The electron in superstrong laser fields

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Contents

A. Particle acceleration in vacuum

Arbitrary travelling plane wave:

Vector potential $\mathbf{A} = \mathbf{e_y} A(\xi)$, $\xi = x - ct$, $\mathbf{E} = -\partial_t \mathbf{A}$, $\mathbf{B} = \nabla \times \mathbf{A}$

Initial conditions: $\mathbf{A}(\xi(t) = 0) = 0$, $\mathbf{v}(t = 0) = 0$

exactly solvable problem owing to

conservation of canonical momentum $p_y = \gamma m v_y + qA = 0$,

and $\gamma m(v_x - c) = \text{const}$

$y(\xi) = \frac{q}{mc} \int_0^{\xi} A(\xi')d\xi'$

$\gamma m v_x = \frac{q^2}{2mc} A^2(\xi)$

$x(\xi) = -\left(\frac{q}{mc}\right)^2 \frac{1}{2} \int_0^{\xi} A^2(\xi')d\xi'$

energy $E = mc^2 \left[1 + \frac{1}{2}\left(\frac{qA}{mc}\right)^2\right] = mc^2(1 + \frac{1}{2}a^2)$, $a = \frac{qA}{mc}$

Maximum acceleration $\triangle E = \frac{1}{2} mc^2 \hat{a}^2$, $\hat{a}$ amplitude

$t = \frac{1}{c}(x - \xi) = -\frac{\xi}{c} + \frac{x(\xi)}{c}$

Special case: Monochromatic wave $\mathbf{A}(\mathrm{x}, t) = \mathbf{e_y} A(\varphi) \sin\varphi$, $\varphi = kx - \omega t$

Energy $E = mc^2 \left(1 + \frac{1}{2}\hat{a}^2 \sin^2\varphi\right)$

Minimum angular spread $\tan\alpha = \left|\frac{v_y}{v_x}\right| = \frac{2}{\hat{a}}$

Lawson-Woodward (LW) theorem:

Net energy gain from smooth em. pulse is zero.

Troha theorem (generalization of LW):

No energy gain from plane em. wave when radiation losses ignored

Clarification by J. X. Wang and W. Scheid et al.

(Phys. Rev. E **65**, 028501 (2002))

Simple explanation

$\frac{d}{dt}\gamma m v_x = q v_y B_z = -\frac{q^2}{2\gamma m}\partial_x A^2 = -\nabla E = 2\mathbf{f}_p \sin 2\varphi$

$\mathbf{f}_p = -\frac{1}{4\gamma_0}\nabla\hat{a}^2$, ponderomotive force

momentum $p_x = \gamma m v_x = \frac{1}{4\gamma_0} m c \hat{a}^2 (1 - \cos 2\phi)$

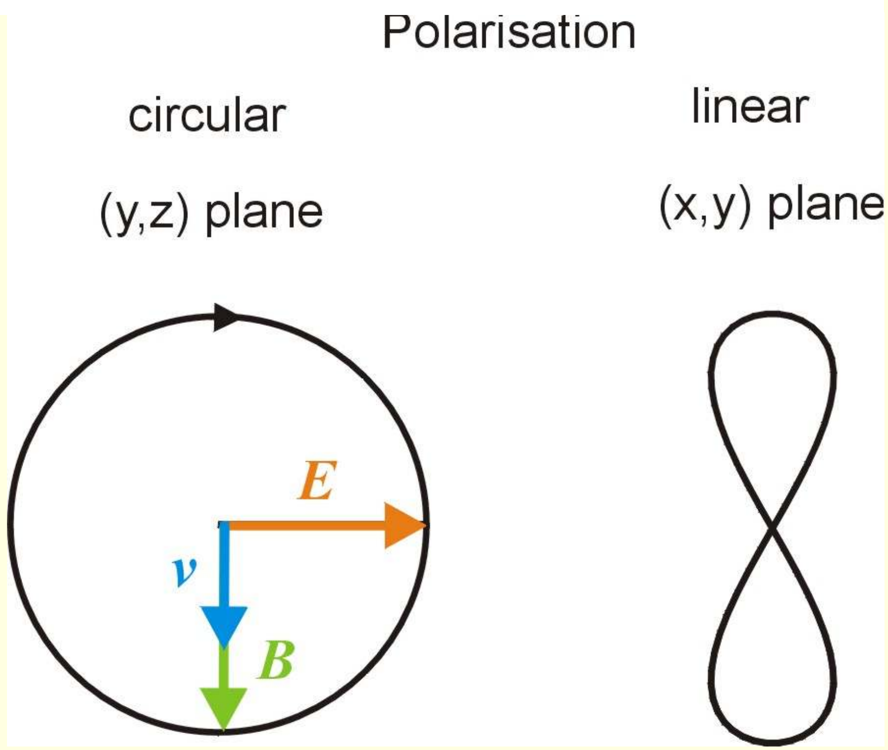
Oscillation center system

linear polarization:

$$E_{os} = mc^2\{\sqrt{1+\hat{a}^2/2} - 1\},\ m_{eff} = \gamma_{oc} m,\ \gamma_{oc} = \sqrt{1+\hat{a}^2/2}$$

circular polarization:

$$\gamma_{oc} = \sqrt{1+\hat{a}^2}$$



CP, electron:

A points into – ***B*** direction

LP: ratio width / hight < 18% for ***l*** = ∞

free particle acceleration	quiver motion in oscillation center frame
$A_y = \hat{A}_y(\varphi)\sin\varphi, \quad \varphi = k(x - ct)$	$A_y = \hat{A}_y(\varphi)\cos\varphi, \quad \varphi = k(x - ct)$
$\gamma\frac{v_x}{c} = \frac{1}{2}\hat{a}^2, \quad \gamma\frac{v_y}{c} = -\hat{a}$	$\hat{x} = -\frac{1}{8k}\frac{\hat{a}^2}{\gamma_0^2}, \quad \hat{y} = \frac{1}{k}\frac{\hat{a}}{\gamma_0}$
$\Delta x = \frac{1}{4k}\hat{a}^2\frac{\pi}{2}$	$\gamma_0 = \gamma_{oc} = \left(1 + \frac{1}{2}\hat{a}^2\right)^{1/2}$
$\gamma_{\text{free}} = 1 + \frac{1}{2}\hat{a}^2$	$W = mc^2\left[\left(1 + \frac{\hat{a}^2}{2}\right)^{1/2} - 1\right]$
$\mathcal{E}_{\text{free}} = \frac{1}{2}mc^2\hat{a}^2$	

