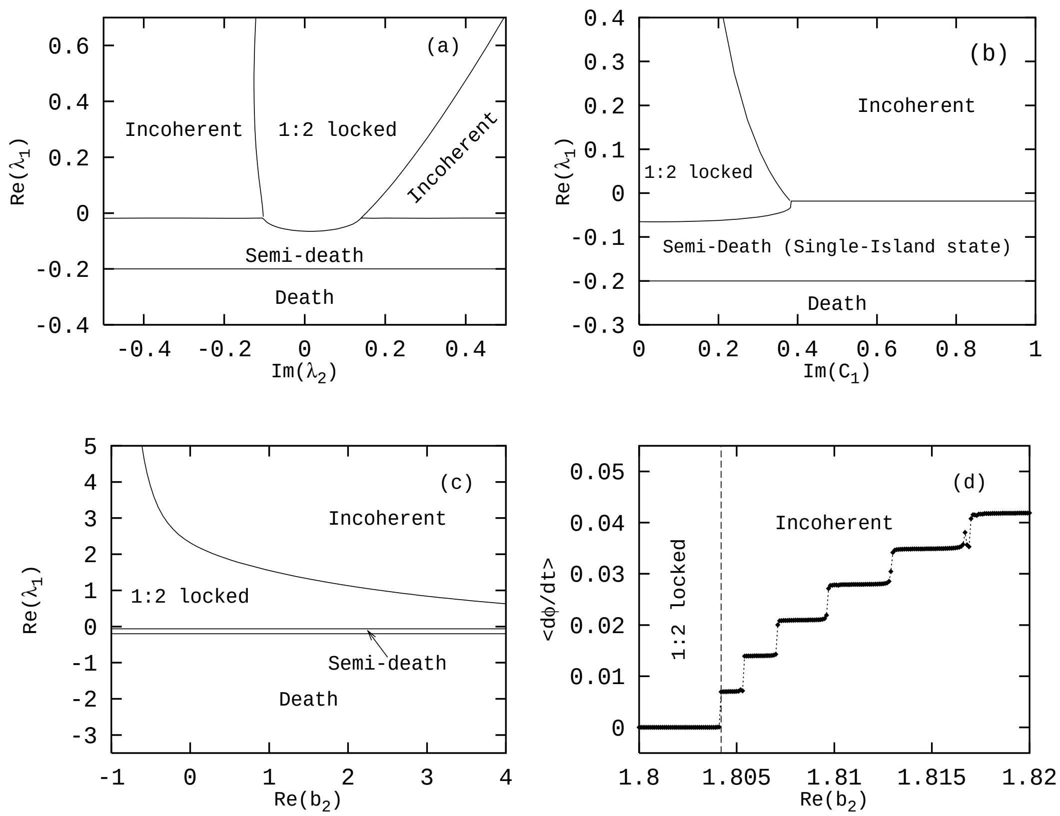

Figure 1.

two regions - the 2 : 1 frequency locked state where $\dot{\phi} = 0$ and an 'incoherent' region where the individual modes continue to oscillate independently leading to a modulated or oscillating island state. The oscillating island states are purely a flow induced state. This is also seen very clearly in Fig. 1(b) where we have obtained a phase diagram in terms of the imaginary coefficient of one of the nonlinear cross coupling terms, namely Imag(c_1). The incoherent region vanishes for values of Imag(c_1) below a critical value. Fig. 1(c) summarizes our findings for the influence of the real part of the cross coupling term, Real(b_2), on the stability diagram. Note that the incoherent region now has no direct access from the single island state but is always mediated by the 2:1 locked region. The transition from the locked region to the incoherent region also shows interesting 'frequency jump' phenomena as shown in Fig.3(d). The time average of $\dot{\phi}$ taken over several periods shows quantum jumps over intervals of Real(b_2) with nearly constant frequency steps. Similar behaviour is also observed with the variation of Real(c_1) in the incoherent region.

To summarize, a center manifold analysis of the nonlinear reduced generalized MHD equations (with bootstrap current contributions and equilibrium sheared flows) shows interesting time asymptotic nonlinear states like single saturated magnetic islands, frequency locked states and oscillating magnetic island states. These solutions could represent possible saturated states of the neoclassical tearing modes in the presence of sheared flows. For a more quantitative comparison and to address questions of accessibility a direct numerical solution of the original nonlinear equations (1- 3) is presently in progress.

