Geodesic Acoustic Modes in Rotating Large Aspect Ratio Tokamak Plasmas

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Abstract. The effect of equilibrium plasma rotation (toroidal and poloidal) on low frequency, electrostatic modes – the geodesic acoustic modes (GAM) and the zonal flows (ZF) – in a large aspect ratio tokamaks is studied in the framework of ideal MHD. It is shown that the plasma rotation results in a frequency up-shift of the ordinary GAM. The new branch of continuum modes induced by the poloidal rotation is found. In the case of slow poloidal rotation its frequency is close to the acoustic frequency c_s/qR_0 (c_s is the speed of sound, q is the safety factor, and R_0 is the major radius of tokamak). The mode disappears in the case of purely toroidal plasma rotation. In the case of larger poloidal angular velocities Ω_P ($\Omega_P \geq 2c_s/qR_0$) the mode becomes unstable and is identified as the unstable ZF.

1. Introduction

Geodesic acoustic modes (GAMs) have been actively studied in recent years. The modes are localized on the flux surfaces and characterized by poloidally symmetric radial electric field and poloidally oscillating perturbations of plasma pressure and mass density induced by plasma compressibility due to geodesic curvature of the magnetic lines of forces, inherently present in toroidal configurations. These modes were predicted a long time ago by Winsor et al. [1] and had been observed in a variety of tokamaks [2]–[5]. Most of theoretical studies of GAMs have been fulfilled for static plasma equilibria. However, tokamak equilibria with essential mass flows may exist. In particular, toroidal plasma velocities comparable to the sound velocity may be reached in discharges with an unbalanced neutral beam injection.The effects of toroidal rotation on GAMs have been investigated recently [6]–[8]. For the case of plasma equilibrium with isothermal magnetic surfaces in addition to an ordinary GAM modified by rotation effect a new, lower frequency branch induced by toroidal plasma flow has been found. The new mode appears as a consequence of the non-uniform equilibrium plasma density and pressure created by the centrifugal force on the magnetic surfaces of tokamak. The frequency of this mode goes to zero in the limit of zero toroidal rotation.

All possible plasma equilibria with a mass flow are not reduced to the equilibria with toroidal rotation only. The tokamak plasmas can also rotate in poloidal direction. Such rotation can be caused by both neoclassical and turbulent effects [9, 10]. In general, tokamak equilibria have both toroidal and poloidal rotation. In this paper we generalize the analytical theory of GAMs for the case of plasma equilibrium with both toroidal and poloidal flows. This problem is analyzed by using the standard ideal MHD equations with adiabatic equation of state.

2. Equilibrium

The equilibrium in axisymmetric tokamak with both toroidal and poloidal plasma flows is considered, so that the tokamak magnetic field and the plasma velocity are described by (see, e.g., [11])

$$\mathbf{B}_0 = F\nabla\varphi + \nabla\psi \times \nabla\varphi, \quad \mathbf{v}_0 = \frac{\kappa(\psi)}{\rho_0}\mathbf{B}_0 + R^2\Omega(\psi)\nabla\varphi. \tag{1}$$

Here ψ is the poloidal flux; φ and θ are the toroidal angle and the poloidal angle, respectively. The straight field line coordinates are used, so that the safety factor q is the flux function, $q = q(\psi)$. Equilibrium plasma entropy is the function of poloidal flux, $p_0/\rho_0^\Gamma = S(\psi)$ (p_0 and ρ_0 are equilibrium pressure and density, Γ is the ratio of specific heats). The poloidal and toroidal angular velocities of plasma are given by

$$\Omega_P \equiv \mathbf{v}_0 \cdot \nabla\theta = \frac{\kappa F}{\rho_0 q R^2}, \quad \Omega_T \equiv \mathbf{v}_0 \cdot \nabla\varphi = \Omega(\psi) + q\Omega_P. \tag{2}$$

