Lagrangian approach to resonant three-mode interaction in magnetohydrodynamics

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Three-mode coupling due to magnetohydrodynamic (MHD) nonlinearity is studied for the understanding of saturation mechanism of the Alfvén eigenmode that is destabilized by energetic particles. An analytic expression of the coupling coefficient among ideal MHD eigenmodes is derived in a general manner, which quantifies the effectiveness of the nonlinear mode coupling (such as the parametric decay rate). This formulation is worked out by extending the MHD Lagrangian theory to nonlinear regime and, for the first time, enables the analysis of the global mode coupling in real geometry. The coupling coefficient is written in terms of the linear displacement vector field, so that it can be analytically and numerically evaluated by utilizing the existing linear stability theories and methods.

1 Introduction

The nonlinear saturation level of the Alfvén eigenmodes (AEs) that is driven by the resonant alpha particles is of significant concern to future magnetic fusion reactor. As for AEs driven by energetic particles and antennas, the recent experiments [1, 2] observe the coupling among three eigenmodes $\omega_a = \omega_b + \omega_c$ or harmonics $\omega_a = 2\omega_b$, which implies that the associated energy transfer can be one of the saturation mechanisms depending on its efficiency. The second harmonic generation [1] is roughly estimated by the quadratic terms of the MHD equations [3]. In this paper, we propose a novel technique to analyze such three (or two) mode interactions induced by the weak MHD nonlinearity and provides a more detailed estimation of the saturation level.

Although the coupling among plane waves or wave packets has been well studied in plasma physics [4, 5], generalization of this "three-wave" theory to global eigenmodes in nonuniform equilibria (like tokamaks) is highly complicated and requires an extensive algebra since the local wavenumber vector $\boldsymbol{k}$ is no longer usable for such non-local oscillations. In this work, we first derive a general weakly-nonlinear equation of motion by developing the Lagrangian theory [6, 7] of ideal MHD, which leads to the amplitude equations of resonant three eigenmodes with the Manley-Rowe symmetry. In contrast to earlier work [8], we can express the coupling coefficient in terms of only the linear displacement vector field. By substituting the AE eigenfunctions into the expression, the energy transfer among the three eigenmodes can be evaluated in a systematic way.

2 Extension of the Lagrangian displacement to nonlinear regime

In this section, we develop a method for solving nonlinear fluctuations based on the Lagrangian formalism of the ideal MHD equations [6]. By adopting an appropriate perturbation expansion of the MHD Lagrangian and invoking Hamilton's principle, we will arrive at a weakly-nonlinear equation of motion for the displacement field, which serves as a natural extension of the well-known linearized equation [9].

Let $V \subset \mathbb{R}^3$ be a domain filled with ideal plasma. For simplicity, we assume that V is surrounded by a rigid ideal wall. Although most experimental devices have vacuum regions between plasmas and walls, we need not consider such free boundary problems in this work since we are concerned with *internal* modes which tend to evanesce at the plasma surface.

In the Lagrangian description of fluids, the dynamical variable is represented by a map $\boldsymbol{x}(t;\boldsymbol{x}_0) \in V$, which denotes the position of an infinitesimal fluid element (or "fluid particles") at time t that leaves an initial position $\boldsymbol{x}_0 \in V$ at $t = 0$. Given such a fluid motion, the Eulerian velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$ is defined by $d\boldsymbol{x}/dt = \boldsymbol{v}(\boldsymbol{x}, t)$, and the ideal MHD equations for the magnetic field $\boldsymbol{B}$, the mass density ρ and the specific entropy s claim that these quantities are frozen to each fluid element.

Now, we consider a perturbed fluid motion; $\boldsymbol{x}(t) \to \boldsymbol{x}(t) + \Xi(\boldsymbol{x}(t), t)$, where $\Xi(\boldsymbol{x}, t)$ denotes the displacements of orbits. Since it is generally difficult to solve the full-nonlinear evolution of $\Xi(\boldsymbol{x}, t)$, we resort to a perturbation method. In this work, an arbitrary small parameter $\alpha(\ll 1)$ is used to measure the amplitude of displacement, and the perturbation expansion is performed in the following specific manner.

