Dynamics of wave-number spectrum of plasma turbulence

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Nonlinear dynamics of a family of reduced plasma turbulence models has been formulated using dimensonal analysis and simple shell models. The steady state solutions of these simple models give power law-like solutions for the wave-number spectrum of plasma fluctuations. Albeit its simplicity in terms of linear physics, such models can be applied to a real plasma in the range of wave-numbers where the linear drive is negligible. It is shown that the simple analytical solution representing the steady state where non-local interactions dominate over the local cascade process, agrees reasonably well with the measured wave-number spectrum from the Tore Supra tokamak.

I. INTRODUCTION

A significant ratio of the energy and particle loss in fusion devices is due to anomalous transport processes. Small scale turbulence, supplied by some form of drift instability is usually the primary cause of such transport. These instabilities are driven unstable by the inevitable inhomogeneities in a confinement device (i.e. "profiles", of temperature, particle density etc.). Focusing at ion scales, the drift instabilities are driven unstable by the free energy source from background gradients from the profiles coupled with either energy dynamics (e.g. ion temperature gradient driven -ITG- mode) or non-adiabaticity in the electron response (e.g. collisional drift instabilities or trapped electron modes -TEM-). Whether the non-adiabaticity is due to collisions or not, collisions in modern fusion devices are sufficiently

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rare that the transport due to collisions is ineffective in getting rid of the inhomogeneities in profiles rapidly. This being the case, the only way the plasma can dispose of the inhomogeneities due to localized deposition and confinement, and thus increase entropy, is by sacrificing some of this free energy to drive certain collective particle motions that can, *on average,* cause transport. This happens when the collective motion of particles and the associated oscillating fields lead to an imbalance between the number of particles (or their energies) going in one direction than the other. Since many kinds of collective motions are possible in a plasma, one can think of the micro-instabilities as being "selected" due to their efficiency in removing the inhomogeneity in heat and particle deposition imposed by the confinement and heating schemes. In particular, the most unstable mode under a given plasma profile, corresponds roughly to the collective motion that is the most efficient in transporting the quantity whose inhomogeneity acts as the free energy source, among all the possible collective motions that the plasma dynamics permit.

Nonlinear interactions limit the efficiency of anomalous transport due to collective motions. This is natural since the energy can not simply keep accumulating at the most unstable mode: it gets transferred to modes that are less efficient in transporting particles and energy via mode coupling. This can be viewed as a transfer of energy (or another conserved quantity) in k-space. The fact that the plasma turbulence reaches steady state, implies that the mode coupling and linear instability drive become roughly comparable. This is true in particular near the most-unstable mode, where the turbulence energy is "injected".

Since the turbulence decorrelation rate is a functional of the intensity of turbulent fluctuations, the turbulence can grow until the local effective decorrelation rate on the scales corresponding to the most unstable mode becomes approximately equal to the linear growth rate. In this stage, the energy injected at that scale at the rate given by the linear instability drive is decorrelated at approximately the same rate, leading to a steady state. While this picture allows us to describe qualitatively the nonlinear dynamics near the most unstable mode, it fails to describe the turbulent energy transfer (i.e. cascade) dynamics far from the scale of the instability drive.

In this paper, we study this nonlinear transfer process far from the energy injection scale which we take roughly to correspond to the scale of the most unstable mode. We will propose simple spectral cascade models that can be used to study the nonlinear dynamics of the spectral transfer. This allows us to describe the temporal evolution of the wave number spectrum of density or electrostatic potential fluctuations under certain assumptions. Wave number spectrum is an important observable, which is widely used in the modelling of similar kinds of nonlinear mode coupling processes in neutral fluid turbulence. It can be directly measured in a tokamak using various different methods. It allows probing the characteristics of the underlying micro-turbulence [1] and gives information about transport and its anomalous nature. In principle this also permits direct comparison to numerical simulations. It was shown recently that when diagnostics are carefully interpreted, direct gyrokinetic simulations can be shown to agree with experimental observations [2], which roughly give an isotropic spectrum in the wave-number range where the measurements were made. (i.e. $0.5 < k_\perp \rho_s < 3.0$).

Wave number spectrum is also important for validation, in that it provides detailed information [3] that can be compared to numerical simulations. Today, direct numerical simulations of gyro-kinetic Vlasov equation appear as the main tools for studying anomalous transport. These simulations are used either directly or as part of multi-scale modelling, for predictions of the transport in parameter regimes that are inaccessible by existing devices.

