# Theory of Low Frequency Instabilities Near Transport Barriers

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**Abstract:** The theory of low frequency instabilities in axisymmetric toroidal plasmas is presented from the point of view of the two-fluids equations, assuming the standard drift wave ordering. Attention is focused on the limit in which neighboring rational surfaces are sufficiently far apart that mode overlapping is non-existent. Owing to field line bending, poloidal side-bands m±1,... coexist with the primary mode m, enhancing noticeably the role of the parallel ion dynamics. The electron and ion branches are investigated accurately under those conditions. It is found that the radial widths of the eigenmodes increase with respect to the slab values; the shear damping rate of the electron branch, respectively the growth rate of the ion branch increases correspondingly. Other interesting results are obtained concerning the frequency, the growth rate and the poloidal variation of the amplitude of the ion mode fluctuations. Those explain the origin of internal transport barriers; they also suggest ways of interpreting fluctuations asymmetries observed in tokamaks and (when collisions are included) the Radiative Improved confinement mode.

## 1. Introduction

We have derived, in the framework of the two-fluids theory, two-dimensional partial differential equations which describe the dynamics of low frequency micro-instabilities elongated along the magnetic field lines; our derivation is free of assumptions regarding the ballooning character of the modes and the stress tensor; ion collisions have furthermore been taken systematically into account. The two-fluids theory is attractive because of its immediate physical content, but it fails to include wave-particle resonant interactions and the trapped particles responses; those can be taken into account via a kinetic extension once the characteristic target instability parameters have been identified by the two-fluids results.

In tokamaks, a crucial parameter in the 2D description of low frequency instabilities is the ratio of the radial width (w) of neighboring eigenmodes (with identical toroidal mode numbers) to the distance (Δ) between the rational surfaces about which they are respectively localized. Strong overlap occurs if $w/\Delta \gg 1$; in that case, new sets of eigenfunctions may be built from linear combinations of the isolated eigenmodes, as proposed by Taylor [1]. Poloidal coupling, which primarily occurs through the magnetic field radial and poloidal gradients, plays a decisive role in the linear stability properties: magnetic shear damping of electron drift waves for example is suppressed for a proper phasing of the isolated eigenmodes.

The inhomogeneity of the magnetic field plays also an important role in the opposite limit $w/\Delta \ll 1$ as the parallel mode number of the side-bands (=1/qR) is larger than that of the primary mode; as a result, the role of the parallel ion dynamics is noticeably enhanced and the radial width and complex frequency are altered significantly. We concentrate here our efforts on this limiting case, which is particularly appropriate to internal transport barriers associated with weak magnetic shear. For the purpose of the analytical theory, the parameter $|\varepsilon_N \hat{s}/q|$ will be assumed small ($\varepsilon_N = L_N/R$ is the ratio of the density length-scale to the major radius).

Our main theoretical results are as follows. The frequency of the ion branch (measured in the ExB rotating frame), $\omega'=\omega-\omega_{ExB}$ , is small compared to the ion diamagnetic frequency ($\omega_i^*$) and has the opposite sign [Eq.(23)]. The growth rate is proportional to the absolute value of the magnetic shear $\hat{s}$ [Eq.(22)]. The normalized density fluctuation is small compared to the normalized ion temperature fluctuation and exhibits an important poloidal variation [Eq.(27)]. The frequency and the radial oscillation and decay lengths are larger in the actual toroidal geometry than in the slab model [Eqs.(23), (21a), (21b) and (24)]; the growth rates are identical [Eq.(22)]. Ion collisions can stabilize the ion branch, especially at high mode numbers [Eq.(28)], but have a negligible effect on the electron branch. The electron branch frequency ω' is about $\omega_e^*$, but dispersive effects are enhanced by the geometry [Eq.(11)]. The radial oscillation scale of the eigenmodes and their damping rate (the latter proportional to $|\hat{s}|$) are larger in the torus than in the slab by fractional powers of $(1+2q^2)$ [Eqs.(13) and (12); the latter result is opposite to that obtained in the strong overlap limit [1]; actually, the balloning formalism cannot grasp the toroidal effects obtained here because of restrictive assumptions]. The experimental relevance of those results is discussed in Section 5.