We restrict ourselves to the case of large aspect ratio tokamaks $R_0/a \equiv 1/\epsilon \gg 1$ (R_0 and a are the major and minor radii of the torus) and of low beta plasma $\beta \equiv 8\pi p_0/B_0^2 \sim \epsilon^2$. Furthermore, we assume that both the poloidal angular velocity and the toroidal angular velocity are sufficiently small, so that $(\Omega_P, \Omega_T) \leq c_s/R_0$. Here c_s is the speed of sound, $c_s^2 = \Gamma p_0/\rho_0$. Under such assumptions the effect of plasma rotation does not exceed the plasma pressure effects. The magnetic surfaces of tokamak are considered to be circular and concentric. To the main order in small parameter ϵ the Shafranov shift can be neglected, so that $R = R_0 + r\cos\theta$ where $r = r(\psi)$ is the label of magnetic surface meaning its radius. One can also expect that, like in the case of plasma equilibrium without rotation, to the main order in ϵ the functions $f = (p_0, \rho_0, \Omega_P, \Omega_T)$ are uniform on the magnetic surfaces. They can be represented in the form $f = \bar{f}(\psi) + \tilde{f}(\psi,\theta)$, $\tilde{f} \sim \epsilon\bar{f}$. The additional applicability condition of this assumption will be given below. Under the above conditions up to the terms of order ϵ^2 the poloidal current stream function F is a function of poloidal flux ψ. The poloidal angle dependent part of plasma pressure p_0 is related to the corresponding part of the mass density by

$$\tilde{p}_0 = \bar{c}_s^2\tilde{\rho}_0, \quad \bar{c}_s^2 \equiv \frac{\Gamma\bar{p}_0}{\bar{\rho}_0}. \tag{3}$$

Then the force balance along the magnetic field results in the following expression of mass density oscillations on the magnetic surfaces caused by the centrifugal effect

$$\lambda_\rho = \frac{M_P^2 - M_P M_T + M_T^2/2}{1 - M_P^2}. \tag{4}$$

Here $M_T \equiv \bar{\Omega}_T/\bar{\omega}_s$ and $M_P \equiv q\bar{\Omega}_P/\bar{\omega}_s$ are the toroidal and poloidal Mach number, respectively, and the dimensionless parameter λ_ρ is introduced according to

$$\frac{1}{\bar{\rho}_0}\mathbf{B}_0 \cdot \nabla\tilde{\rho}_0 \equiv \frac{\lambda_\rho}{R_0^2}\mathbf{B}_0 \cdot \nabla R^2. \tag{5}$$

It follows from Eqs.(4) and (5), that the assumption $\tilde{f} \sim \epsilon\bar{f}$ is valid if

$$\left|1 - M_P^2\right| \geq \left|\frac{M_T^2}{2} - M_P M_T + M_P^2\right|. \tag{6}$$

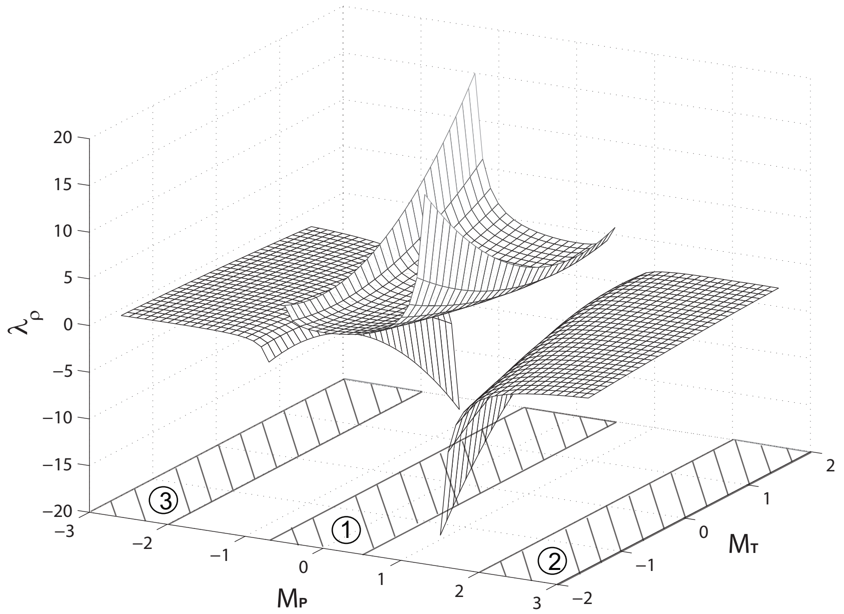

FIG. 1. λ_ρ as a function of M_P and M_T.