$$\boldsymbol{x} + \Xi =(e^{\alpha\boldsymbol{\xi}\cdot\nabla})(\boldsymbol{x}) = \boldsymbol{x} + \alpha\boldsymbol{\xi} + \frac{\alpha^2}{2}\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi} + \frac{\alpha^3}{6}\boldsymbol{\xi}\cdot\nabla(\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi}) + \dots, \tag{1}$$

which means that the nonlinear displacement map Ξ is generated by a displacement vector field $\boldsymbol{\xi}(\boldsymbol{x}, t)$; $\partial\Xi/\partial\alpha = \boldsymbol{\xi}(\boldsymbol{x} + \Xi, t)$. The boundary condition is $\boldsymbol{n}\cdot\boldsymbol{\xi} = 0$ at ∂V. The outstanding feature of this way of expansion is that the perturbed state $\tilde{u}$ of all MHD variables $u = (\boldsymbol{v}, \boldsymbol{B}, \rho, s)^T$ can be expressed by the Lie series expansion, $\tilde{u} = u + \alpha\delta_{\boldsymbol{\xi}}u + (\alpha^2/2)\delta_{\boldsymbol{\xi}}\delta_{\boldsymbol{\xi}}u + (\alpha^3/6)\delta_{\boldsymbol{\xi}}\delta_{\boldsymbol{\xi}}\delta_{\boldsymbol{\xi}}u + \dots$, where an operator $\delta_{\boldsymbol{\xi}}$ is defined by

$$\delta_{\boldsymbol{\xi}}u = \begin{pmatrix} \delta_{\boldsymbol{\xi}}\boldsymbol{v} \\ \delta_{\boldsymbol{\xi}}\boldsymbol{B} \\ \delta_{\boldsymbol{\xi}}\rho \\ \delta_{\boldsymbol{\xi}}s \end{pmatrix} = \begin{pmatrix} \partial_t\boldsymbol{\xi} + \boldsymbol{v}\cdot\nabla\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla\boldsymbol{v} \\ \nabla\times(\boldsymbol{\xi}\times\boldsymbol{B}) \\ -\nabla\cdot(\rho\boldsymbol{\xi}) \\ -\boldsymbol{\xi}\cdot\nabla s \end{pmatrix}, \tag{2}$$

and enjoys the property of the Lie derivative (such as the Jacobi identity).

The MHD Lagrangian density [6] is a function of the Eulerian variables u,

$$\mathsf{L}(u) =\frac{\rho}{2}|\boldsymbol{v}|^2 - \frac{1}{2}|\boldsymbol{B}|^2 - \rho U(\rho, s), \tag{3}$$

where $U(\rho, s)$ denotes the internal energy per unit volume. The Lagrangian evaluated at the perturbed state is therefore expanded with respect to α as follows.

$$\tilde{L} = \int_V \mathsf{L}(\tilde{u})d^3x = \int_V \left(\mathsf{L} + \alpha\delta_{\boldsymbol{\xi}}\mathsf{L} + \frac{\alpha^2}{2}\delta_{\boldsymbol{\xi}}\delta_{\boldsymbol{\xi}}\mathsf{L} + \frac{\alpha^3}{6}\delta_{\boldsymbol{\xi}}\delta_{\boldsymbol{\xi}}\delta_{\boldsymbol{\xi}}\mathsf{L} + O(\alpha^4)\right) d^3x, \tag{4}$$

where $\delta_{\boldsymbol{\xi}}$ acts on a functional $\mathsf{L}(u)$ like $\delta_{\boldsymbol{\xi}}\mathsf{L} = D_u\mathsf{L}\cdot\delta_{\boldsymbol{\xi}}u$ with D_u representing the functional derivative. Let the unperturbed state $u = (\boldsymbol{v}, \boldsymbol{B}, \rho, s)^T$ be a given *equilibrium* state. Since it is an exact solution of the MHD equations, the $O(\alpha)$-term $\delta_{\boldsymbol{\xi}}\mathsf{L}$ vanishes as shown by Newcomb [6]. After some manipulation, we can express the Lagrangian (4) as follows.

$$\tilde{L} = \int_V \left\{\mathsf{L} + \frac{\alpha^2}{2}l^{(2)}(\boldsymbol{\xi}, \boldsymbol{\xi}) + \frac{\alpha^3}{6}\left[3l^{(2)}(\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi}, \boldsymbol{\xi}) - w^{(3)}(\boldsymbol{\xi}, \boldsymbol{\xi}, \boldsymbol{\xi})\right] + O(\alpha^4)\right\}d^3x. \tag{5}$$

Here, the symmetric quadratic form, appearing in the $O(\alpha^2)$-term,

$$l^{(2)}(\boldsymbol{\xi}, \boldsymbol{\eta}) := \rho\frac{D\boldsymbol{\xi}}{Dt}\cdot\frac{D\boldsymbol{\eta}}{Dt} - w^{(2)}(\boldsymbol{\xi}, \boldsymbol{\eta}) \quad \text{for } \forall\boldsymbol{\xi}, \boldsymbol{\eta}$$

corresponds to the Lagrangian density for the linearized system [7], where $D/Dt = \partial_t + \boldsymbol{v}\cdot\nabla$ and the potential energy is

$$\begin{aligned} w^{(2)}(\boldsymbol{\xi}, \boldsymbol{\eta}) :=& \mathcal{B}\boldsymbol{\xi}\cdot\mathcal{B}\boldsymbol{\eta} + \boldsymbol{\eta}\cdot[(\boldsymbol{\xi}\cdot\nabla)\nabla P] + \rho c_s^2(\nabla\cdot\boldsymbol{\xi})(\nabla\cdot\boldsymbol{\eta}) + (\nabla\cdot\boldsymbol{\xi})\boldsymbol{\eta}\cdot\nabla P \\ &+ (\nabla\cdot\boldsymbol{\eta})\boldsymbol{\xi}\cdot\nabla P. \end{aligned} \tag{6}$$