## 2. Methodology

The conventional drift wave ordering defined by

$$\omega \sim \omega_j^*, \quad k_\perp a_i \sim 1, \quad k_\parallel qR \sim 1, \tag{1a}$$

$$\omega_j^* / \Omega_i \sim a_i / L_\perp \sim \mu \ll 1 \tag{1b}$$

is assumed, together with

$$c_i / c_e \sim (m_e / m_i)^{1/2} \sim \mu \tag{1c}$$

(j is the species index, $a_i=c_i/\Omega_i$, $c_i$ and $\Omega_i$ are the ion Larmor radius, thermal velocity and gyro-frequency, respectively). Within this framework, the bulk (in opposition to trapped, not considered here) electrons behave adiabatically (as $\omega \ll c_e/qR$), i.e.,

$$t_e = 0 \qquad \text{and} \qquad \frac{n_e}{N_e} = \frac{e\phi}{T_e}. \tag{2}$$

We consider fluctuations of the form [2]

$$\frac{n(\psi,\chi,\varphi)}{N(\psi)} = \sum_{\ell,m} \hat{n}_{\ell,m}(\psi-\psi_{\ell,m},\chi;\psi_{\ell,m}) \exp\left\{ i\ell\left[ \varphi - \int_0^\chi \nu(\psi_{\ell,m},\chi')\,d\chi' \right]\right\}, \tag{3}$$

where ψ, χ and φ are the standard flux, poloidal and toroidal coordinates [3],

$$\nu(\psi,\chi) = (d\varphi/d\chi)_B = JB_\varphi / R \tag{4a}$$

is the pitch angle of the field lines, J the Jacobian of the transformation **r**→ (Ψ, χ, φ), $\ell$, m are the toroidal and reference poloidal mode numbers [$\sim\mu^{-1}$ according to (1a) and (1b)]. The rational magnetic surfaces are defined by

$$q(\psi_{\ell,m}) = \oint \nu(\psi_{\ell,m},\chi)\,d\chi / 2\pi = m/\ell. \tag{4b}$$

The functions $\hat{n}_{\ell,m}(\psi-\psi_{\ell,m},\chi;\psi_{\ell,m})$ describe the radial structure of the modes in the neighbourhood of the rational surfaces and, owing to the inhomogeneities of the equilibrium, residual poloidal and radial variations; the former plays hereafter an important role. The representation (3) is compatible with the periodicity and the long parallel wavelengths requirements [2]. The summations may be dropped here since $\hat{n}_{i,\ell,m}(\psi-\psi_{\ell,m},\chi;\psi_{\ell,m}) \cap \hat{n}_{i,\ell,m\pm1}(\psi-\psi_{\ell,m\pm1},\chi;\psi_{\ell,m\pm1}) \to 0$, as $w/\Delta \ll 1$. 2D equations are then obtained for the ion density, temperature and parallel flow velocity.

In the following, we consider large aspect ratio tokamaks with circular cross-sections; replacing $\chi$ by the usual poloidal angle $\theta$, we have $B=B_0(1-\varepsilon\cos\theta)$, where $\varepsilon=r/R_0$, r is the minor radius of the nested tori and $\theta=0$ corresponds to the outer equatorial plane. The ion magnetic drift frequency operator can be written as

$$\omega_{B,i} = 2(T_i / e_i R B_\varphi)[(m/r)\cos\theta + i\sin\theta\,\partial_{r-r_{\ell,m}}] + O(\varepsilon), \tag{5}$$

whereas

$$\nu(\psi,\chi) - \nu(\psi_{\ell,m},\chi) = (r - r_{\ell,m})\partial_r q(r) + O(\varepsilon) \tag{6}$$

The residual poloidal dependence of $\hat{n}_{i,\ell,m}$, $\hat{u}_{\|,i,\ell,m}$ and $\hat{t}_{i,\ell,m}$ is described by Fourier series, e.g.,

$$\hat{n}_{i,\ell,m}(r - r_{\ell,m},\theta) = \sum_P \left[\hat{n}_{i,\ell,m}(r - r_{\ell,m})\right]_p e^{ip\theta}, \tag{7}$$

so that the parallel differential operator $\partial_\chi + i\ell[\nu(\psi,\chi) - \nu(\psi_{\ell,m},\chi)]$ leads to the algebraic expressions $i(p + \hat{s}k_\theta x)$, where $\hat{s} = r\,d_r \ln q$, $k_\theta=m/r$ and $x = r - r_{\ell,m}$; we note that $|\hat{s}k_\theta x| << 1$; the index p labels the side-bands. Assuming $|\varepsilon_N \hat{s}/q| << 1$, the infinite system of ordinary differential equations can be truncated to p=0 and p=±1. Finally, the range of poloidal mode numbers for which toroidal and slab terms compete has been identified in order that the most complete equations are obtained (maximum complexity ordering).