The coefficient λ_ρ from Eq.(4) as the function of two variables – the toroidal and poloidal Mach numbers – is represented in Fig. 1. The domains in (M_P, M_T)-plane, where condition (6) is satisfied, are shaded and marked by numbers from 1 to 3.

3. Dispersion relation of GAMs and zonal flows in rotating plasma

We consider the electrostatic perturbations of the above equilibrium and neglect the perturbation of the magnetic field. We assume that the perturbations are axisymmetric, and take their spatio-temporal dependence in the form $f' = f'(\psi,\theta)\exp(-i\omega t)$, where $f'(\psi,\theta)$ is periodic function of θ: $f'(\psi,\theta+2\pi) = f'(\psi,\theta)$. It follows from the induction equation in electrostatic approximation that

$$\mathbf{v}' \times \mathbf{B}_0 = c\nabla\phi', \tag{7}$$

where ϕ' and $\mathbf{v}'$ are the perturbed electrostatic potential and velocity. Then it is evident from Eq.(7) that $\phi' = \phi'(\psi)$. To the main order in ϵ the responses of mass density ρ', of pressure p' and of parallel velocity perturbation $v'_\parallel$ to ϕ' due to the geodesic curvature and non-uniformity of equilibrium mass density and pressure on the magnetic surfaces are oscillating functions of the poloidal angle θ. After straightforward calculations we finally arrive at the following eigenmode equation

$$\frac{d}{d\psi}\left[\frac{\bar{\rho}_0 r^2}{qF}\frac{\Lambda}{D_1 D_{-1}}\frac{d\phi'}{d\psi}\right] = 0, \tag{8}$$

where

$$\Lambda = \hat{\omega}^4 - 2a\bar{\omega}_s^2\hat{\omega}^2 - b\bar{\omega}_s^4, \quad \hat{\omega}^2 \equiv \omega^2 + \Omega_P^2 - \bar{\omega}_s^2/q^2,$$

$$
\begin{aligned}
&D_{\pm 1} = (\omega \mp \bar{\Omega}_P)^2 - \bar{\omega}_s^2/q^2, \\
&a = 1 + \lambda_\rho + \frac{M_T^2}{2}(3+\lambda_\rho) - M_P M_T (1+\lambda_\rho) + \frac{2M_P^2}{q^2}, \\
&b = \frac{4M_P}{q^2}\left\{ M_P\left(\frac{1}{q^2} - 2(1+\lambda_\rho)\right) + M_T(2+\lambda_\rho) - \frac{M_P^3}{q^2} \right. \\
&\qquad \left. - M_P M_T^2 (2+\lambda_\rho) + \frac{M_T^3}{2} + M_P^2 M_T (1+\lambda_\rho) \right\}.
\end{aligned}
\tag{9}
$$

The continuum spectrum is defined by the equation $\Lambda / D_1 D_{-1} = 0$. Let us notice that Λ and $D_{\pm 1}$ are invariant with respect to the transformation $M_P \to -M_P, M_T \to -M_T$.

In the case of purely toroidal plasma rotation, $M_P = 0$, we have $D_{\pm 1} = \hat{\omega}^2$, and the continuum is defined by quadratic equation which gives

$$
\omega^2 = \omega_s^2 \left(2 + \frac{1}{q^2} + 4M_T^2 + \frac{M_T^4}{2}\right). \tag{10}
$$

This mode is an ordinary GAM with the frequency up-shift due to the toroidal rotation. Unlike [6] –[8], in which the plasma temperature has been assumed uniform on the magnetic surfaces, no new branch of low-frequency GAMs due to toroidal rotation arises in the case of equilibrium with isentropic magnetic surfaces.