I [Wcm^{-2}]	10^{18}	10^{20}	10^{22}		10^{24}	
$\hat{a}_e$, $\hat{a}_p$	0.69	6.87	68.67	3.7×10^{-2}	686.7	0.37
v_y/v_x	2.91	0.3	0.03	53.5	2.9×10^{-3}	5.35
$\gamma_{\text{free}} - 1$	0.24	23.6	2.36×10^3	7×10^{-4}	2.36×10^5	0.07
$\mathcal{E}_{\text{free}}$	120.5 keV	12.0 MeV	1.2 GeV	656 keV	120 GeV	65.6 MeV
$\Delta x/\lambda$	0.03	2.95	295	8.7×10^{-5}	2.95×10^4	8.7×10^{-3}
$\gamma_{\text{oc}} - 1$	0.16	3.96	47.6	3.5×10^{-4}	484.6	0.034
W	81.8 keV	2.02 MeV	24.3 MeV	292 keV	247.6 MeV	32.3 MeV
$\hat{x}/\hat{y}$	0.0740	0.1730	0.1766	4.6×10^{-3}	0.1767	0.045

Table 8.1. Maximum achievable acceleration $\mathcal{E}_{\text{free}}$ of electrons and protons in a fourth cycle of a plane Ti:Sa laser pulse (row 5) and corresponding quiver energy W (row 8) in the oscillation center frame. I laser intensity, $\hat{a}$ normalized vector potential amplitude, v_y/v_x ratio of velocities in field and pulse direction, γ_{free}, γ_{os} Lorentz factors, $\Delta x/\lambda$ acceleration distance during a fourth cycle $\Delta\varphi = \pi/2$ in units of Ti:Sa wavelength (λ = 800 nm) during a fourth cycle, $\hat{x}/\hat{y}$ ratio of oscillation amplitudes. First column for a given intensity gives the values for electrons, second column (where listed) the values for protons.

B. Generalized ponderomotive force f_p

The principle

Whenever a high-frequency motion exhibits an asymmetry a drift motion is induced on the slow time scale.

Standard ponderomotive force

Zero frequency (= secular) force from perturbation theory:

$$m[\partial_t \mathbf{v} + (\mathbf{v}\nabla)\mathbf{v}] = q\mathbf{E}[\mathbf{x}_0 + \xi] + \mathbf{v} \times \mathbf{B}$$

$$\Longrightarrow \mathbf{f}_p = -\frac{1}{4} mc\nabla\ \hat{\mathbf{a}}^2 = -\frac{q^2}{4m\omega^2}\nabla\ \hat{\mathbf{E}}^2$$

Properties of $\mathbf{f}_p$:

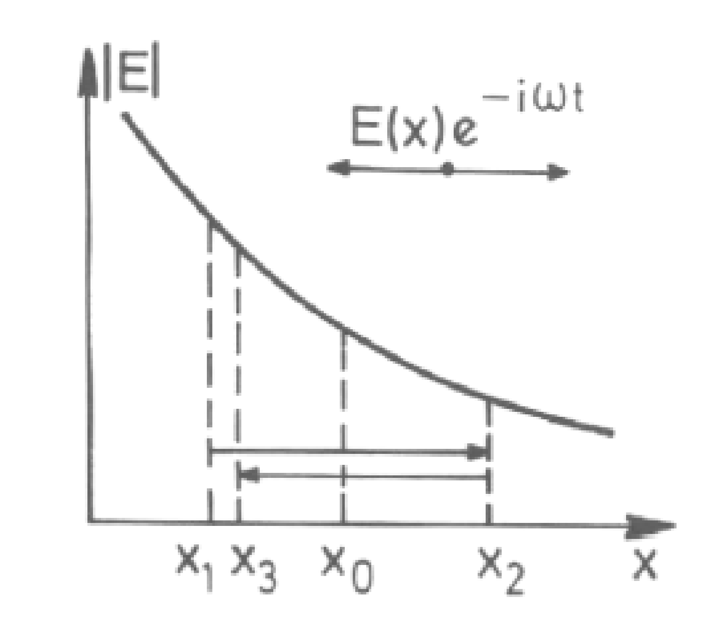
- gradient force
- the same for **transverse** and **longitudinal** waves
- rotational symmetry around beam axis (x direction)

Limits of validity:

- pulse envelope at rest in lab *frame*
- subrelativistic intensities, $|\hat{\mathbf{a}}| < \mathbf{1}$
- oscillation center velocity $|\dot{\mathbf{x}}_0| = |\mathbf{v}_0|/c << 1$

