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VOLUME 3

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Chapter 1

INTRODUCTION

Part F of the Safeguards Technical Manual is being issued in three volumes. Volume 1 was published in 1977 and revised slightly in 1979. Volume 1 discusses basic probability concepts, statistical inference, models and measurement errors, estimation of measurement variances, and calibration. These topics of general interest in a number of application areas, are presented with examples drawn from nuclear materials safeguards. The final two chapters in Volume 1 deal with problem areas unique to safeguards: calculating the variance of MUF and of $\hat{D}$ respectively.

Volume 2 continues where Volume 1 left off with a presentation of topics of specific interest to Agency safeguards. These topics include inspection planning from a design and effectiveness evaluation viewpoint, on-facility site inspection activities, variables data analysis as applied to inspection data, preparation of inspection reports with respect to statistical aspects of the inspection, and the distribution of inspection samples to more than one analytical laboratory.

Volumes 1 and 2 are written in a simplified mode with little provided in the way of statistical bases for the computational procedures set forth in somewhat of a cookbook manner. The volumes indicate how to deal with specific problems with step-by-step computational procedures, but create little understanding of the procedures themselves, their attendant assumptions and possible limitations in applications. Further, the volumes are characterized by a lack of cohesiveness or unity of purpose, consisting of a number of rather isolated procedures with little in the way of a unified development of the statistical applications to Agency safeguards.

Because of these shortcomings in Volumes 1 and 2, the need for preparation of a Volume 3 was identified. Volume 3 covers generally the same material as Volumes 1 and 2 but with much greater unity and cohesiveness. Further, the cookbook style of the previous two volumes has been replaced by one that makes use of equations and formulas as opposed to computational steps, and that also provides the bases for the statistical procedures discussed. Hopefully, this will help minimize the frequency of misapplications of the techniques.

Volume 3 stands alone in the sense that Volumes 1 and 2 need not be read before Volume 3; many examples are common to the volumes but are worked from a different perspective. Having studied Volumes 1 and 2 prior to Volume 3, however, may be helpful in reaching a quicker understanding of the Volume 3 material. Further, a greater appreciation for the material in the first two volumes should follow from studying Volume 3 which is intended to provide the motivation for the statistical procedures covered in the two volumes. Volume 3, of course, also contains more recently developed statistical techniques not present in the earlier volumes.

The 13 chapters of Volumes 1 and 2 have been rearranged and replaced by four chapters in this Volume, identified as Chapters 2-5. Chapter 2 discusses measurement errors in considerable detail (the table of contents is given at the start of each chapter). Chapter 3 is concerned with all aspects of error propagation as it relates to safeguards. Chapters 4 and 5 deal with Agency inspections, first from the design viewpoint, and then with respect to their implementation. The final chapter, Chapter 6, identifies and discusses current developments in the statistical aspects of safeguards, in anticipation of the need to revise Volume 3 periodically to keep the material contained therein current.

Volumes 1 and 2 each contain a glossary of terms. This glossary is omitted in Volume 3 because of the rather exhaustive discussion of measurement errors in Chapter 2. This lengthy discussion effectively replaces the glossary which must be viewed as a limited attempt to summarize a lot of ideas about measurement errors with a few definitions; which is difficult to do effectively. Hence, the need for the full discussion of these ideas in Chapter 2.

Volumes 1 and 2 are somewhat deficient in the completeness of their bibliographies. In Volume 3, a more complete bibliography is included. However, only those works actually cited in Volume 3 are listed in the bibliography. This should not detract from ones ability to perform additional background research in a given topic, however, since the cited articles themselves often contain cross references to other relevant work. Further, in safeguards applications, one can locate most articles of interest in a limited number of places. These include primarily the Institute of Nuclear Materials Management (INMM) Journals and Annual Meeting Proceedings, IAEA Conference Proceedings and related Agency reports, and Proceedings of the recently instituted meetings of the European Safeguards Research and Development Association (ESARDA). Further, in the INMM journals, complete listings are often given of available publications issued at a given facility. Thus, it is a simple matter to locate most articles pertinent to a given topic.