An operator $\mathcal{B}$ is defined by $\mathcal{B}\boldsymbol{\xi} := \boldsymbol{B}\cdot\nabla\boldsymbol{\xi} - (\nabla\cdot\boldsymbol{\xi})\boldsymbol{B}$, and $P = p + |\boldsymbol{B}|^2/2$ denotes the total pressure and $c_s^2 = \partial p/\partial\rho$ the sound speed. Hence, the extremum condition, $\delta\int_{t_1}^{t_2}\int_V l^{(2)}(\boldsymbol{\xi},\boldsymbol{\xi})d^3x dt = 0$, up to $O(\alpha^2)$ recovers the well-known Frieman-Rotenberg (FR) equation $\rho D^2\boldsymbol{\xi}/Dt^2 = \mathcal{F}\boldsymbol{\xi}$ [9].

As displayed in (5), we have derived the 3rd-order potential energy $w^{(3)}(\boldsymbol{\xi},\boldsymbol{\xi},\boldsymbol{\xi})$ which can be written in the following form with cubic symmetry,

$$\begin{aligned} w^{(3)}(\boldsymbol{\xi},\boldsymbol{\eta},\boldsymbol{\zeta}) := &-(\nabla\cdot\boldsymbol{\xi})\mathcal{B}\boldsymbol{\zeta}\cdot\mathcal{B}\boldsymbol{\eta} - (\nabla\cdot\boldsymbol{\eta})\mathcal{B}\boldsymbol{\zeta}\cdot\mathcal{B}\boldsymbol{\xi} - (\nabla\cdot\boldsymbol{\zeta})\mathcal{B}\boldsymbol{\xi}\cdot\mathcal{B}\boldsymbol{\eta} \\ &+ \nabla\cdot(\boldsymbol{\zeta}\cdot\nabla\boldsymbol{\xi} - \boldsymbol{\zeta}\nabla\cdot\boldsymbol{\xi})\delta_{\boldsymbol{\eta}}P + \nabla\cdot(\boldsymbol{\zeta}\cdot\nabla\boldsymbol{\eta} - \boldsymbol{\zeta}\nabla\cdot\boldsymbol{\eta})\delta_{\boldsymbol{\xi}}P \\ &+ \nabla\cdot(\boldsymbol{\xi}\cdot\nabla\boldsymbol{\eta} - \boldsymbol{\xi}\nabla\cdot\boldsymbol{\eta})\delta_{\boldsymbol{\zeta}}P \\ &+ (\nabla\cdot\boldsymbol{\xi})\boldsymbol{\eta}\cdot(\boldsymbol{\zeta}\cdot\nabla)\nabla P + (\nabla\cdot\boldsymbol{\zeta})\boldsymbol{\eta}\cdot(\boldsymbol{\xi}\cdot\nabla)\nabla P + (\nabla\cdot\boldsymbol{\eta})\boldsymbol{\xi}\cdot(\boldsymbol{\zeta}\cdot\nabla)\nabla P \\ &+ \boldsymbol{\eta}\boldsymbol{\zeta}\boldsymbol{\xi} : \nabla\nabla\nabla P - \left(\rho^2\frac{\partial^2 p}{\partial\rho^2} + 2\rho c_s^2\right)(\nabla\cdot\boldsymbol{\xi})(\nabla\cdot\boldsymbol{\eta})(\nabla\cdot\boldsymbol{\zeta}). \end{aligned} \tag{7}$$

for all $\boldsymbol{\xi},\boldsymbol{\eta},\boldsymbol{\zeta}$, where the perturbation of the total pressure is given by $\delta_{\boldsymbol{\xi}}P = \delta_{\boldsymbol{\xi}}p + \boldsymbol{B}\cdot\delta_{\boldsymbol{\xi}}\boldsymbol{B} = \boldsymbol{B}\cdot\mathcal{B}\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla P - \rho c_s^2\nabla\cdot\boldsymbol{\xi}$. It follows that we can derive a weakly nonlinear equation of $\boldsymbol{\xi}$ by invoking Hamilton's principle $\delta\int_{t_1}^{t_2}\tilde{L}dt = 0$ up to $O(\alpha^3)$;