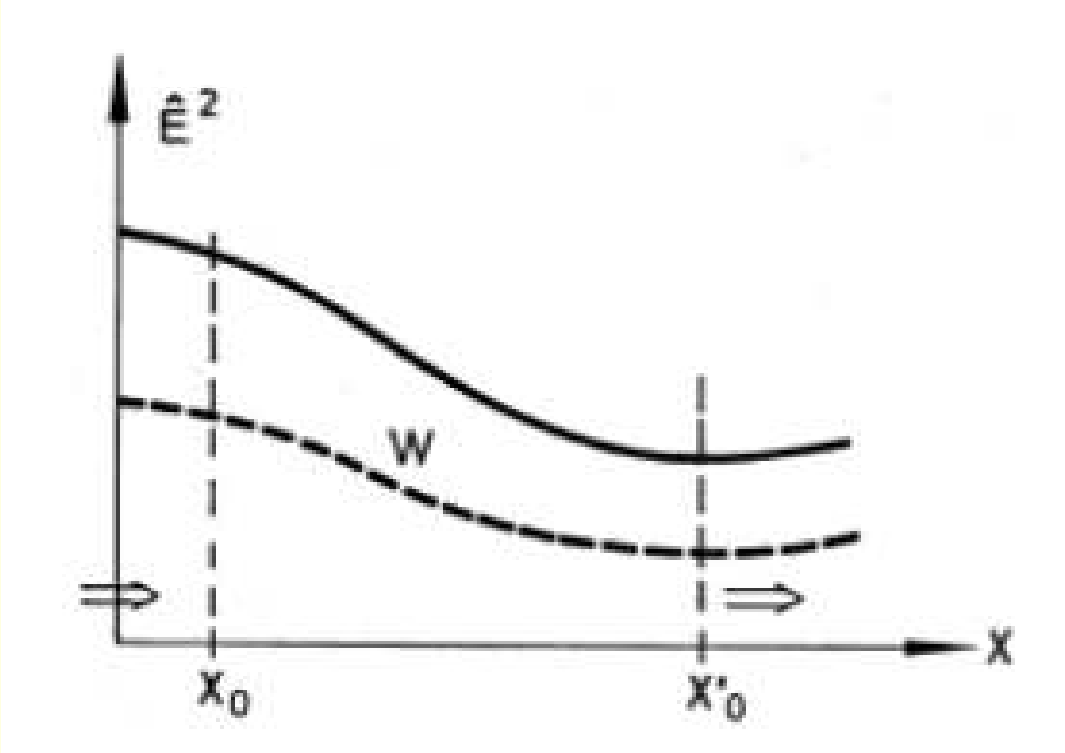
Circular polarization: perturbation procedure above yields $\mathbf{f}_p = 0$

Pictures



Asymmetry produces a drift

A free electron can neither absorb nor emit a photon



To be found:

relativistically correct $\mathbf{f}_p$

Condition: $\exists$ oscillation center $\mathbf{x}_0(t)$ for $\forall\, t$

Step 1: Go to system in which oscillation center $\mathbf{x}_0 = 0$, $\mathbf{x}' = \mathbf{x}_0 + \xi = \xi$.

Step 2: Cycle average Hamiltonian

$$H(\mathbf{p}, \mathbf{x}, t) = q\phi(\varphi') + \{m^2c^2 + [\mathbf{p} - q\mathbf{A}(\varphi')]^2\}^{1/2}$$

over invariant phase φ or, equivalently, proper time t':

$$\Longrightarrow E_{os}(\mathbf{x}_0, t) = m_{eff}c^2$$

lab frame: $\mathbf{p}_0 = m_{eff}\gamma_0\mathbf{v}_0$, $H_0 = E_{os}(\mathbf{x}_0, t) + c[m_{eff}^2c^2 + \mathbf{p}_0^2]^{1/2}$

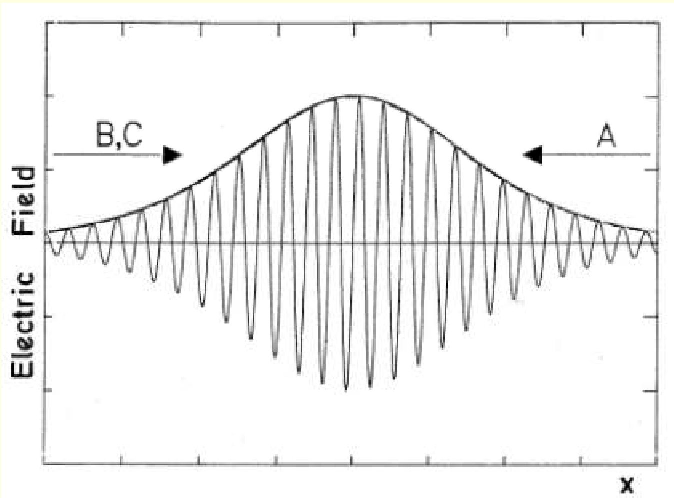
$\mathbf{f}_p = \dot{\mathbf{p}}_0 = -\partial H_0/\partial \mathbf{x}_0$

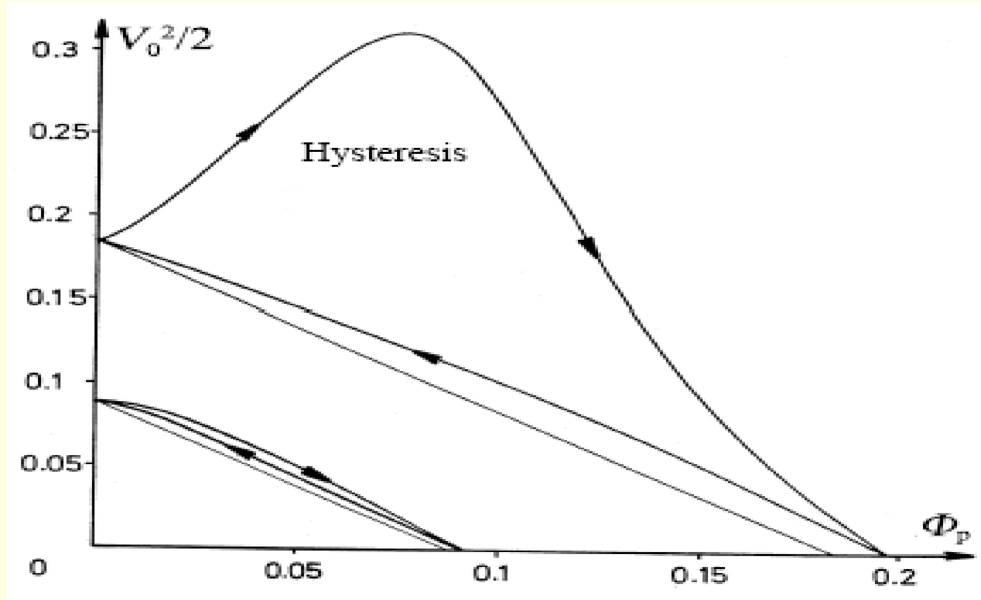
Properties

- Ponderomotive potential $\phi_p = E_{os}(\mathbf{x}_0, t)$ *adiabatic invariant*
- valid in any inertial frame for all intensities
- pulse travelling at any speed, $\mathbf{v}_0 = \dot{\mathbf{x}}_0$ arbitrary
- differs for $\mathbf{E}$ transverse from $\mathbf{E}$ longitudinal, example
- motion oblique to pulse axis breaks rotational symmetry, example