Chapter 2

MEASUREMENT ERRORS

2.1 DEFINITION OF ERRORS

Material balance accounting is an integral part of Agency safeguards. It relies heavily on measured data, which are subject to error. The inferences that are drawn on the basis of accounting data are, as a result, drawn in the presence of errors, and are hence stated in the language of statistics.

This chapter, Chapter 2 of Volume 3, Part F, is concerned with measurement errors, including error sources, error models, kinds of errors, effects of errors, and estimation of errors. Later chapters deal with the effects of errors in drawing inferences on facility performance, based both on the facility's accounting data, and also on the inspection data.

As a starting point in the discussion, it is important to define what is meant by the word, error.

ANSI Standard N15.5[2.1] provides the following definition:

Error of a Measurement--The magnitude and the sign of the difference between the measured value and the true value.

This definition is an attractive one in the context of this Volume since it speaks of a measured value as opposed to a reported or recorded value. The important distinction is that the measured value is the value that would apply to the measurement in question were there no mistakes in recording or reporting the value. The basic assumption behind drawing inferences on the basis of accounting data is that the data are free of mistakes, or defects as they will be called in later chapters. Steps are taken to provide some degree of assurance that this is, in fact, a reasonable assumption. However, whatever may be the concern on the presence of such defects in the data, it is important to keep in mind that the inferences to be drawn on the basis of the measured, quantitative data are based on the definition of an error of measurement given here.

By a simple extension of the definition, a mistake may be said to have occurred if a reported value differs by any amount from a measured value. A synonym for a measured value is an observed value, and these two expressions will be used interchangeably throughout this volume.

2.2 SOURCES OF ERROR

The definition of error given in Section 2.1 is a bit simplistic in that it implies a very simple error structure. In fact, most errors of measurement are not simply structured, and a given error of measurement often represents the combined net effect of many errors.

In this section, a discussion is given of the various sources of error that might affect a given measurement. The narrative discussion of this section is followed by a parallel mathematical model presentation in later sections.

2.2.1 Statistical Sampling Error

Consider a population of individual items, each of which has a true value of some specified characteristic associated with it. If one of these items is selected in some random fashion, then the true value associated with that item will differ from some nominal or base true value (e.g., the average of all true values over all population items) by some amount. Define a statistical sampling error as the difference: true value for randomly selected item minus base true value.

In many Agency safeguards applications, statistical sampling error need not be included when making inferences. This is because: (1) the facility itself will measure 100% of the items involved in the material balance; and (2) the inspection data are analyzed as by-difference data in which the operator's value is compared with the inspector's value for each item in question, the true value of the item in question thus not affecting this difference. That is, the statistical sampling error for the difference is zero.

The following points must, however, be kept in mind. (1) Each item may not have a unique measured value of the characteristic in question associated with it, although it will surely have a unique true value of that characteristic. The effect of this is discussed in 2.2.3.

(2) Suppose the facility does not have measured values for its items so that there is no item by item comparison of the operator's data with that of the inspector. In this event, it is necessary to make inferences about the operator's material balance solely on the basis of the inspector's data for the sampled items. Statistical sampling error must then be included, for the result found in inspection clearly depends on which sub-group of items are selected and measured. This could prove to be a major source of error in some situations.

(3) In the event of attributes inspection of the go, no-go type, the true value associated with each item is conventionally either 1, corresponding to a defect, or 0, corresponding to a non-defect. The nominal or base value for the population in question is a fraction or proportion, equal to the true total number of defects divided by the total number of items. Thus, there is a statistical sampling error committed as each item is selected. The effect of this error on the inferences drawn about the population will disappear only if all items in the population are included in the sample.

2.2.2 Bulk Measurement Error

Material accountancy is based on three measurement operations: (1) determination of the net weight or volume of an item (bulk measurement); (2) sampling of the material; (3) analysis of the sampled material for element and/or isotope concentration. In the event of NDA measurement, the bulk measurement and the sampling of the material are not performed (unless a density correction is applied on the basis of the weight or unless the NDA measurement is made on a sample of material rather than on the whole item).