These predictions affect policy decisions. Therefore it is rather critical that we are sure that these numerical simulations describe the same kind of micro-turbulence as observed in tokamaks. While direct comparisons between experiment and theory is essential in order to increase our confidence in these models (as done in Ref. 2), phenomonological study of the experimental measurements from a theoretical point of view, with an eye on the synthesis of the foremost dynamical mechanisms is also crucial for understanding. Such an approach is useful, in particular when a direct comparison between experiment and numerical simulation is not possible due to various underlying assumptions and limited resolution in direct numerical simulations by contracting the vast amount of observed data in the form of simple physical concepts.

II. WAVE NUMBER SPECTRUM OF DRIFT WAVE TURBULENCE

A. Dimensional Analysis

The simplest way to derive the wave-number spectrum of turbulence is dimensional analysis. The formulation involves finding a conserved quantity, and assuming that there exists a *range* of *scales*, for which it is reasonable to assume the production and dissipation of this conserved quantity can be neglected. Such a range is generally called an *inertial range*, and usually named after the conserved quantity which is transferred indicating also the direction of the transfer (e.g. "energy inverse cascade range", "enstrophy forward cascade range" etc.).

In fusion plasmas, existence of such an inertial range is questionable. This is so, because the injection is not really localized and there exists a multitude of linear instabilities each of which injecting turbulence energy in the vicinity of the spatial scale corresponding to their most-unstable modes. Furthermore, the injection scale itself is not well-localized and it is common to have multiple linear instabilities overlaping in a given region. Dissipation is not truly localized in fusion plasmas either, since the main processes that dissipate turbulence energy are Landau damping and the existance of linearly damped large scale structures that feed on turbulence. Both of these may extract energy from micro-turbulence over a wide range of spatial scales.

Albeit these complications (and others such as intrinsic anisotropy of energy injection), it is nevertheless important to study the non-linear dynamics of turbulence at a basic level in order to develop an understanding of its behaviour. For instance the question, *"what would we expect as the wave-number spectrum if we had a well-defined inertial range?"* is a valid and an important one.

The potential vorticity (PV) defined as

$$h \approx n - \nabla^2 \Phi \tag{1}$$

is approximately conserved by the nonlinear dynamics of fusion plasmas[4]. This means that we can write:

$$\frac{\partial}{\partial t} W(k) = -\frac{\partial}{\partial k} \Pi_W(k) + P_W(k) - \varepsilon_W(k) \tag{2}$$

where $W(k) = \int k d\alpha_k \int_{k-\epsilon}^{k+\epsilon} h_{\mathbf{k}} h_{\mathbf{k}'} d^2\mathbf{k}'$ is the potential enstrophy (PE) density at wave-number k, $\Pi_W(k)$ is the k-space flux, $P_W(k)$ is the production (due to drive) and $\varepsilon_W(k)$ is the dissipation of potential enstrophy. For the forward potential enstrophy cascade, $P(k) = 0$ for $k > k_i$ and $\varepsilon(k) = 0$ for $k < k_d$ (i.e. k_i is the "injection" and k_d is the "dissipation" wave-numbers). In other words, in the potential enstrophy inertial range (i.e. $k_i < k < k_d$), the only non-vanishing term on the right hand side is the nonlinear transfer.

1. *Forward Potential Enstrophy Cascade*

If we integrate (2) from k to ∞, (where k is in the intertial range), we find that in steady state, the flux in k-space is constant [i.e. $\Pi_W(k) = \beta$ where $\beta = \int \epsilon_W(k)\, dk$]. In general, the flux of PE can be estimated using dimensional analysis as:

$$\Pi_W(k) \propto W(k)\, E_K(k)^{1/2} k^{5/2} \tag{3}$$

Note that, in order to write (3) one has to assume that both the flux and the interactions are local in k-space, since the expression involves a single spatial scale k. Here $E_K(k) = \int k d\alpha_k \int_{k-\epsilon}^{k+\epsilon} d^2\mathbf{k}' \left[(\mathbf{k}\cdot\mathbf{k}')\,\Phi_{\mathbf{k}}\Phi_{\mathbf{k}'}\right]$ is the kinetic energy density at wave-number k.

$$W(k) \propto \frac{\beta k^{-5/2}}{E_K(k)^{1/2}} \tag{4}$$

for the forward cascade range where $\Pi_W = \beta$ is constant. Note that the above expression works equally for the 2D Euler turbulence, where $E_K(k) = E(k) = k^{-2}W(k)$ where we get the simple result $W(k) \propto \beta^{2/3}k^{-1}$, for enstrophy or the familiar expression $E(k) = \beta^{2/3}k^{-3}$ for energy[5]. However this implies for instance that $\int_{k-\epsilon}^{k+\epsilon} d^2\mathbf{k}' \left[\Phi_{\mathbf{k}}\Phi_{\mathbf{k}'}\right] \to \left\langle |\Phi_{\mathbf{k}}|^2 \right\rangle \propto \beta^{2/3}k^{-6}$.