Uphill acceleration

non-relativistic





$$f_{p,l} = (1 - V_0)(1 - 3V_0)\, f_{p,t}$$

Electron plasma wave travelling to the right, amplitude fixed in space, electron injected from left with $V_0 > 0$ normalized to constant phase velocity

Broken symmetry: Polarization angle dependence

Assume standing wave $\boldsymbol{A} = \hat{\boldsymbol{A}} \cos \boldsymbol{kx} \exp(-i\omega t)$ non-relativistic,

electron crosses $\boldsymbol{A}$ at arbitrary speed $\boldsymbol{v}_o \perp \boldsymbol{k}$

Decomposing $\boldsymbol{v} = \boldsymbol{v}_o + \boldsymbol{v}_1$ and cycle-averaging the Hamiltonian

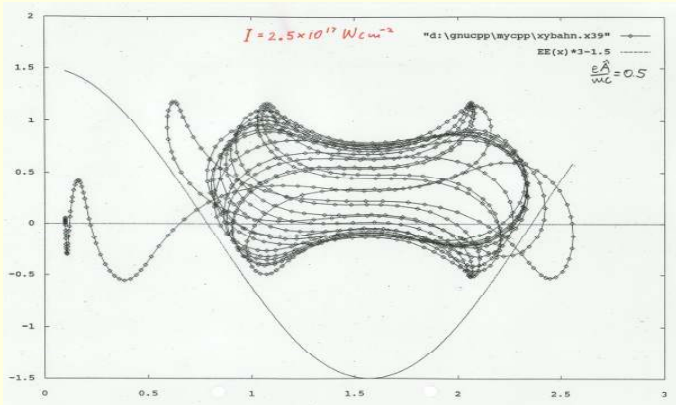
$$H = (\boldsymbol{p}_1 - e\boldsymbol{A})^2 / 2\gamma_0 m = \gamma_0 m\, \boldsymbol{v}_1^2 / 2$$

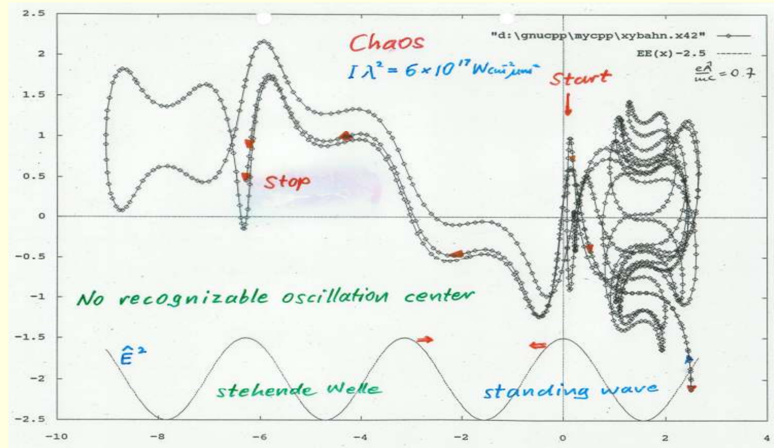
$$\boldsymbol{v}_1 = (e/\gamma_0 m)\, \mathrm{Re}\, \{\boldsymbol{A} - i \text{ nabla } (\boldsymbol{v}_0 \boldsymbol{A}) / \omega + i\, (\boldsymbol{v}_1 \text{ nabla})\, \boldsymbol{A} / \omega\}$$

$$\Rightarrow \boldsymbol{f}_p = -(e^2/4\gamma_0 m) \text{ nabla } [\, \hat{\boldsymbol{A}}^2 - 2\boldsymbol{\beta}_0 \hat{\boldsymbol{A}}\,]^2; \quad \boldsymbol{\beta} = \boldsymbol{v}_0 / c$$

C. Chaos & resonant orbits

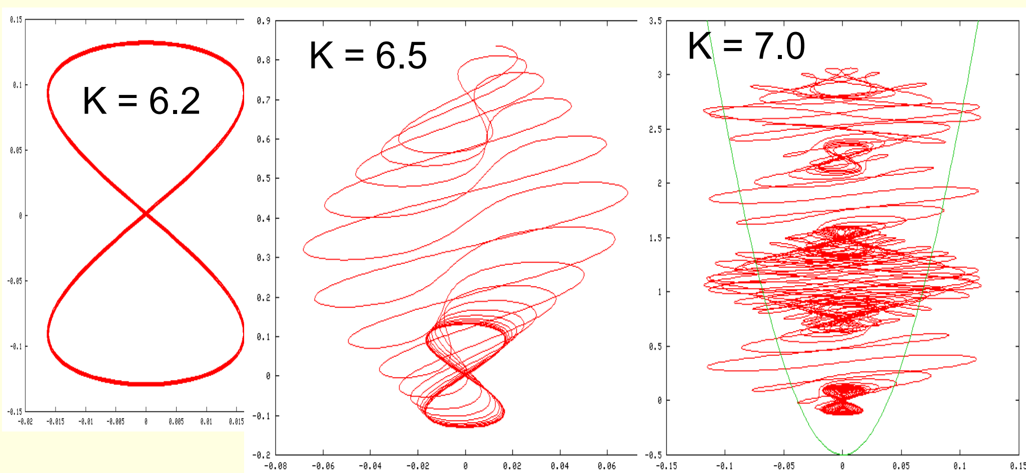
- **Ponderomotively induced chaos** *:* One regular field, two different time scales