When the poloidal plasma rotation takes place ($\bar{\Omega}_P \neq 0$), we have the dispersion relation of the 4th order, $\Lambda = 0$. In the case when both toroidal and poloidal rotations are slow compared to $\bar{\omega}_s$, so that $(M_P, M_T) \ll 1$, with an accuracy up to quadratic terms with respect to these small parameters, we obtain two continuum spectra

$$
\begin{aligned}
\omega_1^2 = \bar{\omega}_s^2 \Bigg[2 + \frac{1}{q^2} + 4M_T^2 - 4M_P M_T \left(1 - \frac{1}{q^2}\right) \\
+ M_P^2 \left(2 - \frac{1}{q^2} + \frac{2}{q^4}\right)\Bigg],
\end{aligned}
\tag{11}
$$

$$
\omega_2^2 = \frac{\bar{\omega}_s^2}{q^2}\left[1 - 4M_P M_T + M_P^2\left(3 - \frac{2}{q^2}\right)\right]. \tag{12}
$$

The first mode is the ordinary GAM modified by plasma rotation. Another mode has a lower frequency which is close to the frequency of acoustic mode $\bar{\omega}_s/q$. The new GAM is intrinsically related to poloidal plasma rotation (despite its weak dependence on poloidal angular velocity!). Namely, it appears as a consequence of the Doppler-shifted response of the side-bands of plasma density, pressure and parallel velocity perturbations to the electrostatic potential perturbation, driven by the curvature of magnetic field lines and by the effect of non-uniformity of equilibrium plasma density and pressure on magnetic flux surfaces created by the centrifugal forces. The Doppler shift of frequency is caused by poloidal rotation and has opposite signs for the $m = 1$ and $m = -1$ side-bands. This mode disappears in the case of purely toroidal rotation, $\bar{\Omega}_P = 0$, due to the above mentioned cancellation of the multiplier $\hat{\omega}^2$ in the dispersion relation.

In the general case we have numerically studied the solutions of the dispersion relation under conditions typical of the tokamak edge, choosing $q = 3$. We have separately considered the domains shown in Fig. 1. The frequencies in the figures are normalized to $\bar{\omega}_s$.

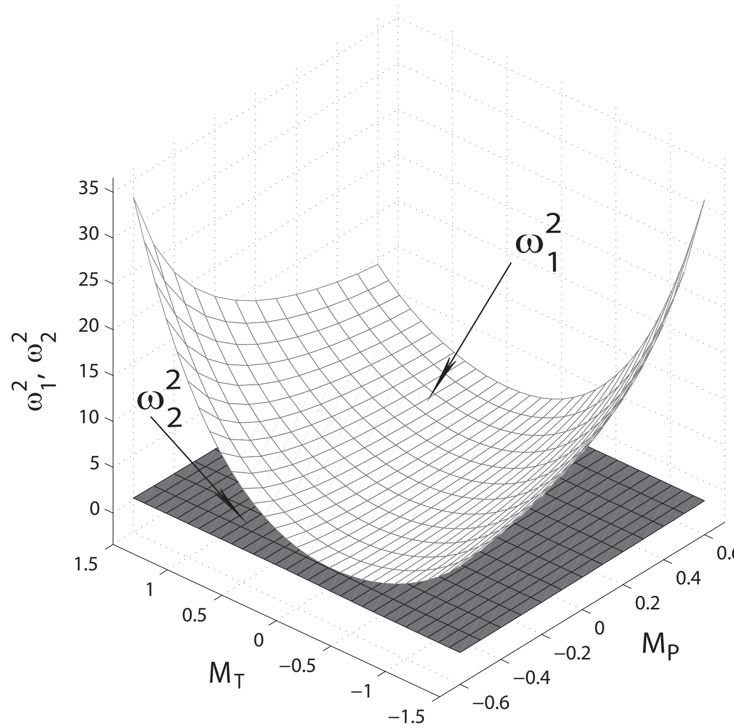

FIG. 2. The squares of mode frequencies ω_1 and ω_2 as the functions of the poloidal and toroidal Mach numbers M_P and M_T in domain 1.