2. *Nonlocal interactions*

Nonlocal interactions with a single dominant mode can be added to (2) in a simple way as:

$$\frac{\partial}{\partial t}W(k) = -\frac{\partial}{\partial k}\Pi_W^{(nl)}(k,q) - \frac{\partial}{\partial k}\Pi_W^{(l)}(k) + P_W(k) - \varepsilon_W(k)$$

where $\Pi_W^{(l)}(k)$ is the local k-space flux as in the previous section, and $\Pi_W^{(nl)}(k,q)$ is the nonlocal flux, driven by the dominant mode, which is now a function of both k , the wavenumber of the scale we look at, and q, the wave-number of the dominant mode . We can write the nonlocal flux in the form:

$$\Pi_W^{(nl)}(k,q) = -\int_0^k dk'k'\left(\hat{\mathbf{z}}\times\mathbf{q}\cdot\mathbf{k}'\right)\left[\overline{h}_{\mathbf{q}}^{*}\widetilde{h}_{\mathbf{k}+\mathbf{q}}\widetilde{h}_{\mathbf{k}}^{*} + \overline{\Phi}_{\mathbf{q}}\widetilde{h}_{\mathbf{k}+\mathbf{q}}^{*}\widetilde{h}_{\mathbf{k}}\right] .$$

Using dimensional analysis (noting that $E(\mathbf{q}) = q^2\overline{\Phi}_{\mathbf{q}}^2$ does not have the same dimensions, in real units, as $E_K(k)$):

$$\Pi_W^{(nl)}(k,q) = E(\mathbf{q})^{1/2}\, W(k)\, k^2$$

And for $\Pi_W^{(nl)} \gg \Pi_W^{(l)}$, we get a limiting form for the enstrophy spectrum given by:

$$W(k) \propto \frac{\beta}{E(\mathbf{q})^{1/2}}k^{-2} .$$

This means that the density fluctuation spectrum far from the dominant mode can be expressed as

$$\left\langle \left|n_{\mathbf{k}}\right|^{2}\right\rangle \propto \frac{\beta}{E\left(\mathbf{q}\right)^{1/2}}\frac{k^{-3}}{\left(1+k^{2}\right)^{2}}$$

under the adiabatic electron assumption.

Note that, here the dominant mode corresponds to the mode that is most efficient in terms on nonlinear interactions. It is assumed that even the non-local interactions with this single, discrete mode is more effective than the local interactions with the neighbouring modes. This could be due to the fact that the spectrum is dominated by a single discrete mode. We expect this mode to correspond to a large scale flow mode such as a zonal flow or a geodesic acoustic mode (GAM).

3. Inverse Energy Cascade

If the PV equation is invertable [i.e. under certain assumptions, we can solve for $\Phi_{\mathbf{k}}\left(t\right)=\mathcal{L}_{k}h_{\mathbf{k}}$], an equation for energy can be written. As in the above case, we can write

$$\Pi_{E}\left(k\right)\sim E_{K}\left(k\right)W\left(k\right)^{1/2}k^{3/2}$$

from dimensional analysis. This gives

$$W\left(k\right)\propto\frac{\varepsilon^{2}k^{-3}}{E_{K}\left(k\right)^{2}}$$

or $W\left(k\right)=\varepsilon^{2/3}k^{1/3}$ for enstropy, and $E\left(k\right)=\varepsilon^{2/3}k^{-5/3}$ for energy for 2D Euler turbulence. Note that the well-known Kraichnan-Kolmogorov spectra that we recover here, i.e. $E\left(k\right)\propto\left\{k^{-3},k^{-5/3}\right\}$, actually imply $\left|\widetilde{\Phi}_{k}\right|^{2}\propto\left\{k^{-6},k^{-14/3}\right\}$ respectively. As can be seen in Figure 3, these don't correspond to the experimental measurements.