$$\rho\frac{D^2\Xi}{Dt^2} - \mathcal{F}\Xi - \frac{1}{2}\mathcal{F}^{(2)}(\Xi,\Xi) = O(\alpha^3), \tag{8}$$

where $\Xi = \alpha\boldsymbol{\xi} + (\alpha^2/2)\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi} + O(\alpha^3)$, and the quadratic force $\mathcal{F}^{(2)}$ is uniquely determined by $-w^{(3)}(\boldsymbol{\xi},\boldsymbol{\xi},\delta\boldsymbol{\xi}) = \delta\boldsymbol{\xi}\cdot\mathcal{F}^{(2)}(\boldsymbol{\xi},\boldsymbol{\xi}) + \nabla\cdot[-2(\delta\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi})\delta_{\boldsymbol{\xi}}P]$. This is a natural extension of the FR equation to nonlinear regime.

3 Nonlinear three-mode interaction

Various nonlinear mode couplings can be discussed by studying the properties of the quadratic force term $\mathcal{F}^{(2)}$ or the associated potential energy $w^{(3)}$. Following the weak turbulence theory, let us naively assume that the solution of (8) takes a form of

$$\alpha\boldsymbol{\xi} = \sum_j [C_j(t)\hat{\boldsymbol{\xi}}_j e^{-i\omega_j t} + \text{c.c.}] \tag{9}$$

where ω_j and $\hat{\boldsymbol{\xi}}_j$ are, respectively, the eigenvalues and eigenfunctions of the FR equation, and the slow variation of the amplitudes $C_j(t)$ $[dC_j/dt \ll \omega_j C_j]$ accounts for weakly nonlinear coupling among the eigenmodes. The amplitude equations may be written as

$$\frac{dC_j}{dt} = \sum_{j',j''}\hat{W}^{(3)}_{jj'j''}C_{j'}C_{j''}\delta(\omega_j + \omega_{j'} + \omega_{j''}), \tag{10}$$

where $\hat{W}^{(3)}_{jj'j''}$ denotes some coupling coefficients that we will focus on later.

However, the above weak turbulence theory is not directly applicable to the Alfvénic disturbance in tokamaks for the following reasons. First, the discrete spectra of AEs are *sparse* especially for low-m,n (poloidal and toroidal) mode numbers, and the considerable range of frequencies is occupied by the Alfvén continuous spectra which are approximated by

$$\omega_A(r) = \left[\frac{m}{q(r)} + n\right]\frac{\bar{B}_T(r)}{R_0\sqrt{\rho(r)}}, \tag{11}$$

where r is the radial position in the flux coordinates, R_0 the major radius of the magnetic axis, $\bar{B}_T(r)$ the flux averaged toroidal magnetic field, and $q(r)$ the safety factor. The discrete AE spectra tend to exist near the minima, maxima and *gaps* of these Alfvén continua. Therefore, the exact resonance condition is rarely satisfied among AEs, but there are many combinations of AEs that satisfy

$$m_a = m_b + m_c, \quad n_a = n_b + n_c, \quad \omega_a = \omega_b + \omega_c + \Delta\omega, \tag{12}$$

$$\text{or} \quad m_a = 2m_b, \quad n_a = 2n_b, \quad \omega_a = 2\omega_b + \Delta\omega. \tag{13}$$

Note that the latter second-harmonic resonance is a special case of (12) with $m_b = m_c$, $n_b = n_c$ and $\omega_b = \omega_c$. The frequency mismatch $\Delta\omega(\ll \omega_j,\ j = a, b, c)$ is closely related to the deviation of eigenvalues from the edges of continua, which sensitively depends on various effects (such as magnetic shear, toroidicity, finite ω/ω_{ci}, hot ions and pressure gradient) [10, 11, 12, 13, 14, 15, 16].

Moreover, in experiments, only the several AEs often gain energy through the resonance with energetic particles and antennas. In this work, we do not try to establish a holistic theory that comprehend all interactions among eigenmodes and continuum modes, the external energy source as well as the non-MHD effects. We will investigate only the interactions between a *large-amplitude* AE (which is already pumped by the external drive) and other AEs satisfying (12). The efficiency of this process is expected to be the bottle-neck of the subsequent cascade of mode-mode couplings.