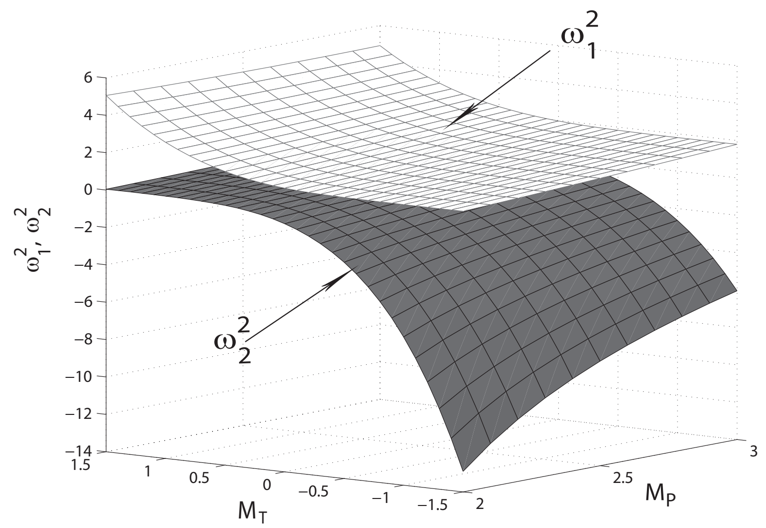

FIG. 3. The squares of mode frequencies ω_1 and ω_2 as the functions of the poloidal and toroidal Mach numbers M_P and M_T in domain 2.

Figure 2 shows the squares of mode frequencies ω_1^2 and ω_2^2 as 2-D functions of M_P and M_T in domain 1, corresponding to sufficiently slow poloidal rotation, $|M_P| \leq 0.6$. Both ω_1^2 and ω_2^2 are positive, which means that two branches of stable continuum modes exist. In the absence of poloidal rotation one of these modes (ω_1) transforms into an ordinary GAM. Another one (ω_2) is the the new found GAM, which is intrinsically related to poloidal plasma rotation. It always has lower frequency than the first mode. In the case of slow poloidal rotation, $(M_P, M_T) \ll 1$, its frequency is defined by Eq.(12).

In Fig. 3 we have presented ω_1^2 and ω_2^2 as 2-D functions of M_P and M_T in domain 2, where the poloidal Mach number M_P is sufficiently large, positive number. The poloidal angular velocity in this domain exceeds the ion sound frequency, $\bar{\Omega}_P > \bar{c}_s/qR_0$. The situation is qualitatively different from domain 1. The mode described by ω_1 remains stable ($\omega_1^2 > 0$), but in some subdomain ω_2^2 becomes negative. It means that the mode is aperiodically unstable. In this case the mode can be identified as an unstable zonal flow. According to the figure, the most unstable are the flows with negative M_T, such that $M_P \cdot M_T < 0$.

In Fig. 4 the dependence of ω_2^2 on the poloidal Mach number M_P is given for different values of M_T. When the toroidal plasma rotation is sufficiently fast ($M_T \simeq 1$) the mode described by ω_2 is stable and can be identified as the GAM induced by the poloidal plasma rotation. The instability of this mode appears when the toroidal rotation is relatively slow, so that $M_T \leq 0.78$. The function ω_2^2 grows with M_P. If it is negative for some M_T in the interval $2.0 < M_P < M_0$, it passes through zero at some point $M_P = M_0(M_T)$, which depends on M_T. At this point the mode transforms into the stable zonal flow. With further increase of M_P the mode transforms into a stable, oscillating mode, which can be identified as the GAM induced by the poloidal plasma rotation. Its frequency is smaller than the frequency of the ordinary GAM. As far as dispersion relation is invariant with respect to transformation $M_P \to -M_P, M_T \to -M_T$, in domain 3 we will have the mode picture similar to Fig. 3.

4. Conclusion

In this paper, a theory of low frequency, electrostatic ideal modes – GAMs and zonal flows – in tokamaks has been developed to include the effects of both the toroidal and poloidal equilibrium plasma flows. The analysis is based on ideal MHD equations with the adiabatic equation of state. It has been assumed that the aspect ratio of tokamak is large ($R_0/a \gg 1$), that the plasma pressure is low ($\beta \sim \epsilon^2$), that the poloidal plasma rotation is relatively slow ($\Omega_P \leq c_s/R_0$), and that the magnetic surfaces of tokamak are circular. The electrostatic, axisymmetric perturbations of the rotating plasma have been considered. The perturbations are characterized by the electrostatic potential, which is uniform on the magnetic surfaces (the principal harmonic), and the mass density, pressure and parallel velocity, which are oscillating functions of the poloidal angle to the main order in ϵ (the side-bands). The responses of plasma mass density, pressure and parallel velocity to the electrostatic potential perturbation are driven by two effects: by the curvature of the magnetic field lines (which is responsible for the ordinary GAMs) and by non-uniformity of equilibrium plasma mass density and pressure on the magnetic surfaces, created by the centrifugal forces.