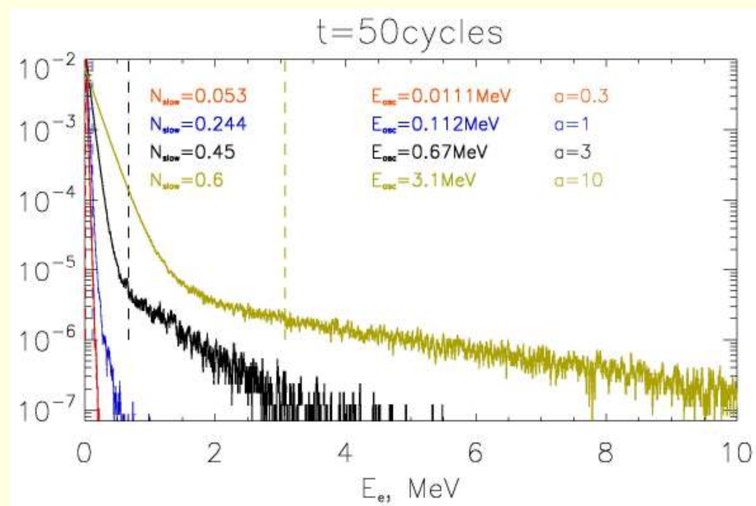




- ***Chaos in crossed regular fields*** *:* Static *E* – field

$$V = m_e c^2 \hat{a} \sqrt{1 + \kappa^2 x^2}$$





- ***Fast resonant*** **electrons :** Do not undergo regular motion

D. Relativistic transparency and intense pulse propagation

Relativistic critical density increase relevant for • fast ignition, • advanced electron and ion acceleration schemes, • fast pulse switching and pulse tailoring

Standard formula $n_{cr} = [1 + \hat{a}^2/(2)]^{1/2}$, $\qquad A = (\boldsymbol{A}, \Phi/c), \; J = (\boldsymbol{j}, -en_e c)$

needs closer inspection

$$\Box A = -J/\varepsilon_0 c^2, \;\; \partial_\alpha A^\alpha = 0, \;\; \mathrm{j} = -en_e\mathbf{u}.$$

$$\mathrm{j}(x,t) = -\frac{e}{\gamma_{\mathrm{th}}\gamma(\mathbf{u})}\sum \gamma(\mathrm{v}_i)\mathrm{v}_i = -e\sum\frac{\mathrm{p}_i}{\gamma_i m_e},$$

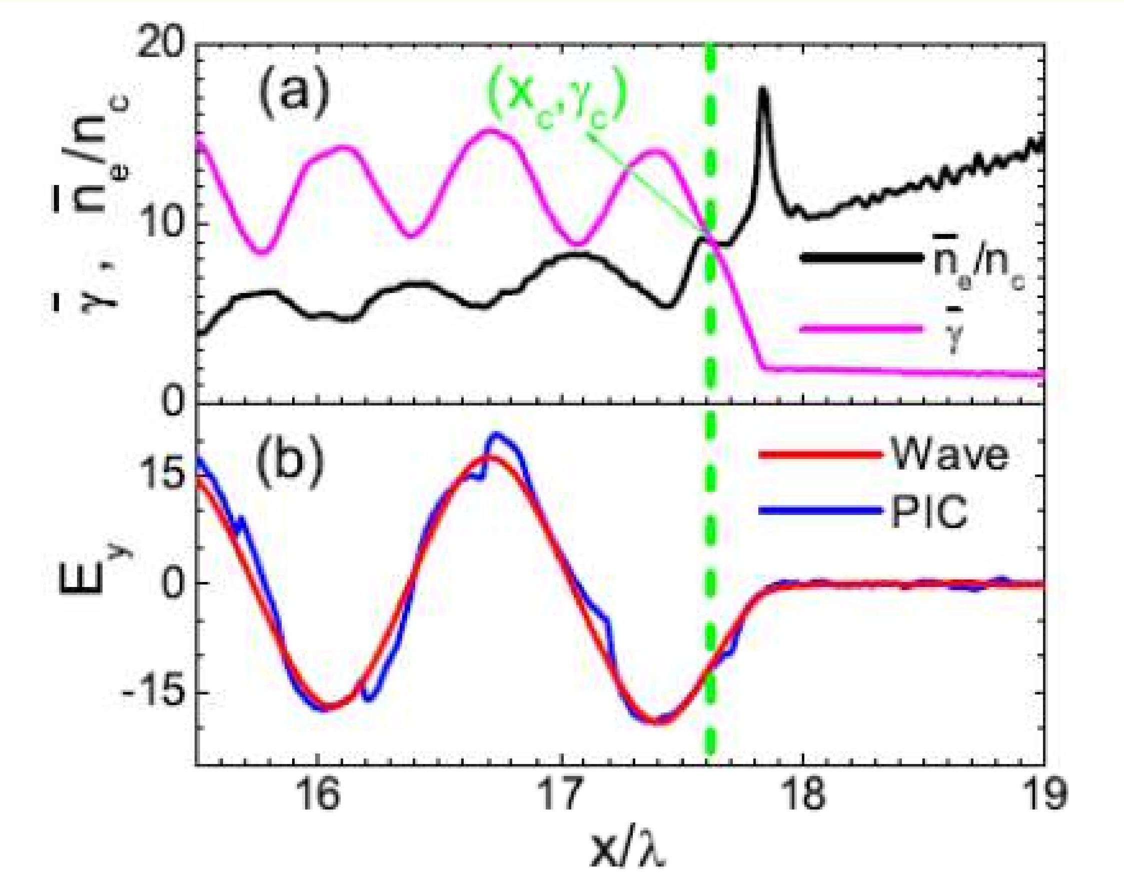
To be confirmed

$$\partial_{xx}E + \frac{\omega^2}{c^2}(1 - \frac{n_e}{n_{\mathrm{cr}}})E = 0; \qquad n_{\mathrm{cr}} = \overline{\gamma}(x)n_c.$$

What we can hope: **fundamental frequency ω, cycle averaged**

$$\mathrm{j}_\omega = -e\overline{n}_e\mathbf{u}_\omega = -\frac{e}{\overline{\gamma}}\sum\gamma(\mathrm{v}_i)\mathrm{v}_i|_\omega;$$

$$\overline{\gamma} = \overline{\gamma_{\mathrm{th}}\gamma(\mathbf{u})} = \overline{\sum\gamma(\mathrm{v}_i)/n_e}.$$



Su-Ming Weng (2011):

