Nonlinearly driven second harmonics of Alfvén Cascades

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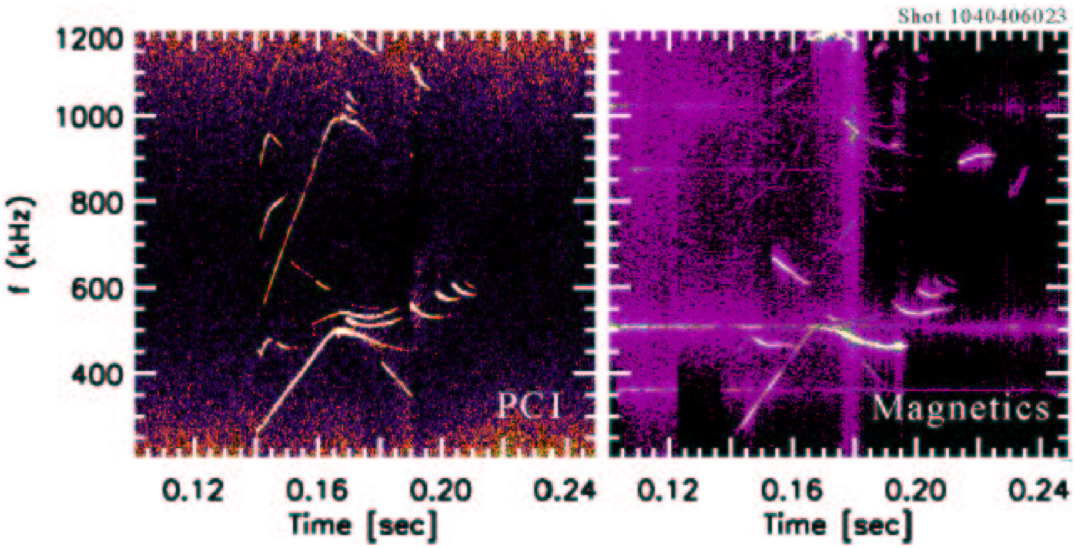
Alfvén Cascades (reminder)

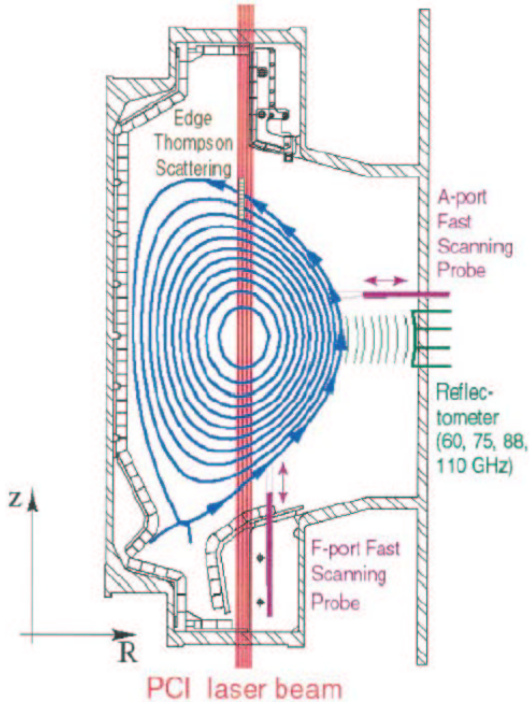
- Alfvén Cascades (ACs) are shear Alfvén eigenmodes localized at the minimum q surface ($r = r_*$) in plasmas with magnetic shear reversal.
- The eigenmode frequency $\omega = \omega_{\mathrm{A}m,n} + \Delta\omega$ is slightly higher than the local shear Alfvén wave frequency : $\omega^2_{\mathrm{A}m,n} = k^2_{\|*} v^2_{\mathrm{A}} = (m - nq_*)^2 v^2_{\mathrm{A}} / (Rq_*)^2$.
- During current ramp-up q_* decreases with time which leads to the characteristic sweeping of the AC frequency.
- The frequency sweeps up to $\omega_{\mathrm{TAE}} = v_{\mathrm{A}}/(2Rq_*)$, where transition to TAE mode occurs.