Let us choose three eigenmodes $\alpha\boldsymbol{\xi} = \sum_{j=a,b,c}[C_j\hat{\boldsymbol{\xi}}_j e^{-i\omega_j t} + \text{c.c.}]$ labeled by a, b, c, which satisfy the near resonance condition (12). Here, $\omega_j > 0$ is assumed without loss of generality. The amplitude equations for these eigenmodes can be derived from (8) or the Lagrangian (5) as follows.

$$\frac{dC_a}{dt} = -i\frac{W^{(3)}_{a,b,c}}{\mu_a}C_bC_ce^{i\Delta\omega t}, \quad \frac{dC_b^*}{dt} = i\frac{W^{(3)}_{a,b,c}}{\mu_b}C_a^*C_ce^{i\Delta\omega t}, \quad \frac{dC_c^*}{dt} = i\frac{W^{(3)}_{a,b,c}}{\mu_c}C_a^*C_be^{i\Delta\omega t}, \tag{14}$$

where * denotes complex conjugate and we have defined

$$\mu_j = \int_V \left[\hat{\boldsymbol{\xi}}_j^* \cdot \rho(\omega_j + i\boldsymbol{v}\cdot\nabla)\hat{\boldsymbol{\xi}}_j\right] d^3x \in \mathbb{R} \quad (j = a, b, c) \tag{15}$$

$$W^{(3)}_{a,b,c} = \int_V w^{(3)}(\hat{\boldsymbol{\xi}}_a^*, \hat{\boldsymbol{\xi}}_b, \hat{\boldsymbol{\xi}}_c)d^3x. \tag{16}$$

The system (14) is known to be solvable in the classical nonlinear theory [4, 5]. The Manley-Rowe symmetry is obviously satisfied by the fact that a common coupling coefficient $W^{(3)}_{a,b,c}$ (= 3rd-order potential energy) appears in the three equations.

In the absence of the equilibrium flow $\boldsymbol{v} = 0$, the wave actions $N_j = \mu_j|C_j|^2$ $(j = a, b, c)$ are always positive and only the periodic energy exchange among the three modes occurs. As illustrative examples, we exhibit the following two cases of energy transfer.

Case (i). $N_a(0) \gg N_b(0), N_c(0) \sim 0$ (higher freq. $\Rightarrow$ lower freqs.)

If $N_a(0) > \Delta N := \mu_a\mu_b\mu_c\Delta\omega^2/4|W^{(3)}_{a,b,c}|^2$ or $|C_a(0)| > \Delta C_a := \sqrt{\Delta N/\mu_a}$ the parametric decay occurs and the subsequent solution oscillates like

$N_a(t)$	$N_a(0)$	$\searrow$	ΔN	$\nearrow$	$N_a(0)$
$N_{b,c}(t)$	0	$\nearrow$	$N_a(0) - \Delta N$	$\searrow$	0

Case (ii). $N_b(0) = N_c(0) \gg N_a(0) \sim 0$ (lower freqs. $\Rightarrow$ higher freq.)

Lower-frequency eigenmodes ω_b and ω_c cooperatively drive higher harmonics ω_a.

$N_a(t)$	0	$\nearrow$	$N_{b,c}(0) - \Delta N'$	$\searrow$	0
$N_{b,c}(t)$	$N_{b,c}(0)$	$\searrow$	$\Delta N'$	$\nearrow$	$N_{b,c}(0)$

where $\Delta N' = \sqrt{\Delta N\,(N_b(0) + \Delta N/4)} - \Delta N/2$. While there is no clear threshold in this case, the initial amplitudes $C_{b,c}(0)$ that are comparable to $\Delta C_{b,c} := \sqrt{\Delta N/\mu_{b,c}}$ will decrease about 20%, and about 40% of the energy will be transferred to the higher harmonics. The limit of large $\Delta\omega$, i.e., $|C_{b,c}(0)| \ll \Delta C_{b,c}$, corresponds to the nonresonant forced oscillation, $N_{b,c}(0) - \Delta N' \simeq N_b(0)N_c(0)/4\Delta N$.

Note that the second-harmonic resonance $\omega_a = 2\omega_b + \Delta\omega$ can be discussed in the same manner by identifying the mode b with c. Although the solutions are periodic in both cases, such the recurrent character of the solution would be destroyed if this coupling is further accompanied by other secondary and tertiary mode couplings like (10). The larger coupling coefficient and the smaller frequency mismatch, the more effective energy transfer occurs, and they predict how much and how quickly the large-amplitude AE (driven by energetic particles) loses energy via mode coupling.

4 Low-β approximation of the coupling coefficient among GAEs

In this section, we analytically estimate the coupling coefficient $W^{(3)}_{a,b,c}$ for the global Alfvén eigenmodes (GAEs) [10] in cylindrical geometry (r, θ, z). We assume that cylindrical equilibria satisfy the low-β ordering; by introducing a small parameter ϵ,

$$\boldsymbol{v} = 0, \quad B_z = B_0 = \text{const.}, \quad B_\theta(r) \sim \epsilon B_0, \quad p(r) \sim \epsilon^2 B_0^2/2, \quad \rho(r) \sim \rho_0. \tag{17}$$

Let a and $2\pi R_0$ be the radius and length of the cylinder and impose the periodic boundary condition in the z direction. The eigenmodes are written in the form of

$$C\hat{\boldsymbol{\xi}}e^{-i\omega t} = C(\hat{\xi}_r(r), \hat{\xi}_\theta(r), \hat{\xi}_z(r))\exp(im\theta + ikz - i\omega t) \quad (k = n/R_0), \tag{18}$$

where we normalize the eigenfunctions by $\|\hat{\boldsymbol{\xi}}\|^2 := \int_V \rho|\hat{\boldsymbol{\xi}}|^2 d^3x = 4\pi^2 R_0\rho_0 a^4$ such that $\hat{\boldsymbol{\xi}} \sim a$ and the amplitude becomes small $C \sim \alpha \ll 1$.