Linear polarization,

$\hat{a} = 10$,

density profile length

$L = 3\ \lambda$

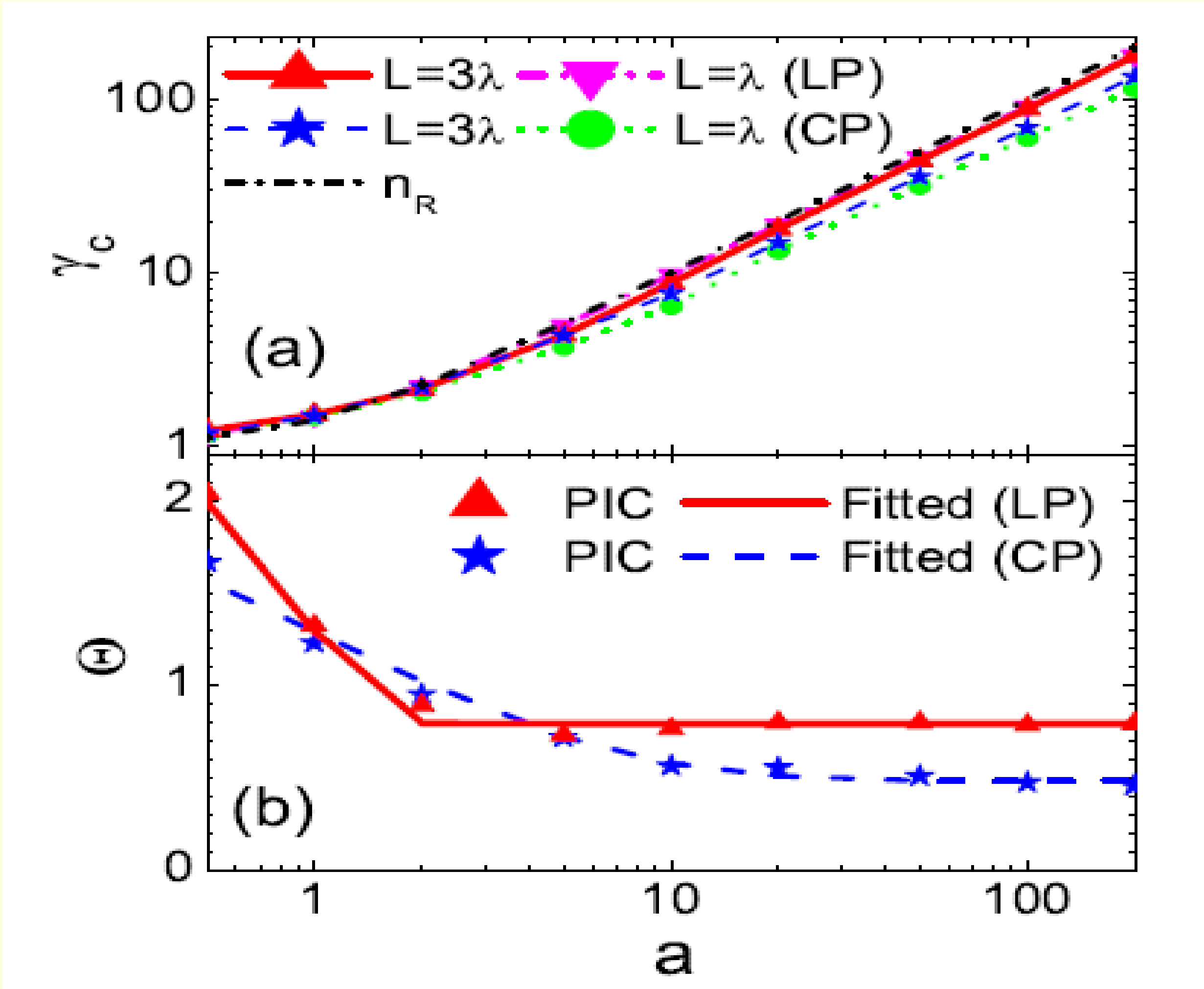
Wave equation well satisfied on laser cycle average

Thermal mass increase included

Remember: U four flow velocity (current);

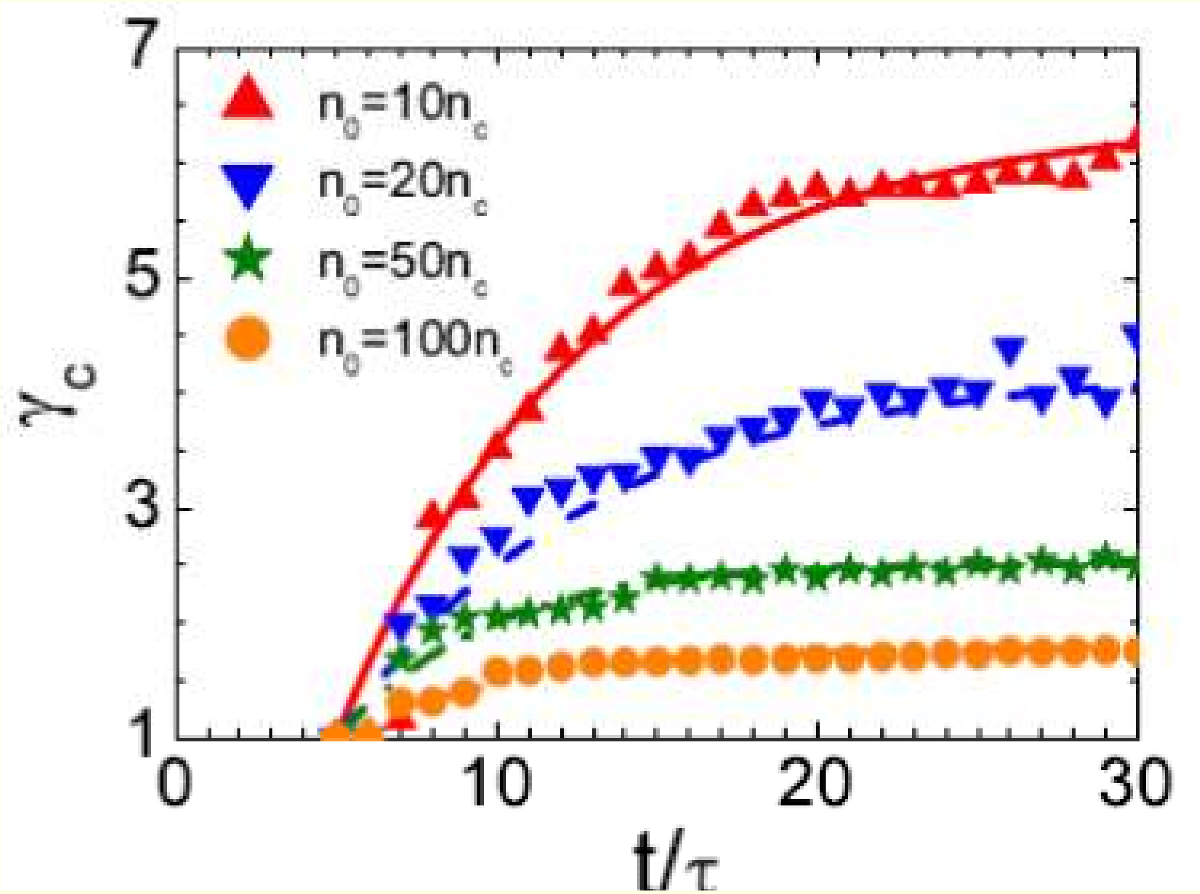
W individual particle thermal four velocity defined by $UW = u_\alpha w^\alpha = 0$