It is convenient to divide the total error of a measurement into component parts, the parts corresponding to these three basic measurement operations. The bulk measurement error is defined as the magnitude and sign of the difference between an item true weight (or volume) and its measured or observed weight. With this definition, it is implied that the error is a single quantity, and as far as its effect is concerned, it is possible to regard it as such. However, in actuality, the bulk measurement error may be, and quite likely will be, the net effect of many errors associated with the bulk measurement, some of which may tend to cancel one another in their effect. See the further discussion in 2.2.5 and 2.4.

2.2.3 Material Sampling Error

Material sampling error is defined with respect to the characteristic being measured. This may be uranium concentration, U-235 concentration, plutonium concentration, etc.

Material sampling error is the magnitude and sign of the difference between the true value of the characteristic in question for the sampled material and the corresponding true value for the totality of material represented by the sample. It is important to keep in mind just what is this totality of material. To illustrate, if the characteristic in question is uranium concentration, and if the concentration is to be uniquely determined for a given container, then the sampling error is the difference between the uranium concentration for the (presumably) small sample drawn from the container, and the average concentration in that container. This may be called the "within-container sampling error." On the other hand, if a sample is drawn from a given container with the concentration to be applied to other nominally like containers, then the variability in concentration from one container to another is included in the sampling error, along with the variability within containers.

In this latter context, it is noted that material sampling error is closely related to statistical sampling error for that part of the error that occurs because of differences in concentration from item to item. Some prefer to not make a distinction between statistical and material sampling errors. Others find it convenient to do so; it is largely a matter of personal preference, but it seems convenient in the context of material accountancy to make that distinction. This is because when concentrations are uniquely determined for different containers (or groups of containers), and the value in question is the difference between the inspector's and the operator's measured concentrations, the statistical sampling error, as defined here, has no effect. On the other hand, there will still be a material sampling error, assuming that both parties did not analyze the same sample of material.

2.2.4 Analytical Measurement Error

As with material sampling error, analytical error is defined with respect to a specified characteristic. Analytical error is the magnitude and sign of the difference between the true value of the characteristic for the sampled material and the corresponding measured or observed value. Note that this error is defined with respect to the material sampled, and not to the totality of material to be characterized by that sample. It is, of course, the combined effects of sampling and analytical that is important.

In the event of measurement by NDA rather than by the bulk measurement-sampling-analytical route, then the error in the NDA measurement may for convenience be labeled an analytical error (although, as pointed out in Section 2.2.2, sampling error will also be introduced if the NDA measurement is performed on a sample of material rather than on the entire item).

2.2.5 Other Errors

As was indicated in 2.2.2, a given identified error is actually the net effect of potentially many errors. For example, in the weighing operation, the error in weighing could be the combined effect of how the item was positioned on the scale, the scale type, the particular scale of that type, the operator, and the environment (temperature, humidity) to name a few obvious potential sources of error. The extent to which specific error sources are identified and studied individually depends on the circumstances. For example, if the weighing error for the operation in question has little impact on the quality of the accountancy data, then there is little need to identify each source that contributes to the error. On the other hand, if the observed weights at the measurement point in question are judged to have larger than desirable errors of measurement, studies might well be initiated to ascertain why. In conducting these studies, at least some of the potential individual measurement error sources would be identified and evaluated as to their individual effects.

Regardless of the degree to which the error structure is decomposed into individual sources, in the accountancy applications to be discussed in this volume, the principal breakdown of errors will be limited to bulk measurement, material sampling, and analytical errors, always keeping in mind the more complex underlying error structure.

2.2.6 Statistical Sampling Distributions

Each time a measurement of some kind is made, there is a corresponding measurement error associated with the observed or measured value. Obviously, one does not know the value of the error; if it were known, then the observed value could be "corrected" for the known error, leaving the true value.