B. Shell model approach

Shell models are used in neutral fluid and magnetohydrodynamic (MHD) turbulence to describe the nonlinear dynamics of the wave-number spectrum in a cascade process. They correspond to a severe reduction of the initial physics model to a system of ordinary differential equations that can be treated as a simple dynamical system. They respect the initial conservation laws of the physics model, and thus possess similar characteristics with it. In particular, they have fixed points associated with the power-law spectra of the original physics model.

The shell models have not been very popular for fusion plasmas historically. In fact, the simple Hasegawa-Mima shell model as derived by Ottinger and Carati [6] is the first example of a shell model for fusion plasmas that the current authors are aware of. Here we present a generalization of this model to the case with non-local interactions (see [7, 8]):

$$\frac{\partial\Phi_{n}}{\partial t}+\overline{\alpha}\frac{qk_{n}\overline{\Phi}}{1+k_{n}^{2}}\left[g\left(1+g^{2}k_{n}^{2}-q^{2}\right)\Phi_{n+1}-\left(1+g^{-2}k_{n}^{2}-q^{2}\right)\Phi_{n-1}\right]=C\left(\Phi_{n},\Phi_{n}\right) \qquad (5)$$

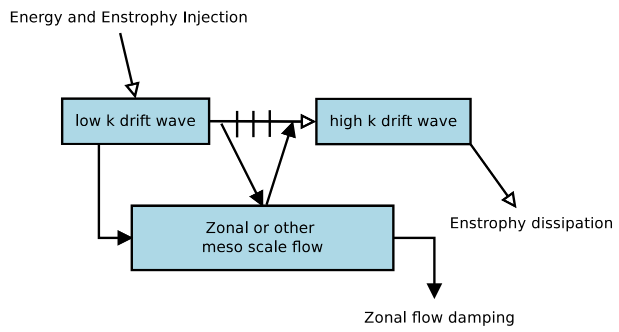

Figure 1: Schematic description of energy injection, enstrophy dissipation and predator-prey dynamics between meso-scale flows and the drift-wave spectrum.

$$\frac{\partial}{\partial t}\left(q^2\overline{\Phi}\right) = \overline{\alpha}\sum_n qk_n^3 g\left(g^2-1\right)\Phi_n\Phi_{n+1} - \nu_F q^2\overline{\Phi}\ , \tag{6}$$

where

$$C\left(\Phi_n,\Phi_n\right) \equiv \alpha\frac{k_n^4\left(g^2-1\right)}{1+k_n^2}\left[g^{-7}\Phi_{n-2}\Phi_{n-1} - \left(g^2+1\right)g^{-3}\Phi_{n-1}\Phi_{n+1} + g^3\Phi_{n+1}\Phi_{n+2}\right]$$

This is a coupled system, describing the evolution of drift wave turbulence undergoing local cascade and interacting with a large scale mode evolving self-consistently with drift wave turbulence.

1. *Stationary spectrum with disparate scale interactions*

We can obtain the steady state fluctuation spectrum when disparate scale interactions dominate using (5) and considering only the second term on the left hand side. This gives:

$$\left[g\left(1-q^2+g^2k_n^2\right)\Phi_{n+1} - \left(1-q^2+g^{-2}k_n^2\right)\Phi_{n-1}\right] = 0$$

which has the solution:

$$\Phi_n \sim \frac{k_n^{-1/2}}{\left(1-q^2+k_n^2\right)}$$

or when written in terms of the fluctuation intensity, this implies (for $q \ll k$):

$$\left|\widetilde{\Phi}_k\right|^2 \sim \frac{k^{-3}}{\left(1-q^2+k^2\right)^2} \sim \frac{k^{-3}}{\left(1+k^2\right)^2}\ . \tag{7}$$

This is the steady state spectrum when there is a large scale mode and the disparate scale interactions with this mode are dominant. Note that large scale here is effectively defined as a $k_{\|} \approx 0$ with a $|k|$ smaller compared with the range for which we observe the turbulence. Generally these are zonal flows. However other meso-scale modes or external mean flows may become dominant if zonal flows are weak or artificially suppressed.