Experimental motivation (I)

- Measurements with Phase Contrast Imaging (PCI) on Alcator C-mod show a second harmonic density perturbation. The second harmonic signal is faint at the edge magnetic probes, suggesting a narrow radial profile of the perturbation.



[J. A. Snipes *et al.* 31st EPS Conference, London, 2004]



Experimental motivation (II)

- Our goal is to calculate the second harmonic density perturbation driven by quadratic terms in the MHD equations.
- The analysis can potentially be used to estimate the AC amplitude at the mode center.
- Instrumental non-linearities may be a factor in interpretation of PCI measurements. This issue needs to be addressed separately.

Preliminary remarks

- The shear Alfvén wave has a relatively weak nonlinearity. The quadratic nonlinearities vanish in a uniform plasma with straight magnetic field lines.
  $\Rightarrow$ Consider coupling to compressional Alfvén and acoustic perturbations.

- The double AC eigenmode frequency

$$2\omega = 2(\omega_{\mathrm{A}m,n} + \Delta\omega) = \omega_{\mathrm{A}2m,2n} + 2\Delta\omega$$

  is near the double mode number branch of the Alfvén continuum. This can lead to resonant enhancement of the 2ω perturbation.

Mode representation

Magnetic field representation: $\mathbf{B} = \mathbf{B}_0 + (\mathbf{B}_1 + \mathbf{B}_2 + \text{c.c.})$, where $\mathbf{B}_1 \propto e^{-i\omega t}$, $\mathbf{B}_2 \propto e^{-2i\omega t}$ etc.

Use the plasma velocity representation

$$\begin{aligned}
\mathbf{v}_1 &= \frac{\mathbf{b}_0}{B_0} \times \nabla \dot{\Phi}_1 + \frac{1}{B_0} \nabla_\perp \dot{\Psi}_1, \\
\mathbf{v}_2 &= \frac{\mathbf{b}_0}{B_0} \times \nabla \dot{\Phi}_2 + \frac{1}{B_0} \nabla_\perp \dot{\Psi}_2 + \dot{\xi}_2 \mathbf{b}_0,
\end{aligned}$$

where the shear (Φ), compressional (Ψ) and acoustic (ξ) functions represent the three degrees of freedom of the plasma.

The first harmonic acoustic perturbation ξ_1 is negligible for the main part of the AC. We exclude it by assuming $\beta = 0$.

Derivation path

- The known linear AC eigenfunction (Φ_1 and Ψ_1) produces a quadratic nonlinear force in the momentum balance equation.
- We calculate the driven compressional (Ψ_2), acoustic (ξ_2) and shear Alfvén (Φ_2) perturbations to Φ_1 from the momentum balance equation.
- We use Φ_1, Φ_2, Ψ_1, Ψ_2, and ξ_2 in the second harmonic continuity equation

$$\frac{\partial \rho_2}{\partial t} + \nabla \cdot (\rho_1 \mathbf{v}_1) + \nabla \cdot (\rho_0 \mathbf{v}_2) = 0,$$

to determine the 2ω density perturbation

$$\rho_2 = \rho_{\Phi_1^2} + \rho_{\Psi_1^2} + \rho_{\Phi_1 \Psi_1} + \rho_{\Phi_2} + \rho_{\Psi_2} + \rho_{\xi_2}.$$

Momentum balance equation (I)

Consider the three "projections" of the momentum balance equation:

- Acoustic equation: $\mathbf{b}_0 \cdot \{\text{Momentum balance eq.}\}$
- Compressional equation: $\nabla \cdot \left[\frac{1}{B_0}\{\text{Momentum balance eq.}\}_\perp\right]$
- Vorticity equation: $\nabla \cdot \left[\frac{\mathbf{B}_0}{B_0^2} \times \{\text{Momentum balance eq.}\}\right]$

The acoustic and compressional equations give

$\rho_{\Psi_1^2},\ \rho_{\Phi_1\Psi_1},\ \rho_{\xi_2},\ \rho_{\Psi_2} \ll \rho_{\Phi_1^2},\ \rho_{\Phi_2}.\ \Rightarrow$

$\boxed{\text{The dominant contribution to } \rho_2 \text{ arises from shear Alfvén perturbations.}}$

- The $\rho_1\mathbf{v}_1$ term in the continuity equation generates

$$\rho_{\Phi_1^2} \simeq \rho_0 \left[\frac{\mathbf{b}_0}{B_0} \times \nabla\Phi_1\right] \cdot \nabla \left[\nabla\Phi_1 \cdot \left(\nabla \times \frac{\mathbf{b}_0}{B_0}\right)\right] \sim m^2\rho_0\Phi_1^2/(r^3RB^2)$$

- ρ_{Φ_2} is determined by quadratic terms in the vorticity equation.