When the Alfvén continuous spectrum $\omega_A(r)$ [given by (11) with $\bar{B}_T(r) \equiv B_0$] takes a minimum at $r = r_0$, we consider an AE eigenvalue ω just below the minimum, $|\omega - \omega_A(r_0)| \sim \epsilon|\omega_A(r_0)|$. The eigenvalue problem is then solved approximately by the local analysis [11]. In terms of $\phi = r\hat{\xi}_r$, it is reduced to

$$\frac{d}{dy}\left[(1-y^2)\frac{d\phi}{dy}\right] - \frac{m^2}{r_0^2}L_\omega^2(1-y^2)\phi + g\phi = 0 \tag{19}$$

in the local coordinate $y = (r - r_0)/L_\omega$, where

$$L_\omega^2 = L^2\frac{\omega_A^2(r_0) - \omega^2}{\omega_A^2(r_0)}, \quad g = -\frac{L^2}{r_0}\frac{\rho'}{\rho}\bigg|_{r=r_0}, \quad L^2 = 2\frac{\omega_A^2}{(\omega_A^2)''}\bigg|_{r=r_0}.$$

The variational method [11] shows that, for $g > 3/4$, there exists an eigenvalue $\omega^2 \simeq \omega_A^2(r_0)\,(1 - r_0^2\epsilon_0/m^2L^2)$ whose eigenfunction is approximated by the Gaussian,

$$r\hat{\xi}_r \simeq e^{-\frac{(r-r_0)^2}{\Delta r^2}} \quad \text{with} \quad \Delta r^2 = 2\frac{r_0^2}{m^2}\sqrt{\epsilon_0}, \quad \epsilon_0 = g - \frac{1}{4} - \sqrt{g - \frac{1}{2}}. \tag{20}$$

Once $r\hat{\xi}_r$ is solved in this way, the other components of the eigenfunction are related to $r\hat{\xi}_r \sim a^2$ in the following manners.

$$\delta_{\hat{\boldsymbol{\xi}}} P \simeq 2\frac{\sqrt{\rho}\omega_A B_\theta}{mr} r\hat{\xi}_r \sim \epsilon \frac{B_0^2}{2}, \qquad \nabla \cdot \hat{\boldsymbol{\xi}} \simeq -2\frac{kB_\theta}{mrB_z} r\hat{\xi}_r \sim \epsilon \tag{21}$$

$$\hat{\xi}_\theta \simeq i\frac{1}{m}(r\hat{\xi}_r)' \sim a, \qquad \hat{\xi}_z \simeq -i\frac{B_\theta}{mB_z}(r\hat{\xi}_r)' \sim \epsilon a \tag{22}$$

This low-β ordering significantly simplifies the coupling coefficient. When a triplet of AEs are substituted into (7), we find that the first three terms [e.g., $-(\nabla \cdot \boldsymbol{\xi})\mathcal{B}\boldsymbol{\zeta} \cdot \mathcal{B}\boldsymbol{\eta}$] are dominant ($\sim \epsilon$), which originate from the nonlinearity of the $\boldsymbol{J} \times \boldsymbol{B}$ force, that is, the Maxwell stress. Using $\mathcal{B}\hat{\boldsymbol{\xi}} \simeq i\omega\sqrt{\rho}\hat{\boldsymbol{\xi}}$, we obtain an insightful expression,

$$\begin{aligned} W^{(3)}_{a,b,c} = \int_V \Big[&(\nabla \cdot \hat{\boldsymbol{\xi}}^*_a)\omega_b\omega_c\rho\hat{\boldsymbol{\xi}}_b \cdot \hat{\boldsymbol{\xi}}_c - (\nabla \cdot \hat{\boldsymbol{\xi}}_b)\omega_a\omega_c\rho\hat{\boldsymbol{\xi}}^*_a \cdot \hat{\boldsymbol{\xi}}_c - (\nabla \cdot \hat{\boldsymbol{\xi}}_c)\omega_a\omega_b\rho\hat{\boldsymbol{\xi}}^*_a \cdot \hat{\boldsymbol{\xi}}_b \Big] d^3x \\ &+ O(\epsilon^2) \end{aligned} \tag{23}$$

which implies that $W^{(3)}_{a,b,c}$ is roughly the inner products between the eigenfunctions weighted by $\nabla \cdot \hat{\boldsymbol{\xi}}_j (\sim \epsilon)$ and $\omega_j\omega_{j'}$. The spatial overlapping of three eigenfunctions is obviously essential for the strong coupling.

