Anomalous particle transport and self-consistent wave evolution of magnetized electrons interacting with electron-cyclotron waves

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Abstract. The nonlinear interaction of relativistic electrons with electron-cyclotron waves in a constant magnetic field is studied. The electron diffusion across the magnetic field is analyzed over and near the local threshold to chaos for ionosphere plasma parameters assuming that the amplitude of the wave is constant. The diffusion is found to obey simple power law in time and the scaling exponent is indicative of sub-diffusion. The anomalous diffusion is caused by the effect of the resonant phase-space islands in the particle motion. The self-consistent treatment uses a closed set of nonlinear equations consisting from the equations of motion under the electromagnetic field as well as the wave equation for the evolution of the vector potential. The electron motions drive the evolution of the wave amplitude and frequency mismatch through the current. We use the above model to study the relativistic electron-cyclotron absorption in the ionosphere and in fusion plasma.

1. Introduction

The nonlinear interaction of electrons with electron-cyclotron waves is of great importance for the laboratory and astrophysical plasmas. This problem has been investigated in detail for the last forty years [1][2][3]. Under conditions of resonance between the electron's cyclotron motion and the Doppler-shifted wave frequency, the wave-particle interaction is characterized by a significant energy exchange and an electron acceleration [3]. This effect has been considered in the study of electron-cyclotron instabilities [4] as well as for the interpretation of radiation observations in the ionosphere, and it is widely applied in fusion experiments for plasma heating and current drive. The linear theory for the wave absorption and the quasilinear theory for the electron distribution function are currently the main tools for the study of wave-particle interactions. However, in cases where nonlinear effects are important, the validity of these theories becomes questionable. In a recent work, electron-cyclotron heating simulations were performed using a nonlinear treatment, in contrast with the linear and quasilinear theories [5]. The results show that the deviation can be strong for present day fusion experiments. Furthermore, in numerous publications it is shown that the quasilinear theory breaks down due to the presence of resonant islands in the system phase-space [6][7]. These formations cause large time-space scaling of the particle kinetics, and thus non-Gaussian diffusion. In this report, we focus on the interaction of magnetized relativistic electrons with electron-cyclotron waves. We present the results of a recent analysis on the anomalous diffusion of electrons in the presence of a monochromatic electron-cyclotron wave of constant amplitude, without using a quasilinear approximation for the phase space [8]. The complex formation of the phase-space is underlined, which is strongly connected to the anomalous particle diffusion. We also perform a self-consistent analysis of the wave-electron interaction. A set of nonlinear and relativistic equations is derived, which account for the effects of electron motions on the temporal evolution of the wave. As an application, the problem of electron-cyclotron absorption for the cases of ionosphere and fusion plasma is studied.

2. Anomalous electron diffusion under a constant-amplitude wave

We briefly discuss the results of Ref. [8] on the interaction of magnetized relativistic electrons with an electron-cyclotron wave ($\omega, k$), which propagates at an angle $\theta$ with respect to a uniform magnetic field $B=B_0z$. We assume that the wave does not have a temporal evolution other than the phase-term $e^{i(kr-\omega t)}$. 

1
The knowledge obtained from this simplified approach are a guide towards a more complete, self-consistent treatment. The wave field is described by the normalized vector potential

\[
A = A_0 \left( \cos \theta \sin \phi \hat{x} + \cos \phi \hat{y} - \sin \theta \sin \phi \hat{z} \right)
\]  

(2.1)

where \( \phi = k \cdot r - \omega t \) is the wave phase. In (2.1), the amplitude \( A_0 \) is normalized with \( m_e c^2 / e \), where \( c \) is the speed of light and \( e, m_e \) are the electron charge and rest mass. A value \( A_0 \) corresponds to a power flux \( S = 30 N_p \omega \omega^2 A_0^2 \) (W/cm²). The spatial coordinates are normalized with \( c/\omega_e \), the time with \( \omega_e^{-1} \), the frequencies with \( \omega_e \) and the wave-vectors with \( \omega_e / c \), where \( \omega_e = e B_0 / m_e c \) is the cyclotron frequency. The wave-particle interaction can be described by the two-dimensional, autonomous Hamiltonian [9]

\[
H = \left[ 1 + (p_x + A_0 \cos \theta \sin \phi)^2 + (p_y + x + A_0 \cos \phi)^2 + (p_z - A_0 \sin \theta \sin \phi)^2 \right]^{1/2} \left[ -p_y / N_0 \cos \theta \right]
\]