Mass increase by $W = w^\alpha$



Su-Ming Weng (2011):

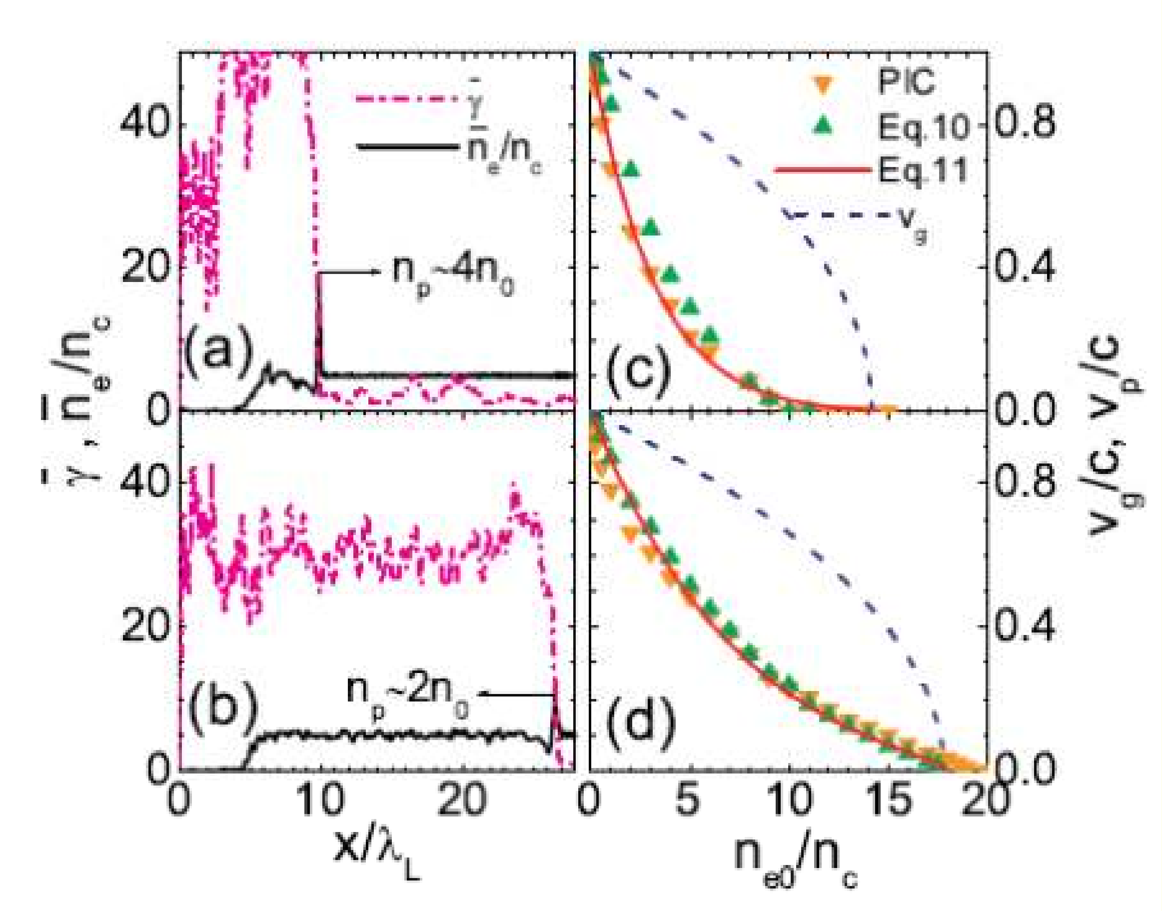
$$n_{cr} = [1 + \Theta \hat{a}^2/(2)]^{1/2}\, n_c$$



Su-Ming Weng (2011)

Density step, â = 10, LP

Pulse propagation in relativistically undeerdense plasma



$\hat{a} = 20.$ Remember: $v_g v_\varphi = c^2$

$$v_p = v_g \exp(- n_p/n_{cr})$$

Radiation reaction

Radiation loss/sec of accelerated particle in rest frame is $\kappa\, a^2$, $\kappa = \mu_0 q^2/6\pi c$

non-rel.: $d\mathbf{p}/dt = \mathbf{f}_{ex} - \boldsymbol{\sigma}$, $d(m\mathbf{v}^2/2)/dt = \mathbf{f}_{ex}\mathbf{v} - \boldsymbol{\sigma}\mathbf{v} = -\kappa\, \mathbf{a}^2$

rel.:

$$\kappa \mathrm{a}^2 \Rightarrow \kappa a^\alpha a_\alpha, \sigma \mathrm{v} \Rightarrow \sigma^\alpha v_\alpha, a^\alpha v_\alpha = 0 \Rightarrow \dot{a}^\alpha v_\alpha = -a^\alpha a_\alpha \Rightarrow \underline{\sigma^\alpha = -\kappa \dot{a}^\alpha}$$