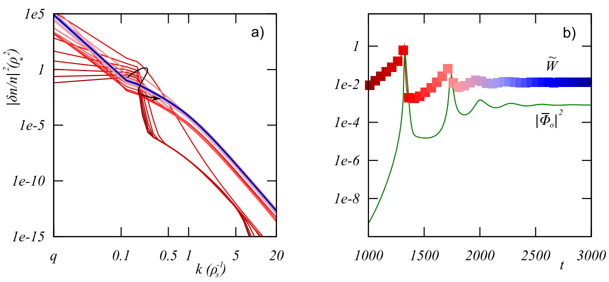

Figure 2: Predator-prey oscillations between zonal flows and the drift wave spectrum. Here the green line in b) is the zonal flow level, and colored points are the drift-wave amplitude, colored in such a way that time advances from dark red for the initial state to dark blue in the final state. The same colors are used in a), so that the oscillations of the spectrum can be seen as we go from dark red to dark blue.

2. Predator-Prey Oscillations

Since the model in its general form (e.g. Eqns 5-6) incorporates coupling between zonal flows and drift-wave turbulence, it displays a character of predator prey oscillations[9, 10]. However in the shell model formulation the drift wave level is not reduced to a single variable (the amplitude) and is treated as a spectrum. An example of such oscillations that saturate to a fixed point and the change in the spectrum during these oscillations can be seen in Figure In fact, since the zonal flow regulates turbulence by refracting large scale structures to smaller scales the qualitative picture is better represented in a model with two or more drift wave scales. In such a model the low-k drift waves would be driven by the background gradients, they would transfer their enstrophy to high-k drift waves as they interact with the zonal flow and the high-k drift waves would dissipate potential enstrophy. The system might reach a quasi-steady state if the zonal flow damping is relatively low, and otherwise would oscillate with a frequency linked to zonal flow damping. In the limit where drive and damping can be neglected, an exact analytical solution of these oscillations can be given in terms of the Jacobi elliptic functions.

C. Comparison with Tore-Supra measurements

The turbulence spectrum in tokamaks can be measured using different methods. Here, we demonstrate a standard, ohmic, L-mode shot from Tore-Supra tokamak measured using the Doppler reflectometer system, DifDop. The figure shows a reasonably good agreement between the measured spectrum and the analytical expression $k^{-3}/\left(1+k^2\right)^2$. Of course, one can find better fits, with different functional forms. For instance, as one goes to higher k, the spectrum will ultimately take the form of an exponential, which suggests that we observe a "dissipation" range. For instance in the case of disparate scale interactions and a physical mechanism of damping that goes as $\gamma_d\left(k\right)\sim-\lambda_d k^2$, one does indeed recover a scaling: [7]$\left\langle\left|\widetilde{n}_k\right|^2\right\rangle\propto k^{-3}e^{-\lambda k}/\left(1+k^2\right)^2$. One can also simply use an exponential function. However, the expression $k^{-3}/\left(1+k^2\right)^2$, does not have any fitting parameters (apart from the fluctuation level), and can be linked to the physical process of disparate scale interactions. Therefore its agreement with experiment is remarkable and maybe more informative than a fit with an exponential function.

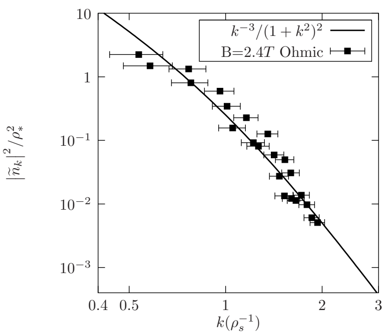

Figure 3: Measurement of density fluctuation spectrum in a standard Ohmic discharge in Tore Supra tokamak. Note that since there is no absolute calibration of the experimental setup for the fluctuation level, the y axis is in arbitrary units.

III. RESULTS & DISCUSSIONS

It was shown that simple intuitive modelling of the strongly nonlinear wave-number spectrum of density fluctuations in tokamak plasmas is possible by the use of shell models, or turbulence cascade models. When the disparate scale interactions are dominant a simple analytical form $\left\langle|\widetilde{n}_k|^2\right\rangle \propto k^{-3}/\left(1+k^2\right)^2$ is shown to be possible, which agrees reasonably well with the experimental results from Doppler reflectometry measurements. It was also shown that predator-prey oscillations can be described by shell models that include nonlocal interactions with a single dominant mode, which can be a zonal flow, GAM or a similar meso scale structure with a well defined wave-number. Direct comparison of the fluctuation measurements, gyrokinetic models and the simple cascade model presented here, suggest that the strongly nonlinear nearly isotropic high-k part of the spectrum can be described by the physical picture of a cascade among different scales either by local processes or mediated by nonlocal interactions to meso scale flow structures.

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