Although one may not know the particular error involved in a given measurement, one must know something about the possible magnitude of the error so that some statement can be made about the true value in question. The information about the error is conveyed through its known (or estimated) probability distribution. Specifically, one might have knowledge that an error, say ε, is distributed according to the normal distribution with zero mean and variance σ_ε^2.

As has been indicated before, a given measured value is affected by many errors of measurement. By appropriately propagating errors (this topic to be covered in Chapter 3), and by applying results from mathematical statistics theory, one can describe in some defined way the effect of the combined errors on the measured value. Carrying this one step further, one can find similar results for specified functions of a number of measured values. The specified functions of interest in safeguards applications are, for example, MUF, total inventory, operator minus inspector value, etc.

Given any measured value or any specified function of measured values (called a statistic) a goal in statistical inference is to make probability statements about some parameter on the basis of the observed or measured data. To do this, one must know the probability density function of the statistic in question. The probability density function enables one to compute a probability of occurrence for each possible set of outcomes of the statistic. The function in question can be derived from statistical theory given a set of input assumptions. The resulting function is often referred to as a sampling distribution. (In statistical theory, there is a distinction between a density function and a distribution function, the latter being the integral of the former. More correctly, the statistical function in question should be called a sampling density function.)

The foregoing discussion is pertinent to the discussion on errors because a sampling distribution provides for probability statements about the size of the error that might have been committed in a given application. For example, an important statistic in safeguards applications is the material unaccounted for (MUF). Each time a MUF is calculated, an error is made because the calculated MUF will differ from the true MUF by some amount due to all the errors of measurement that were unavoidably committed when calculating the MUF. Thus, given the sampling distribution of the MUF statistic, one can make inferences about the true MUF on the basis of the calculated MUF, for even though the size and sign of the error is not known, one does have knowledge of how it behaves in a probabilistic sense. Precisely how this knowledge is derived is the subject of the next chapter.

2.3 ERROR MODELS

In the foregoing discussion, it is indicated that there are many potential sources of error that might affect an observed result. In Section 2.4 to follow, it will be shown that these errors do not all behave in the same way. Although one is interested ultimately in the net effect of all errors of different kinds as they jointly affect a result, it is often helpful, and sometimes essential, to write down an appropriate mathematical model to identify the errors and how they relate to one another. There are several reasons for doing this:

(1) Writing the model aids in propagating the errors, i.e., in finding the net effect of the errors acting jointly.

(2) It identifies which are the important sources of error so that corrective steps can be taken if necessary and possible.

(3) It helps to insure that potentially important errors are not overlooked.

(4) It leads one to question the assumptions inherent in the model, and thus leads to more realistic models.

On this latter point, it should be understood that a model is a mathematical description of reality. When faced with the choice, one of course prefers simple models, even if they may depart a bit from reality. The model builder has an important task: to write the model that provides an adequate description of reality; and, at the same time to derive a model that is not too difficult to use. Proper attention must be devoted to model-building because the model may impact heavily on the results of the analysis. This very important point is sometimes overlooked.

In the next two sections, two kinds of models are considered: the additive model and non-additive model.

2.3.1 Additive Model

The additive model is the simplest model with which to work, and is also one which often provides a close approximation to reality. It is the basis for many common statistical techniques, such as the analysis of variance, and is widely employed in practice.

A very simple additive model is

$$x = \mu + \varepsilon \qquad \text{(eq. 2.3.1)}$$

where, for example,

x = observed gross weight of can of UO_2 powder, in grams

μ = true gross weight of can

ε = the error

The additive nature of the model is clear. The error, ε, selected in some as yet unspecified way, is simply added to the true weight to give the observed weight. Of course, one only has knowledge of x, and not μ or ε. On the basis of the observed x and some knowledge about the probability distribution for ε, one can make inferences about the size of μ (i.e., assign a value to μ along with some probability statement.) In another context, one might know μ (e.g., assigned value of a standard) and observe x, and use this information to make inferences about ε.