Momentum balance equation (II)

The second harmonic vorticity equation has the form

$$\frac{\bar{B}_0 r}{m}\left[4\frac{1}{r}\frac{d}{dr}\left(r\frac{d\tilde{\Phi}_2}{dr}\left(\frac{\omega^2}{\bar{v}_{\mathrm{A}}^2}-\bar{k_{\|}}^2\right)\right)-16\frac{m^2}{r^2}\tilde{\Phi}_2\left(\frac{\omega^2}{\bar{v}_{\mathrm{A}}^2}-\bar{k_{\|}}^2\right)\right]=$$

$$2\frac{dD}{dr}\left(\left(\frac{d\tilde{\Phi}_1}{dr}\right)^2-\frac{m^2}{r^2}\tilde{\Phi}_1^2\right)+D\left(\frac{d\tilde{\Phi}_1}{dr}\frac{d^2\tilde{\Phi}_1}{dr^2}-\tilde{\Phi}_1\frac{d^3\tilde{\Phi}_1}{dr^3}\right)-\bar{k}_{\|}\frac{d^2\bar{k}_{\|}}{dr^2}\frac{d\tilde{\Phi}_1^2}{dr}$$

where $D=\omega^2/\bar{v}_{\mathrm{A}}^2-\bar{k}_{\|}^2$,

$\Phi_1=\tilde{\Phi}_1(r)\exp\left[i(n\phi-m\theta-\omega t)\right]$,

$\Phi_2=\tilde{\Phi}_2(r)\exp\left[2i(n\phi-m\theta-\omega t)-i\frac{\pi}{2}\right]$.

The derivation involves flux surface averaging (denoted by bar).

Recall AC eigenmode theory

Eigenmode equation with normalized coordinate $x \equiv (r - r_*)m/r_*$:

$$\frac{d}{dx}(S + x^2)\frac{d\tilde{\Phi}_1}{dx} - (S + x^2)\tilde{\Phi}_1 + Q_1\tilde{\Phi}_1 = 0. \tag{1}$$

Effects of hot ions and toroidicity are included in $Q_1 = Q_{\mathrm{hot}} + Q_{\mathrm{tor}}$:

$$Q_1 \equiv \frac{\omega_{\mathrm{A}}^2 q_*^2 \bar{R}^2}{\bar{v}_{\mathrm{A}}^2 (m - nq_*)} \frac{q_*}{r_*^2 q_*''} \left(\frac{\omega_{\mathrm{c,hot}}}{\omega_{\mathrm{A}}} \left(-\frac{r}{\rho_0} \frac{d\bar{\rho}_{\mathrm{hot}}}{dr} \right)_{r=r_*} + \frac{2m\epsilon_*(\epsilon_* + 2\Delta_*')}{1 - 4(m - nq_*)^2} \right).$$

Solutions to (1) exist if $Q_1 > 1/4$. The distance $\Delta\omega$ to the Alfvén continuum is determined by the eigenvalue S:

$$S \equiv \frac{2\omega_{\mathrm{A}}\Delta\omega}{\bar{v}_{\mathrm{A}}^2} \frac{mq_*}{r_*^2 q_*''} \frac{\bar{R}^2 q_*^2}{m - nq_*}.$$

Variational solution for $Q_1 = 1$ gives the lowest order radial eigenmode $S = 0.10$, $\tilde{\Phi}_1 = Ae^{-x^2/1.76^2}/\sqrt{S + x^2}$