If we assume three eigenmodes have the same helicity $m_a/n_a = m_b/n_b = m_c/n_c$, the associated Alfvén continua take a minimum at the common position $r = r_0$. Then, all eigenfunctions ($\phi = r\hat{\xi}_r$) are peaked at $r = r_0$ and their coupling is expected to be strong. Using the approximation of $\nabla \cdot \hat{\boldsymbol{\xi}}$ in (21) and extracting the values at $r = r_0$, the strength of coupling can be estimated by

$$\sqrt{\omega^{(3)}_{a,b,c}} := \sqrt{4\pi^2 R_0\rho_0 a^4}\frac{|W^{(3)}_{a,b,c}|}{\sqrt{|\mu_a\mu_b\mu_c|}} = \sqrt{\frac{|\omega_a|\rho_0}{|m_a|\rho(r_0)}\frac{a^2}{R_0^2}\left|\frac{n_a}{q(r_0)m_a}\right|} F(\Delta r/r_0, m_a, m_b), \tag{24}$$

where a non-dimensional function F depends on the modal structures. If the Gaussian approximation (20) is further employed, it is greatly simplified into

$$F(\Delta r/r_0, m_a, m_b) = \left(\frac{2}{\pi}\right)^{1/4}\sqrt{\frac{2(m_a^2+m_b^2+m_c^2)}{\frac{\hat{r}}{r_0}+\frac{r_0}{\hat{r}}}}, \qquad \frac{\hat{r}}{r_0} = \sqrt{2}\epsilon_0^{1/4}. \tag{25}$$

5 Numerical evaluation

To make best use of the general expression derived in (7) and also validate the analytic formula (24), we have numerically solved the full eigenvalue problem in the cylindrical geometry (called the Hain-Lüst equation, see [10]) and calculated the coupling coefficient in a direct manner. We choose the same equilibrium profiles as Ref. [10],

$$J_z(r) = (rB_\theta)'/r = J_0(1 - r^2/a^2)^\nu, \qquad \rho(r) = (\rho_0 - \rho_a)(1 - r^2/a^2)^\kappa + \rho_a. \tag{26}$$

Owing to the positive shear, from $q_0 = 2B_0/(R_0J_0)$ to $q_a = q_0(\nu + 1)$, and the density gradient, from ρ_0 to ρ_a, there exist minima of Alfvén continua.

For $\omega_{A0} = B_0/a\sqrt{\rho_0} = 1.0$, $q_0 = 1.0$, $q_a = 5.0$, $\rho_a/\rho_0 = 0.05$, $\kappa = 1.0$ and $R_0/a = 3.0$, the numerically calculated GAE spectra are shown in Fig. 1 for $(m, n) = (1, 2), (2, 4), (3, 6)$. They satisfy the near resonance condition (12) with $\Delta\omega = 0.0350$. Since the maximum plasma beta is found to be $\beta_0 = 0.033$ at $r = 0$, the low-β approximation is not sufficiently but modestly satisfied; $\epsilon \sim 0.1$. The coupling coefficient among the GAEs is calculated as

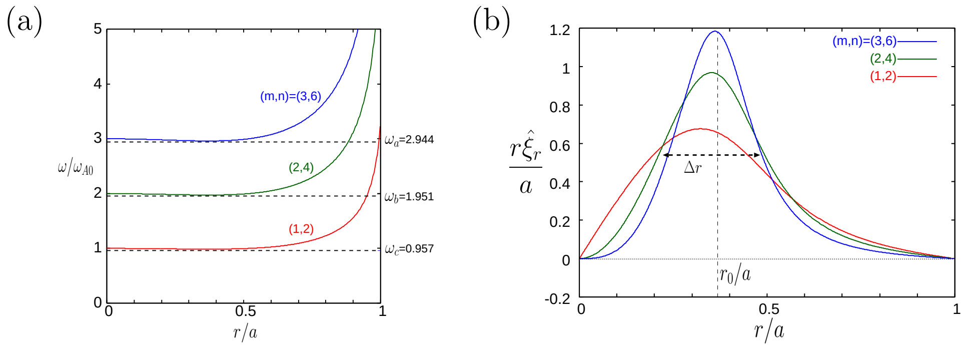

Figure 1: (a) Discrete GAE spectra $\omega_a, \omega_b, \omega_c$ below Alfvén continuous spectra $\omega_A(r)$ and (b) corresponding eigenfunctions with respect to $r\hat{\xi}_r$.