This simple additive model can be extended to include additional terms. For example, suppose that a difference between scales exists. Then, letting

θ_i = error for scale i

the previous model might be written

$$x_i = \mu + \theta_i + \varepsilon \qquad \text{(eq. 2.3.2)}$$

As this model is written, if θ_i were, say, 3 grams, then the model would indicate that items weighed on this scale would consistently read high by 3 grams, not counting the additional error, ε, associated with any given reading.

The additive model is considered further in Section 2.4.

2.3.2 Other Models

Although the additive model provides an adequate description of reality in many instances, this is not always the case. As a very simple example, even though individual errors may be described by additive models, it does not follow that a statistic of interest will have an additive model. To illustrate, keeping in mind (eq. 2.3.1,)(but letting the weights be net weights rather than gross weights), let

$$y = \alpha + \eta \qquad \text{(eq. 2.3.3)}$$

where y = observed ratio of uranium to UO_2

α = true ratio

η = the error

Suppose now that one is interested in the observed net weight of uranium, and not UO_2. The model for this is found by multiplying each side of (eq. 2.3.1) by each side of (eq. 2.3.3)

$$xy = \mu\alpha + \mu\eta + \alpha\varepsilon + \varepsilon\eta \qquad \text{(eq. 2.3.4)}$$

which is no longer a simple additive model.

As an extension of this, suppose that the model for the uranium to UO_2 ratio is not additive as in (eq. 2.3.3), but is rather of the multiplicative form:

$$y = \alpha\eta \qquad \text{(eq. 2.3.5)}$$

as would be the case if the error, η, were expressed as a multiplier, e.g., η = 1.01 would represent a 1% relative error. Then, the model for the net weight of uranium is, from (eq. 2.3.1) and (eq. 2.3.5),

$$xy = \mu\alpha\eta + \varepsilon\alpha\eta \qquad \text{(eq. 2.3.6)}$$

which is another non-additive model that might apply.

To summarize, although additive models are often adequate, it does not follow that they apply in all situations. One must be aware of the model before errors can be appropriately propagated and inferences drawn.

As a final comment, non-additive models may at times be appropriately transformed to result in additive models. For example, upon using logarithms, (eq. 2.3.5) may be written

$$\ell n y = \ell n \alpha + \ell n \eta \qquad \text{(eq. 2.3.7)}$$

which is now additive in the logarithms.

When errors are propagated in Chapter 3, the model will be kept in mind, if not explicitly written in each instance.

2.4 KINDS OF ERRORS

It has already been noted that there are potentially many sources of error that might affect a given measured value. It is also important to note that not all error sources will behave the same way in their effects. This fact is especially important in safeguards applications, as will be noted time and time again in future chapters.

There are three broad categories or kinds of errors that will be identified. These are random errors, systematic errors or biases, and errors that fall in neither category, which are usually called short-term systematic errors in safeguards applications. The different kinds of errors are perhaps best understood in the context of an example. The example will be developed further in each of the next three sections until the three basic kinds of errors will have been discussed, along with variations on them.

2.4.1 Random Errors

The example to be developed is as follows. Six sintered UO_2 pellets of nominally the same composition are to be analyzed for percent uranium. Let

x_i = measured percent uranium for pellet i

μ = nominal (or true) percent uranium

ρ_i = deviation from the nominal value for pellet i

ε_j = deviation due to analytical for measurement j

For simplicity in exposition, an additive model is assumed. (The distinction to be made among the kinds of errors is independent of this assumption.) The model representing the six measured values may be written:

$$\left.\begin{aligned} x_1 &= \mu + \rho_1 + \varepsilon_1 \\ x_2 &= \mu + \rho_2 + \varepsilon_2 \\ & \\ x_6 &= \mu + \rho_6 + \varepsilon_6 \end{aligned}\right\} \qquad \text{(eq. 2.4.1)}$$

Consider ρ_i. Since this differs for each of the six observations in the data set, ρ_i is called a random error. Further, with reference to the discussion in Section 2.2.1, ρ_i is a statistical sampling random error. If ρ_i is regarded as a random variable with zero mean and with variance σ_ρ^2, then σ_ρ^2 is called the statistical sampling random error variance. Note the important distinction between ρ_i and σ_ρ^2; ρ_i is an error while σ_ρ^2 is an error variance.