## 3. Results for the electron drift branch

The electron drift branch is characterized by $\omega' \sim \omega^*$ and $\Delta_\perp \ll 1$, where $\Delta_\perp = a_i^2(\partial_x^2 + k_\theta^2)$ is the normalized Laplacian. The radial eigenvalue equation is

$$[\tau_e(1+2q^2)\Delta_\perp - \tau_e(1+\tau_e+\eta_i^{-1})(\omega' - \omega_e^*)/\omega_e^* + (\varepsilon_N \hat{s} x / q a_s)^2](\hat{n}_i)_o = 0, \tag{8}$$

where $\eta_i=L_{Ni}/L_{Ti}$, $\tau_e=T_e/T_i$ and $a_s^2=\tau_e a_i^2$. Equation (8), already obtained in [4], differs from the slab equation [5] by the (neoclassical) factor $(1+2q^2)$. The solutions are

$$\hat{n}_i^{(0)} \propto H_n(K_t^{1/2} x / a_s)\exp(-K_t x^2 / 2a_s^2) \tag{9}$$

with

$$K_t = i(1+2q^2)^{-1/2} |\varepsilon_N \hat{s}/q| \,\mathrm{sign}\,\omega_e^* \tag{10}$$

and

$$\Re\omega' = [1 - (1+2q^2)k_\theta^2 a_s^2 (1+\tau_e+\eta_i)\tau_e^{-1}]\omega_e^*; \tag{11}$$

$$\Im\omega' = \gamma_n = -(2n+1)(1+2q^2)^{1/2}(1+\tau_e+\eta_i)\tau_e^{-1}|k_\theta a_s||\hat{s}|\,c_s/qR \tag{12}$$

(The $H_n$ 's are Hermite polynomials).Damping of electron drift waves by transiting ions in a sheared magnetic field is a consequence of the wave energy being radiated away from the rational surface; as Eq.(12) shows, shear damping is larger in a toroidal plasma than predicted by the slab model. The scale of the radial oscillation is also larger:

$$w = a_s |K_t|^{-1/2} = a_s(1+2q^2)^{1/4} |\varepsilon_N \hat{s}/q|^{-1/2}. \tag{13}$$

The range of poloidal mode numbers over which both conditions of negligible overlap and negligible wave-particle resonant interaction are fulfilled is given by

$$\tau_e^{-1} q^{-2} \varepsilon_N^2 < k_\theta^2 a_s^2 < (1+2q^2)^{-1/2} q^{-1} |\varepsilon_N / \hat{s}|; \tag{14}$$

compatibility requires that

$$|\varepsilon_N \hat{s}/q| \ll \tau_e(1+2q^2)^{-1/2} \tag{15}$$

The ratio of the amplitudes of the side-bands $(\hat{n}_i)_{\pm1}$ and $(\hat{t}_i)_{\pm1}$ to the p=0 components is of order $(\varepsilon_N \hat{s}/q)^{1/2}$ if $k_\theta a_s$ is in the range (14). We note finally that

$$(\hat{t}_i)_0 = \eta_i(\hat{n}_i)_0 \tag{16}$$

## 4. Results for the ion drift branch

The ion drift branch is characterized by $\omega' \ll \omega^*$ and $\Delta_\perp \ll 1$. The radial eigenvalue is

$$\left[ D\Delta_\perp + \frac{1}{(2/3)-\eta_i}\frac{\omega'}{\omega_i^*} - \left(\frac{\varepsilon_N \hat{s}}{q}\right)^2 (x/a_i)\frac{\omega_i^*}{\omega' - 1.2 i \nu_i \Delta_\perp}(x/a_i)\right](\hat{t}_i)_0 = 0 \qquad (17a)$$

where

$$D = 1 + 12.5(1+\tau_e^{-1})[\varepsilon_N/(1-1.5\eta_i)k_\theta a_i]^2 - 2i\nu_i/(1-1.5\eta_i)\omega_i^* \qquad (17b)$$

[We note that, also in the slab (D=1), Eq.(17a) differs from the equation obtained by Coppi et al. [6], who failed to take finite Larmor radius corrections to the perpendicular components of the fluctuation velocity systematically into account; the latter are essential in the ion energy equation. The effect of collisions will be neglected at first. The solutions of Eq.(17a) are then

$$(\hat{t}_i)_0 \propto H_n(K_t^{1/2} x/a_i)\exp(-K_t x^2/2a_i^2) \qquad (18)$$

where $K_t$ is any root of

$$D_0^2 K_t^2 [k_\theta^2 a_i^2 + (1+2n)K_t] = [(2/3)-\eta_i]^{-1}(\varepsilon_N \hat{s}/q)^2 \qquad (19)$$

satisfying Re $K_t>0$; the complex eigenvalues $\omega'$ are given by

$$\omega'/\omega_i^* = [(2/3)-\eta_i]D_0[k_\theta^2 a_i^2 + (1+2n)K_t]; \qquad (20)$$

here, $D_0$ is given by Eq.(17b) with $\nu_i=0$.