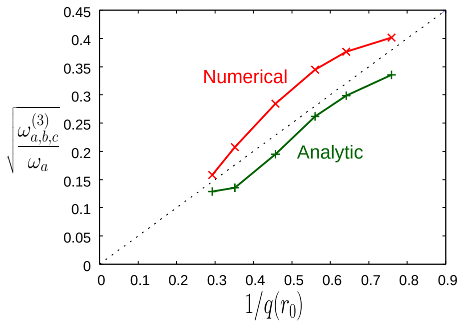

Figure 2: Dependence on the safety factor q at $r = r_0$

	Numerical result	Analytic formula
$\sqrt{\omega^{(3)}_{a,b,c}}$	0.689	0.576

where the numerical result is obtained from the direct integration of (7) using the genuine eigenfunctions whereas the analytic result is derived by (24) with the Gaussian model (25). Even though the parameter ϵ is not so small, our analytic formula shows a good agreement with the numerical result. The remaining discrepancy is mainly stemming from the use of the Gaussian model. In Fig. 1(b), the $(1, 2)$-mode does not fit into the Gaussian in comparison with other two modes and its modal width Δr is regulated by the boundaries rather than the local analysis. When the parameter ν (namely, q_a) is increased, both the numerical and analytic results show the same tendency to decline in proportion to $1/q(r_0) \sim \epsilon$ (Fig. 2), as predicted by the analytic formula (24).

The values of $\omega^{(3)}_{a,b,c}$ and $\Delta\omega$ can predict the effect of the nonlinear coupling according to Cases (i) and (ii) in Sec. 3. The energy exchange becomes active when the amplitude $|C_j|$ exceeds the level of $\Delta C_j := \sqrt{\Delta N/\mu_j} = \sqrt{\Delta\omega^2/4\omega_j\omega^{(3)}_{a,b,c}}$, which is numerically calculated as $\Delta C_a = 0.0132$, $\Delta C_b = 0.0182$ and $\Delta C_c = 0.0260$.

6 Summary

Based on the MHD Lagrangian theory, we have formulated a weakly-nonlinear equation of motion (8) for the displacement field $\boldsymbol{\xi}$. The new quadratic force term $\mathcal{F}^{(2)}$ is responsible for nonlinear mode-mode couplings, which lead to various phenomena such as parametric decay, second harmonic generation, explosive instability (if negative-energy mode exists) and weak turbulence (if many modes are coupled with each other). The associated potential energy $W^{(3)}(\boldsymbol{\xi}, \boldsymbol{\xi}, \boldsymbol{\xi})$ with cubic symmetry serves to quantify the efficiency of individual three-mode (and also second-harmonic) coupling. This formalism, for the first time, enables us to analyze nonlinear coupling of "global" MHD eigenmodes, which has been unexplored in the conventional wave-kinetic theory that rests on the eikonal approximation.

For the purpose of predicting nonlinear dynamics of AEs in tokamaks, we have examined nonlinear coupling of AEs in cylindrical geometry both analytically and numerically. The low-β approximation is used to obtain an analytic formula of the coupling coefficient (24), which shows a good agreement with the direct numerical calculation of $W^{(3)}$. Besides the spatial overlapping of three eigenfunctions, the finite compressibility $\nabla \cdot \hat{\boldsymbol{\xi}} (\sim \sqrt{\beta})$ of the eigenfunction is essential for the coupling of AEs to be effective. The coupling coefficient is found to be proportional to $\sqrt{\beta}$ (or $1/q$) and square of the inverse aspect ratio a^2/R_0^2.

Since the exact resonance condition is rarely satisfied among AEs, the frequency mismatch $\Delta\omega$ must be also taken into account. The mode coupling is inhibited by $\Delta\omega$ in such a way that the nonlinear effect works only for large-amplitude AEs satisfying $C_j \gtrsim \Delta C_j$. The value ΔC_j can be interpreted as a nonlinear saturation level of the energetic-particle-driven AE because the energy transfer among AEs becomes enhanced beyond this level. For the three GAEs studied in Sec. 5, the saturation level is about $\Delta C_j \sim 0.01$, where C_j is related to the magnetic field fluctuation via $\delta \boldsymbol{B}/B_0 \simeq C_j(\omega_j/\omega_{A0})\sqrt{(\rho/\rho_0)}\hat{\boldsymbol{\xi}}_j/a$.

In tokamaks, AEs emerge due to various effects (including the toroidicity) and the eigenvalue equation [especially g in (19)] should be modified to reflect them [11, 12, 13, 14, 15, 16]. We expect that $\Delta\omega$ is sensitive to these modifications whereas the low-β approximation of the coupling coefficient given here is still valid. Since the general expression of the coupling coefficient is obtained in this work, it can be evaluated numerically by substituting eigenfunctions of the existing eigenvalue solver for tokamak. This quantitative evaluation would be more relevant to the experimental observation of the saturation level.

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