Consider ε_j. Since this also differs for each of the observations in the data set, ε_j is also a random error. More specifically, with reference to Section 2.2.4, ε_j is an analytical random error and, analagous with σ_ρ^2, the quantity σ_ε^2 is called the analytical random error variance.

It is noted from (eq. 2.4.1) that since ρ and ε have the same subscripts for all six observations, it is not possible to distinguish between the sampling and analytical errors. One might wish to combine them in the model, replacing $(\rho_1 + \varepsilon_1)$ by m_1, etc. The quantity m_1 might then be called the measurement random error, and σ_m^2 the measurement random error variance.

With respect to this last point, it is recognized by modelers that there are many potential sources of error that affect a given result, some identified and others not. It is common practice to group effects in a model especially when the effects cannot be distinguished, as in this model. If, say, duplicate analyses were performed on each pellet, then ρ_i and ε_j would not be combined; their effects are then distinguishable.

The characteristic feature of a random error in a model is that its subscript changes for each observation in the data set. The safeguards significance of random errors is that their effect on measurement uncertainty can be reduced in a relative sense by making additional measurements. A random error is said to propagate to zero in a relative sense with an increasing number of measurements. For this reason, random errors are controllable and, given sufficient resources, can be made to have little importance in many safeguards applications.

2.4.2 Systematic Errors; Biases

The model (eq. 2.4.1) is extended. Let

Δ = deviation from the nominal due to the analytical method, for all measurements in the data set

Then write

$$
\begin{aligned}
x_1 &= \mu + \Delta + \rho_1 + \varepsilon_1 \\
x_2 &= \mu + \Delta + \rho_2 + \varepsilon_2 \\
x_3 &= \mu + \Delta + \rho_3 + \varepsilon_3 \\
x_4 &= \mu + \Delta + \rho_4 + \varepsilon_4 \\
x_5 &= \mu + \Delta + \rho_5 + \varepsilon_5 \\
x_6 &= \mu + \Delta + \rho_6 + \varepsilon_6
\end{aligned}
\qquad \text{(eq. 2.4.2)}
$$

Note that Δ differs from ρ_i and ε_j in that there is no subscript (or, equivalently, the subscript may be the same for all members of the data set). The quantity Δ is called a systematic error or a bias, terms which are often used interchangeably. Some users make a distinction between these two terms in the situation where the quantity Δ is estimated in some way. The distinction made is that if observations in the data set are corrected on the basis of the estimate of Δ, then Δ is called a bias. However, since one cannot know Δ precisely, but can only estimate it, it is clear that the observations cannot be completely corrected for the bias Δ. There is a residual bias, consisting of the difference between Δ and its estimate, and this residual bias is then called a systematic error. This distinction between bias and systematic error is not made by all modelers. The important idea to keep in mind is that whatever the Δ quantity is called, the assumptions concerning Δ must be stated or implied so that errors can be properly propagated corresponding to the assumed model.

In modeling, the distinction between the systematic error and the random error is that the subscript on the systematic error is the same for all members of the data set (or, equivalently, there is no subscript). If Δ is a random variable with zero mean and variance σ_Δ^2, then σ_Δ^2 is called a systematic error variance. As a random variable, Δ is selected at random from some population just as was the random error, ρ_i or ε_i, the distinction being that, once selected, Δ is the same for all members of the data set.

In many safeguards applications, the effect of the systematic error is of dominant importance when compared with that of the random error. This is because, unlike the random error, the effect or impact of the systematic error cannot be reduced by taking additional measurements. The systematic error, as will be seen in later chapters, limits the effectiveness of safeguards from the material accounting point of view, unless steps can be taken to reduce its effect in some way. Merely making more measurements will not help.