(2.2)

where \( H \) is normalized with \( m_e c^2 \) and the canonical momenta with \( m_e c \). We study the system for parameters corresponding to radio-wave heating of the night-time ionosphere: the frequency is \( \omega / 2\pi = 3 \) MHz, the magnetic field is \( B_0 = 3.5 \times 10^{-5} \) T, the plasma density is \( n_e = 10^{2} \) cm\(^{-3} \) and the initial electron distribution is monoenergetic with \( E_0 = 1.279 \) MeV. For this case, the dispersion relation for circularly polarized waves, \( N_0^2 = 1 - \omega_p^2 / \omega (\omega - \omega_e) \), where \( N_0 \) is the refraction index and \( \omega_p = (4\pi n_e e^2 / m_e)^{1/2} \) the plasma frequency, is a good approximation. An extensive study on the dynamics of the Hamiltonian (2.2) has been performed in Ref. [9]. It is found that significant chaos exists only for amplitudes larger than a critical value \( A_{0,c} \), which depends on the other wave parameters \( \omega, \theta \). A local estimate of \( A_{0,c} \) can be found by utilizing the fact that acceleration comes together with chaos. The complexity of the phase space is visualized in Fig. 1(a), where we present a Poincaré surface-of-section for \( \theta = 40^\circ \) and \( A_0 = 0.1 \). Clearly, the phase space is a highly-complex, inhomogeneous mixture of periodic and stochastic behavior. The time-scaling of the diffusion is determined by the exponent \( \alpha \) of the power-law \( \langle (r - r_0)^2 \rangle \sim t^\alpha \). In Fig. 1(b) we show the exponent \( \alpha \) as a function of \( A_0 \) for \( \theta = 40^\circ \). We observe that for all \( A_0 > A_{0,c} \approx 0.03 \) it is \( \alpha < 1 \), which corresponds to sub-diffusion. This is connected to the resonant islands of the phase-space seen in Fig. 1(a). These formations cause particle trapping, which suppresses the diffusive behavior. Obviously, this is a case where the quasilinear theory breaks down because chaos is not complete. Also, the wave slows down the radial transport of the electrons, acting as a barrier, and this may have important consequences for the overall particle transport.

![Fig. 1. (a) Poincaré surface-of-section \((x,p_x)\), (b) Scaling exponent \( \alpha \) as a function of amplitude \( A_0 \).](image)

3. Self-consistent model for wave-particle interaction

In this section, a self-consistent treatment for the nonlinear interaction of electron-cyclotron waves with magnetized relativistic electrons is presented. The model relies on the coupling of the relativistic equations of motion under the electromagnetic field with the wave equation. The vector potential is
given again by (2.1), but in the self-consistent model the amplitude \( A \) and frequency \( \omega \) have a time dependence. The electron motions drive the temporal evolution of the wave amplitude and frequency through the current density. The normalized equations of motion are

\[
\dot{\mathbf{p}} = -\mathbf{A} + \frac{\dot{\mathbf{z}} \times \mathbf{p}}{\gamma} + \mathbf{p} \times \left( \nabla \times \mathbf{A} \right) / \gamma, \quad \dot{\gamma} = -\mathbf{p} \dot{\mathbf{A}} / \gamma
\]  

(3.1)

where \( \mathbf{p} \) is the relativistic mechanical momentum and \( \gamma \) the Lorentz factor. The normalizations applied in (3.1) are the same as in Sec. 2. Using (2.1) for the vector potential, the equations of motion become

\[
\dot{p}_\| = -A \left( \cos \theta \cos \psi \sin \phi + \sin \psi \cos \phi \right) - A \omega \left( \cos \theta \cos \omega \sin \psi \cos \phi \right) / \gamma
\]

(3.2.a)

\[
\dot{\psi} = \frac{1}{p_\|} \gamma + A \left( \cos \psi \cos \theta \sin \phi \right) / p_\perp + A \omega \left( \cos \psi \sin \theta \cos \phi \right) / p_\perp
\]