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Appendix: Expressions for the Coefficients

$$\lambda_{j=1,2} = Z_0 \int\int \left(\phi^A_{jc} \frac{\partial L}{\partial Z_0} \phi_{jc} + \psi^A_{jc} \frac{\partial L}{\partial Z_0} \psi_{jc} + p^A_{jc} \frac{\partial L}{\partial Z_0} p_{jc} \right) dxdy,$$

$$a_1 = \int\int \left[\phi^A_{1c} \left(-\{\phi_{1c}, \nabla^2_\perp \phi_{2c}\} - \{\phi_{2c}, \nabla^2_\perp \phi_{1c}\} - \{\psi_{1c}, \nabla^2_\perp \psi_{2c}\} - \{\psi_{2c}, \nabla^2_\perp \psi_{1c}\} \right) \right.$$
$$\left. + \psi^A_{1c} \left(-\{\phi_{1c}, \psi_{2c}\} - \{\phi_{2c}, \psi_{1c}\} \right) + p^A_{1c} \left(\{\phi_{1c}, p_{2c}\} + \{\phi_{2c}, p_{1c}\} \right) \right] dxdy + O(\omega).$$

$$a_2 = \int\int \left[\phi^A_{2c} \left(\{\phi_{1c}, \nabla^2_\perp \phi_{1c}\} + \{\psi_{1c}, \nabla^2_\perp \psi_{1c}\} \right) + \psi^A_{2c} \{\phi_{1c}, \psi_{1c}\} - p^A_{2c} \{\phi_{1c}, p_{1c}\} \right] dxdy + O(\omega).$$

$$b_1 = \int\int \left[\phi^A_{1c} \left(\{\phi_{11}, \nabla^2_\perp \phi_{3c}\} + \{\phi_{13}, \nabla^2_\perp \phi_{1c}\} + \{\psi_{11}, \nabla^2_\perp \psi_{3c}\} + \{\psi_{13}, \nabla^2_\perp \psi_{1c}\} \right) \right.$$
$$\left. + \psi^A_{1c} \left(\{\phi_{11}, \psi_{3c}\} + \{\phi_{13}, \psi_{1c}\} \right) + p^A_{1c} \left(-\{\phi_{11}, p_{3c}\} - \{\phi_{13}, p_{1c}\} \right) \right] dxdy,$$

$$b_2 = \int\int \left[\phi^A_{2c} \left(\{\phi_{12}, \nabla^2_\perp \phi_{3c}\} + \{\phi_{23}, \nabla^2_\perp \phi_{1c}\} + \{\phi_{13}, \nabla^2_\perp \phi_{2c}\} + \{\psi_{12}, \nabla^2_\perp \psi_{3c}\} \right. \right.$$
$$\left. + \{\psi_{13}, \nabla^2_\perp \psi_{2c}\} + \{\psi_{23}, \nabla^2_\perp \psi_{1c}\} \right) + \psi^A_{2c} \left(\{\phi_{12}, \psi_{3c}\} + \{\phi_{23}, \psi_{1c}\} + \{\phi_{13}, \psi_{2c}\} \right)$$
$$\left. + p^A_{2c} \left(-\{\phi_{12}, p_{3c}\} - \{\phi_{23}, p_{1c}\} - \{\phi_{13}, p_{2c}\} \right) \right] dxdy,$$

$$c_1 = \int\int \left[\phi^A_{1c} \left(\{\phi_{12}, \nabla^2_\perp \phi_{4c}\} + \{\phi_{24}, \nabla^2_\perp \phi_{1c}\} + \{\phi_{14}, \nabla^2_\perp \phi_{2c}\} + \{\psi_{12}, \nabla^2_\perp \psi_{4c}\} \right. \right.$$
$$\left. + \{\psi_{24}, \nabla^2_\perp \psi_{1c}\} + \{\psi_{14}, \nabla^2_\perp \psi_{2c}\} \right) + \psi^A_{1c} \left(\{\phi_{12}, \psi_{4c}\} + \{\phi_{24}, \psi_{1c}\} + \{\phi_{14}, \psi_{2c}\} \right)$$
$$\left. + p^A_{1c} \left(-\{\phi_{12}, p_{4c}\} - \{\phi_{24}, p_{1c}\} - \{\phi_{14}, p_{2c}\} \right) \right] dxdy,$$

$$c_2 = \int\int \left[\phi^A_{2c} \left(\{\phi_{22}, \nabla^2_\perp \phi_{4c}\} + \{\phi_{24}, \nabla^2_\perp \phi_{2c}\} + \{\psi_{22}, \nabla^2_\perp \psi_{4c}\} + \{\psi_{24}, \nabla^2_\perp \psi_{2c}\} \right) \right.$$
$$\left. + \psi^A_{2c} \left(\{\phi_{22}, \psi_{4c}\} + \{\phi_{24}, \psi_{2c}\} \right) + p^A_{2c} \left(-\{\phi_{22}, p_{4c}\} - \{\phi_{24}, p_{2c}\} \right) \right] dxdy$$

The superscript A refers to functions which are the solutions of the adjoint form of eq.(4).