2.4.3 Short-term Systematic Errors

The model (eq. 2.4.2) is further extended. Suppose that the six pellets are not all distributed to the same laboratory for analysis. Let

ℓ_k = deviation from the nominal due to the analysis being performed in laboratory k

Also suppose that within laboratory k, conditions change from one time-frame (day, shift, week, etc.) to the next so that

$t_{m(k)}$ = deviation from the nominal due to the analysis being performed in time frame m within laboratory k

Note that in the case of $t_{m(k)}$, the subscript is written to indicate that the "time" effect is peculiar to a given laboratory. That is, time frame 1 in laboratory 1 does not correspond to time frame 1 in laboratory 2, say.

With ℓ_k and $t_{m(k)}$ defined, suppose that the model now becomes

$$
\begin{aligned}
x_1 &= \mu + \Delta + \ell_1 + t_{1(1)} + \rho_1 + \varepsilon_1 \\
x_2 &= \mu + \Delta + \ell_1 + t_{1(1)} + \rho_2 + \varepsilon_2 \\
x_3 &= \mu + \Delta + \ell_1 + t_{1(1)} + \rho_3 + \varepsilon_3 \\
x_4 &= \mu + \Delta + \ell_2 + t_{1(2)} + \rho_4 + \varepsilon_4 \\
x_5 &= \mu + \Delta + \ell_2 + t_{2(2)} + \rho_5 + \varepsilon_5 \\
x_6 &= \mu + \Delta + \ell_2 + t_{3(2)} + \rho_6 + \varepsilon_6
\end{aligned}
\qquad \text{(eq. 2.4.3)}
$$

The model indicates that three of the pellets were sent to one laboratory where all three analyses were performed in the same time frame, and three were sent to a second laboratory where one analysis was performed in each of three time frames. Both laboratories used the same analytical technique and the random error variances due to analytical are assumed to be identical (indicated by use of ε_j for all six measurements).

The quantities ℓ_k and $t_{m(k)}$ differ from both the random error (ρ_j and ε_j) and the systematic error (Δ) in that for each error, the subscript is the same for some members of the data set, but not for all. Thus, ℓ_k and $t_{m(k)}$ are neither random nor systematic errors, but are some kind of intermediate type error.

In this particular application, ℓ_k may be called a laboratory error or effect, and $t_{m(k)}$ may be called a time effect, or a laboratory condition effect. In more general terminology, this kind of error that is intermediate to a random and a systematic error has been rather commonly referred to as a short-term systematic error in safeguards applications. In making a distinction between this and the systematic error, the latter is sometimes called a long-term systematic error.

It should be noted here that the distinction that is made between random errors, systematic errors, and short-term systematic errors is with respect to the particular set of data under discussion. For example, if the data set under consideration were to consist of only the first three observations rather than all six, then ℓ_k and $t_{m(k)}$ would both be (long-term) systematic errors rather than short-term systematic errors, for then the subscript would be the same for all members of the data set.

Before leaving this section on short-term systematic errors, an important side-issue comment is made with respect to the error $t_{m(k)}$. This point is made because of its importance both with respect to the interpretation of data from interlaboratory experiments (see Section 2.6.5), and also as it affects the analysis of inspection data.

For laboratory 1, since all measurements are performed in the same time frame, one cannot distinguish between the time effect and the laboratory effect. This is an important point because one professed aim of interlaboratory experiments is to remove the effects of differences between laboratories by correcting all results to some base value, that is, by obtaining estimates of the ℓ_k's and correcting for the laboratory effects. However, this approach does not recognize the importance of the time effect, $t_{m(k)}$ which is usually confounded with or indistinguishable from the laboratory effect ℓ_k. Thus, when one attempts to remove laboratory biases in this way, the results are only applicable to the given time frame that existed at the time of the interlaboratory experiment. The between-time variance, σ_t^2, may well be a dominant effect when compared with σ_ℓ^2, in which case it would be misleading to conclude that one can correct for differences between laboratories. Rather, in most instances, one would use interlaboratory data to obtain the combined estimate, $\sigma_\ell^2 + \sigma_t^2$, which becomes a systematic error variance, long term, when applied only to a given laboratory, and short term otherwise. In this instance, one usually calls this simply a between laboratory variance, it being understood that the time effect is implicitly included in that variance component. Note that for laboratory 2, the measurements are made at three different times, but this may not be the usual mode in inter-laboratory experiments.