(3.2.b)

\[
\dot{\omega} = k p_\perp \sin \theta \cos \psi / \gamma + k p_\perp \cos \theta / \gamma - \omega
\]

(3.2.c)

\[
\dot{p}_\perp = -A \psi \sin \psi + A \omega p_\perp \left( \cos \psi \cos \theta \sin \phi \right) / p_\perp
\]

(3.2.d)

\[
\dot{\psi} = A p_\perp \left( \cos \psi \sin \psi \right) / \gamma - A \omega p_\perp \left( \cos \psi \cos \theta \sin \phi \right) / \gamma
\]

(3.2.e)

In (3.2), \( p_\|, p_\perp \) are the parallel and perpendicular momenta with respect to the magnetic field and \( \psi \) is the phase of the perpendicular momentum, \( \psi = \tan^{-1}(p_\perp / p_\|) \), which depends on time. The normalized wave equation for the evolution of the vector potential reads

\[
\nabla^2 \mathbf{A} - \dot{\mathbf{A}} = -\omega^2 \mathbf{j}
\]

(3.3)

where \( \mathbf{j} \) is the current density, normalized with \( e n_c \). Using again the representation (2.1) for \( \mathbf{A} \), we obtain equations for the temporal evolution of the amplitude \( A_0 \) and the frequency \( \omega \)

\[
\dot{A}_0 + (k^2 - \omega^2) A_0 = -\omega^2 \left[ j_\| \left( \cos \theta \cos \psi \sin \phi + \sin \psi \cos \phi \right) - j_\psi \sin \theta \sin \phi \right]
\]

(3.4.a)

\[
A_0 \dot{\omega} + 2 \omega A_0 = -\omega^2 \left[ j_\| \left( \sin \psi \sin \phi \cos \theta \cos \phi \right) + j_\psi \sin \theta \cos \phi \right]
\]

(3.4.b)

where \( j_\|, j_\psi \) are the parallel and perpendicular current densities. Assuming an initial electron distribution function \( f_0(\mathbf{r}_0, \mathbf{p}_0) \), the normalized current density is given by

\[
\mathbf{j} = -\int d^3 \mathbf{r}_0 d^3 \mathbf{p}_0 f_0(\mathbf{r}_0, \mathbf{p}_0) \delta(\mathbf{r} - \mathbf{r}_0) \mathbf{p}.
\]

The right-hand sides in the equations (3.4), which represent the effect of the particle motions on the wave, are spatially-dependent through the wave phase and the current densities. This dependence is periodic in space, because the wave-number \( k \) is constant, and since we are interested in the temporal evolution of \( A_0, \omega \), we may average (3.4) over space in order to obtain equations only over time. Taking into account the form of the current density, the result after the averaging is

\[
\dot{A}_0 + (k^2 - \omega^2) A_0 = -\omega^2 \left[ p_\| d \mathbf{p}_\| d \mathbf{p}_\perp \int dp_0 f_0 \left[ \left( j_\| \left( \cos \theta \cos \psi \sin \phi + \sin \psi \cos \phi \right) - j_\psi \sin \theta \sin \phi \right) / \gamma \right] \right]
\]

(3.5.a)

\[
A_0 \dot{\omega} + 2 \omega A_0 = -\omega^2 \left[ p_\| d \mathbf{p}_\| d \mathbf{p}_\perp \int dp_0 f_0 \left[ \left( j_\| \left( \sin \psi \sin \phi \cos \theta \cos \phi \right) + j_\psi \sin \theta \cos \phi \right) / \gamma \right] \right]
\]