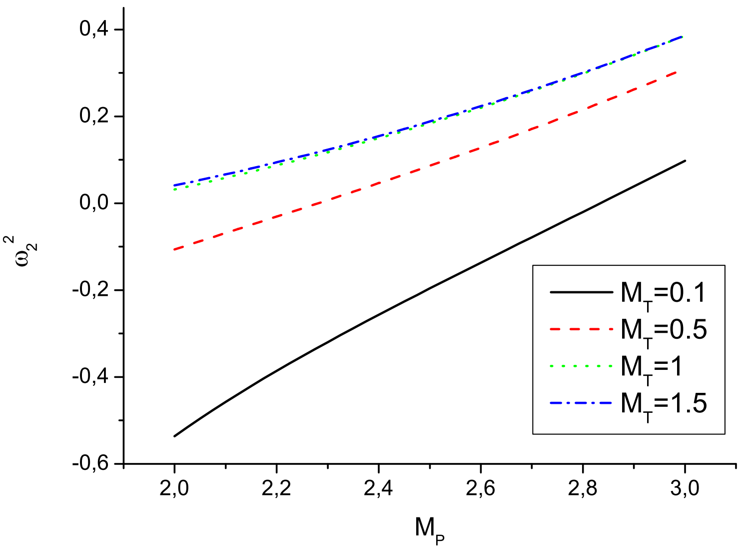

FIG. 4. The square of ω_2 as the function of the poloidal Mach number M_P in domain 2 for different M_T .

In the limiting case of purely toroidal rotation ($\Omega_P = 0$) only the ordinary GAM with the frequency shifted up by the toroidal rotation exists in the case of plasma equilibrium with the isentropic magnetic surfaces. Due to the poloidal rotation the frequencies of the responses of side-bands to the electrostatic potential perturbation are shifted (the Doppler effect). The Doppler shift has the opposite signs for the $m = \pm 1$ side-bands. This difference in the Doppler shift of side-band frequencies results in a new type of GAMs-zonal flows (in addition to the rotation modified ordinary GAM). In the case of slow plasma rotation $(M_P, M_T) \ll 1$ its frequency is close to the acoustic mode frequency c_s/qR_0. The new GAM is intrinsically related to poloidal plasma rotation and disappears in the case of purely toroidal plasma rotation.

Numerical analysis of the continuum spectra has been performed for $q = 3$ and finite poloidal and toroidal Mach numbers. It has shown that in the case of relatively slow poloidal rotation such that $\bar{\Omega}_P \leq \bar{c}_s/qR_0$ both branches of the continuum spectrum are stable. The branch, which is an ordinary GAM modified by plasma rotation, has higher frequency compared to the new found mode induced by poloidal rotation. When the angular velocity of poloidal rotation is larger, $|M_P| \geq 2$, the situation becomes qualitatively different. While the branch, corresponding to the standard GAM, remains stable and its frequency is mainly a growing function of $|M_P|$ for $M_T =$ const, a new branch of the continuum modes becomes unstable in some range of the poloidal and toroidal Mach numbers. The instability is aperiodic, so that the modes are non-oscillating, and in a natural way are identified as unstable zonal flows. The most unstable (with larger growth rates) are the flows with $M_P \cdot M_T < 0$. For $M_T =$ const the instability is suppressed with the increase of the poloidal Mach number $|M_P|$, and passing through some point $M_P = M_0$, the mode transforms into the oscillating mode – the marginally stable GAM.

This work was fulfilled (partly) in the framework of realization of the Federal program

"Scientific and pedagogical personnel of innovative Russia" for 2009-2012, and also was partly supported by the Russian Foundation for Basic Research (Grant No. 10-02-01302).

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