There must hold: $v_\alpha \sigma^\alpha = 0$ => simplest ansatz: $\sigma^\alpha = -\kappa da^\alpha/d\tau - \kappa a^\beta a_\beta/c^2$

➔ Abraham-Lorentz-Dirac equation

$$m\frac{dv^\alpha}{d\tau} = f^\alpha_{ex} + \kappa\left(\dot{a}^\alpha - \frac{a^\beta a_\beta}{c^2} v^\alpha\right)$$

Runaway solution: $\tau_0 = \kappa/m = 6.26 \times 10^{-24}$ s

$$v^\alpha = (\sinh \exp[\tau/\tau_0], 0, 0, \cosh \exp[\tau/\tau_0]), \; a^\alpha = \frac{\exp[\tau/\tau_0]}{\tau_0}(\cosh \exp[\tau/\tau_0], 0, 0, \sinh \exp[\tau/\tau_0])$$

ALD equation in three-form

$$\frac{d\mathbf{p}}{dt} = \mathbf{f}_{ex} + \tau_0\left[\frac{d}{dt}\left(\gamma\frac{d\mathbf{p}}{dt}\right) - \frac{\mathbf{a}^2}{\gamma c^2}\mathbf{p}\right],\ \mathbf{a} = \frac{d\mathbf{v}}{dt}$$

$$\frac{d\gamma}{dt} = \mathbf{f}_{ex}\frac{\mathbf{v}}{c} + \tau_0\left[\frac{d}{dt}\left(\gamma\frac{d\gamma}{dt}\right) - \frac{\mathbf{a}^2}{c^2}\right]$$

Exact solution by A. Di Piazza in Landau's version of ALD

From the equations above follows that in circular polarization classical effects of radiation reaction are irrelevant at all laser intensities in the optical domain.

In linear polarization classical effects become significant at intensities exceeding 10^{23} W/cm^2 and 1 µm wavelength (see following picture by A. Di Piazza)

Solution by A. Di Piazza

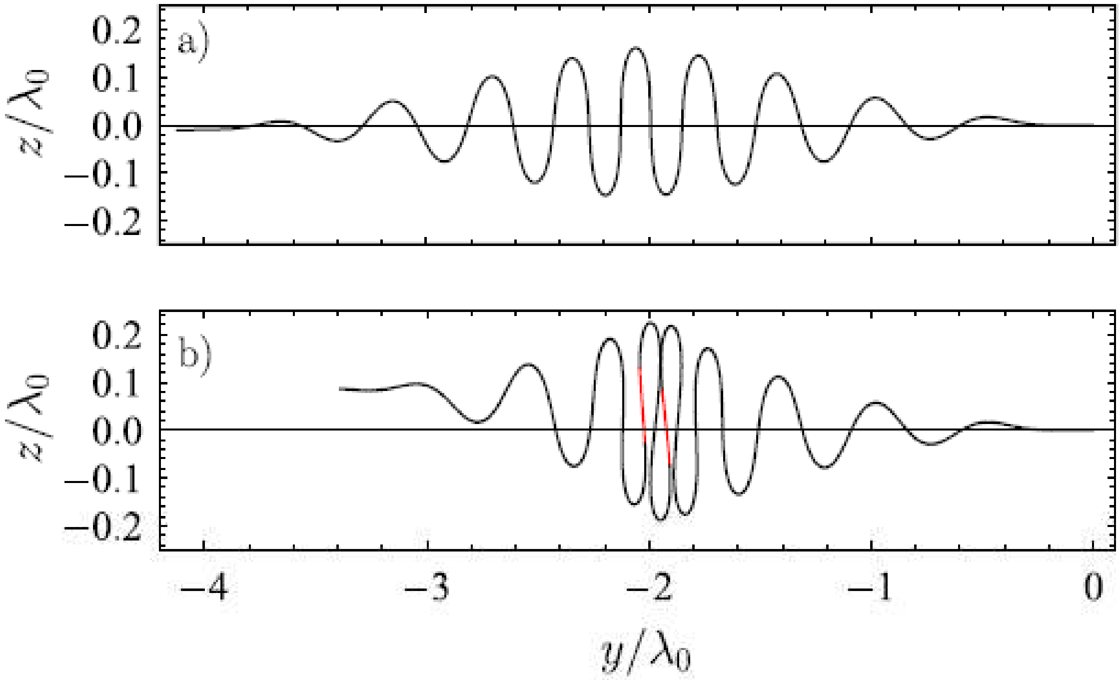


FIG. 1 (color online). The electron trajectories in units of the laser wavelength λ_0 calculated (a) by removing and (b) by keeping the RR terms in Eq. (3). The initial electron energy is 40 MeV, the laser field intensity $I_0 = 5 \times 10^{22}$ W/cm^2, the wavelength $\lambda_0 = 0.8$ μm, the pulse duration 27 fs and the waist size $\sigma_0 = 2.5$ μm. The red (gray) portions of the trajectory are those in which the longitudinal velocity of the electron is positive.

Summary

The motion of a charged particle in a plane wave can be solved exactly. In circular polarization radiation reaction plays no role at any laser intensity; in linear polarization self-field reaction in the IR becomes significant beyond $I = 10^{23}$ W/cm².

When an oscillaion center exists a ponderomotive force $\boldsymbol{f}_p$ can be defined. Covariance is preserved if the cycle averaging is done in proper time along the orbit. Under a strong particle drift $\boldsymbol{f}_p$ shows a polarization dependence; uphill acceleration is observed in the longitudinal ectric field.

In presence of a static or oscillatory field in addition to the laser wave the particle orbits show chaotic behaviour.

In the relativistic regime the critical density increase exists on a cycle average for the fundamental wave, however its magnitude depends on the density profile and on polarization, circular vs linear.

A new time scale has been found for building up of the critical density.

At relativistic intensities laser pulse propagation is no longer of dispersive character, i.e., propagation at group velocity; rather is it slowed down considerably and is determined by partial pulse reflection from the relativistically transparent plasma.