### 4.1 The Small Magnetic Shear Limit

If $\hat{s}<<1$, the condition of negligible overlap can be satisfied either when $k_\theta^2 a_i^2 < K_t$ or when $k_\theta^2 a_i^2 > K_t$. One can show that under no circumstances is wave-particle resonant interaction negligible in the former case. In the latter, Eq.(19) yields $K_t = K_t^{(0)} + K_t^{(1)}$ with

$$K_t^{(0)} = \pm i(\eta_i - 2/3)^{-1/2}|\varepsilon_N \hat{s}/q k_\theta a_i D_0| \qquad (21a)$$

$$K_t^{(1)} = -(1+2n)(K_t^{(0)})^2/2k_\theta^2 a_i^2 . \qquad (21b)$$

$\Re e\, K_t$ is therefore positive corresponding to bounded eigenmodes ($\eta_i>2/3$ is required for bounded unstable solutions). Further, the growth / the damping rates of the modes are

$$\gamma_n = (\mp \mathrm{sign}\,\omega_i^*)(1+2n)(\eta_i - 2/3)^{1/2}|\hat{s}|c_i/qR \qquad (22)$$

whereas their angular frequency is

$$\Re e\, \omega' = -(\eta_i - 2/3)D_0 k_\theta^2 a_i^2 \omega_i^* . \qquad (23)$$

$\Re e\, \omega'$ and $\omega_i^*$ have opposite signs. The characteristic radial oscillatory scale of non overlapping toroidal eigenmodes,

$$w = a_i|K_t^{(0)}|^{-1/2} = a_i(\eta_i - 2/3)^{1/4}|q k_\theta a_i D_0/\varepsilon_N \hat{s}|^{1/2}, \qquad (24)$$

is again larger than in the corresponding slab model (factor $D_0^{1/2}$).

One can show that the range of poloidal mode numbers over which both conditions of negligible overlap and negligible wave particle resonant interaction are fulfilled is given by

$$[\eta_i - (2/3)]^{-1}q^{-1}|\varepsilon_N| < D_0|k_\theta a_i|^3 < [\eta_i - (2/3)]^{-1/2}|\varepsilon_N \hat{s}/q| ; \qquad (25)$$

compatibility requires that

$$|\hat{s}| < [\eta_i - (2/3)]^{1/2}. \qquad (26)$$

The toroidal eigenmodes are given by series similar to Eq.(7). Of particular interest is the expression of the density fluctuation:

$$\hat{n}_i(x,\theta) = (\tau_e \omega_i^*)^{-1}\{(\hat{s}k_\theta x)^2(\omega_{t,i}^2/\omega') + (5/3)[\eta_i - (2/3)]^{-1}\omega_{B,i}\}(\hat{t}_i)_0 \qquad (27)$$

where $\omega_{t,i}=c_i/qR$ and the operator $\omega_{B,i}$ is defined in (5); the amplitude of the density side-bands $(\hat{n}_i)_{\pm 1}$ are typically larger than the amplitude of the main component $(\hat{n}_i)_0$ by $(\varepsilon_N \hat{s}/q)^{-1/3}$, whereas $(\hat{t}_i)_{\pm 1}/(\hat{t}_i)_0 \sim (\varepsilon_N \hat{s}/q)^{1/3}$; moreover, the ratio $(\hat{n}_i)_0/(\hat{t}_i)_0$ is of order $(\varepsilon_N \hat{s}/q)^{2/3}$.

### 4.2 The Role of Collisions

In the limit $k_\theta^2 a_i^2 > K_t$, ion collisions tend to stabilize the ion branch at the rate $\Im m\,\omega' = \Delta\gamma = -(4/3) k_\theta^2 a_i^2 \nu_i$ ; (28)

## 5. Summary and Experimental Relevance

### 5.1 Internal Transport Barrier

The growth rate of the ion drift mode [Eq.(22)] is proportional to the absolute value of the magnetic shear parameter. **This result brings a simple explanation to the formation of internal transport barriers with minimum q profiles**. The assumption of weak overlap is particularly appropriate here (a contrario, the opposite limit considered in most other works is not valid). Since the growth rate is furthermore independent of $|k_\theta a_i|$, Landau damping (not considered here) is expected to suppress the instability at long wavelenghts.

### 5.2 Radiative Improved Mode

At the other end of the spectrum, i.e. for finite $|k_\theta a_i|$ values, ion collisions may stabilize the system if $\nu_i$ is large enough, as Eq.(28) shows. **Shrinking of the instability range owing to both Landau and collisional damping may explain the reduction of conductive/convective anomalous transport at the edge of the high density Radiative Improved confinement mode discharges** (the above analysis requiring both $\hat{s} \ll 1$ and negligible mode overlap is however not directly applicable).

### 5.3 Asymmetry of the fluctuations

The amplitude of the ion drift mode density fluctuations are characterized by important poloidal asymmetries, cf. Eq.(27). Under the above assumptions ($K_t < k_\theta^2 a_i^2$ and low shear), two maxima arise near the equatorial plane, respectively on the low and on the high field sides. **Those results are relevant to observations in the core of TEXT-U**. The ion temperature fluctuations are larger than the density fluctuations and only weakly $\theta$ dependent.

### References


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