(3.5.b)
where the mean values are taken over the plasma electrons. Equations (3.2), (3.5) form a closed self-consistent set of equations describing the wave-particle system, where $A_0$, $\omega$ in (3.2) are calculated as solutions of (3.5), while the integrals involved in (3.5) are determined from the electron orbits, which are solutions of (3.2). We apply this model to the case of electron-cyclotron absorption in the ionosphere and in fusion plasma. We reconsider the case of Sec. 2, in order to underline the connection of the self-consistent system to the simplified model. In Fig. 2(a), the amplitude $A_0$ is shown as a function of time. The evolution of the amplitude is in accordance with the behaviour presented in Sec. 2. Wave power is absorbed by the plasma particles, which gain significant amounts of energy, until the amplitude reaches the threshold to chaos. After this point, the absorption procedure saturates due to electron trapping in regions of phase-space where the motion is regular. For small amplitudes, the phase-space is dominated by the islands and the absorption of the electromagnetic radiation is not possible. In Fig. 2(b) the evolution of the average electron energy is shown. The energy gain of the plasma electrons due to absorption of the electron-cyclotron wave is obvious. The results presented in Fig. 2 imply a consistency with the energy conservation law. This consistency can be verified quantitatively as follows: the energy conservation theorem in the plasma volume occupied by the test particles reads 
\[ \omega_p^2 \Delta \langle \gamma \rangle + \Delta (A_0^2 \omega^2) = 0, \]
where the first term stands for the total particle energy and the second for the wave energy. During the test-particle simulations, we numerically followed the validity of this relation, and the resulting accuracy was of the order $10^{-4}-10^{-5}$.

**FIG. 2.** (a) Amplitude $A_0$ and (b) mean electron energy $\langle \gamma \rangle$ vs time for absorption in the ionosphere.

We further consider a simple model for the absorption region in a fusion plasma. In our simulation, we consider the absorption of the 2nd harmonic X-mode injected in the infinite slab at an angle $\theta=60^\circ$, which corresponds to a toroidal angle $\theta'=300^\circ$. We follow the wave-particle interaction for $t\approx 1500 \Omega_e^{-1}$, which is approximately the time needed by the wave, moving with velocity $c/N_0$, to cover the minor diameter $d_T=1$ m of the tokamak. The magnetic field is $B_0=2.5$ T, the plasma density is $n_e=10^{12}$ cm$^{-3}$ and the initial wave-power is $P_{\text{wav}}=400$ KW. We assume that the initial distribution is Maxwellian with temperature $2.55$ KeV. The refraction index is calculated using the cold plasma dispersion relation for the X-mode [10]. In Fig. 3(a), the vector potential amplitude $A_0$ is plotted as a function of time, together with the result predicted by the linear theory of absorption. The nonrelativistic linear absorption coefficient reads [5]

\[ \Gamma_L = \pi/8 \omega_p^2 k^2 \sin^2 \theta v_{th} \left( \sin^2 \theta N_0^2 + B_1 N_0^2 C_1 \right) / \omega \cos \theta \left( B_2 N_0^2 + C_2 \right) \]

where $v_{th}$ is the thermal velocity and the coefficients $B_1, B_2, C_1, C_2$ are functions of the wave and plasma parameters $\theta, \omega, \omega_p$ and $\omega_e$. The relation (3.6) is a good approximation for our case, where the Doppler effect is dominant over the relativistic corrections. This is verified by the fact that the relevant condition $N_0 \cos \theta > v_{th} / c$ [10] is valid during the entire simulation. The disagreement of the nonlinear result with the prediction of the linear theory is evident. The nonlinear calculations show a significant
reduction of the absorption. This is in accordance with recent results on the importance of nonlinear effects during electron-cyclotron heating [5]. In fig 3(b) the evolution of the wave frequency is shown. The frequency, despite its nonlinear variation, remains confined near the initial second-harmonic value.

**FIG. 3.** (a) $A_0$, together with the linear prediction, and (b) $\omega$ vs $t$ for absorption in a fusion plasma.

4. Conclusions

In this report, we studied the nonlinear interaction of magnetized relativistic electrons with electron-cyclotron waves. Using a simple model where the wave has constant amplitude, we demonstrated that the quasilinear theory breaks down because chaos is not complete and the phase space is an inhomogeneous mixture of periodic and stochastic orbits. The wave slows down the radial transport of the electrons, which may be of importance for the overall particle transport. The self-consistent analysis showed that the main characteristics of the constant wave amplitude particle dynamics are preserved, leading the absorption of the electromagnetic wave to a minimum value in a relatively short time. For the case of a fusion plasma, the disagreement with the linear theory is significant. We feel that there is a need to reconsider the importance of nonlinear effects during electron-cyclotron heating. The configuration used in this report is relatively simple and our current work focuses on more realistic tokamak slab geometries.

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