Second harmonic shear Alfvén sideband

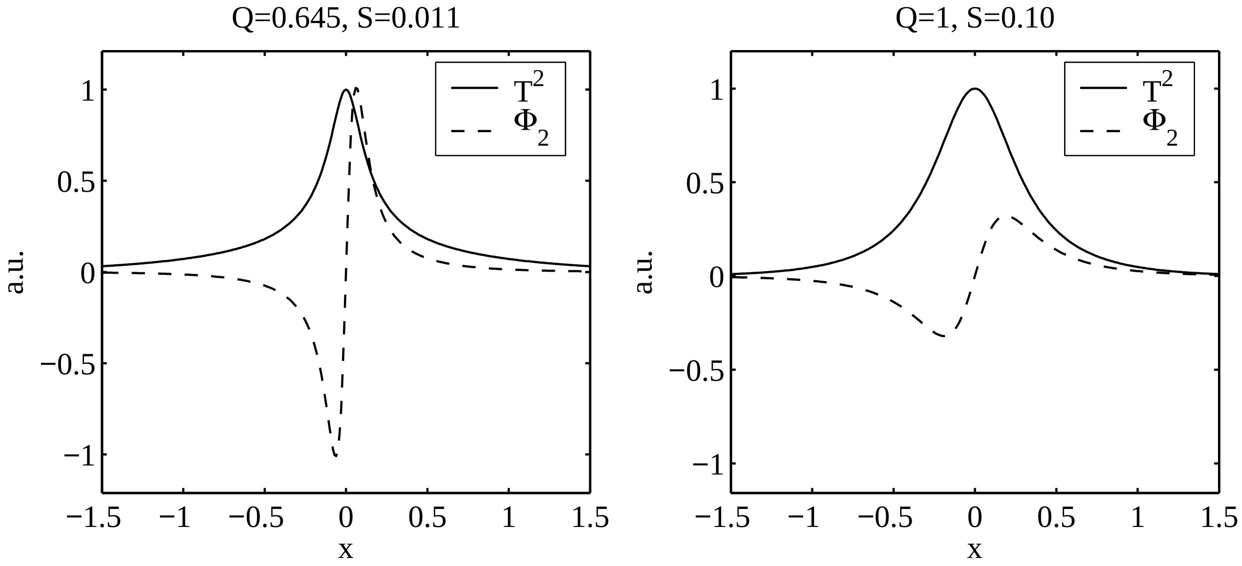
Denote $T(x) \equiv m\tilde{\Phi}_1/(r_* \sqrt{\bar{B}_0})$. The equation for second harmonic sideband reduces to

$$
\begin{aligned}
&4\frac{d}{dx}(S+x^2)\frac{d\tilde{\Phi}_2}{dx} - 16(S+x^2)\tilde{\Phi}_2 + 4Q_2\tilde{\Phi}_2 = \\
&= 4x((T')^2 - T^2) + (S+x^2)\left(T'T'' - TT'''\right) + (T^2)', \quad (2)
\end{aligned}
$$

where S is the AC eigenvalue and Q_2 accounts for the hot ion and toriodicity effects.

In the special case $Q_{\mathrm{hot}} \gg Q_{\mathrm{tor}}$, Q is independent of m and n and $Q_1 = Q_2 = Q$.

Numerical solution for shear Alfvén sideband



- The amplitude of Φ_2 decreases with increasing Q. For $Q \simeq 0.65$ we find

$$\Phi_2 \sim T^2 \sim \frac{m^2}{r_*^2 \bar{B}_0} \quad \Rightarrow \quad \rho_{\Phi_2} = \rho_0 \nabla \Phi_2 \cdot \left(\nabla \times \frac{\mathbf{B}_0}{B_0^2} \right) \sim \frac{m^3 \rho_0}{r_*^3 R \bar{B}_0{}^2} \tilde{\Phi}_1^2.$$

- For large values of Q, $\rho_{\Phi_1^2}$ can compete with ρ_{Φ_2}.

Summary

- The ratio of density perturbations at the second and first harmonics can be estimated as

$$\frac{\rho_2}{\rho_1} \sim \frac{m^2 \Phi_1}{r^2 B_0} \sim \frac{mq}{\epsilon} \frac{|\mathbf{B}_1|}{B_0}.$$

- Second harmonic shear Alfvén sideband dominates over compressional Alfvén and acoustic perturbations.

- Comparison of the first harmonic and second harmonic PCI signals can provide important information about internal perturbations at the mode location, assuming that instrumental nonlinearities are insignificant.