2.5 EFFECTS OF ERRORS

Much of this Volume deals with the effects of errors on quantities of safeguards importance in a very detailed way. The discussion in this section anticipates the more detailed presentations to follow in later chapters, and is intended to provide an overview of the role played by errors of measurement. First, the effect of errors on facility MUF is discussed, and then the effect on operator and inspector comparisons is considered.

2.5.1 Effect on Facility MUF

A given facility reports a MUF at the end of each material balance period, i.e., upon completion of a physical inventory. The MUF is affected by errors of measurement. It is also affected by unmeasured inventories, unmeasured loss streams, and mistakes in the recording, transmittal, and reporting of data. It would also be affected, of course, by any thefts or diversions. As a first step in the evaluation of the facility MUF, only the effect of measurement errors is taken into account. That is, one wants to make probability statements about the true MUF on the basis of the observed MUF and its calculated standard deviation due to errors of measurement. The true MUF, which includes the effects of all factors other than the errors of measurement, may then be further evaluated, but this further evaluation may be largely non-statistical in nature.

Associated with each MUF is an actual standard deviation describing its uncertainty. There is also a calculated or reported standard deviation. It is highly unlikely that these agree exactly, although the extent to which they disagree may be difficult to characterize. The disagreement comes about because of one or more of a number of reasons:

(1) The actual input measurement error variances will not be the same as their estimated values used in the error propagation.

(2) There are approximations used in the error propagation.

(3) The errors may be improperly propagated, even though the approximate nature of the propagation may not lead to a serious discrepancy. Improper propagation could occur because of a large number of reasons. Some common ones being:

(a) Treating constant element or isotope factors as having no associated error

(b) Failing to account for items that are identically a part of two cancelling components of the MUF calculation (e.g., in receipts and ending inventories)

(c) Making arithmetic mistakes

While keeping in mind that the calculated standard deviation of MUF is not the actual standard deviation (a quantity that will not be known), nevertheless the calculated standard deviation is used in making judgements about the significance of the MUF. The inspector need not proceed on blind faith and accept the facility's calculated MUF. Experience with similar plants provides guidelines as to what is a reasonable value for the standard deviation. Any large differences between calculated and guideline values can be investigated as to cause, and the calculated value appropriately corrected if found to be in error.

Assuming that the calculated standard deviation of MUF is a reasonably correct value, a diagnostic look at the calculations will reveal what are the major sources of error. An identification can then be made of possible steps that might be taken to reduce its size. On the other hand, study may also reveal that excessive measurement effort is being made in some instances; measurement effort may well be better directed elsewhere. In short, it is worthwhile to go beyond the simple calculation of the standard deviation of MUF, and use the calculations to redirect measurement effort as judged desirable.

This diagnostic look may well reveal that the standard deviation of MUF is limited in size by systematic errors. Unfortunately, it is not a simple matter to obtain estimates of systematic error variances in all cases, nor is it possible to reduce their effects without extensive effort, if at all. (It is faulty reasoning to suppose that extensive system recalibrations will eliminate systematic errors, although it is a step in the right direction.) One conclusion that might follow is that too much effort in the facility measurement control program is being directed at obtaining current estimates of random error variances whose effects on the standard deviation of MUF may be negligible in a relative sense. This information, if put to use, may be quite important to a facility burdened by measurements made solely for safeguards purposes. The facility and the inspectorate can jointly benefit by careful study of the calculations affecting the standard deviation of MUF.

2.5.2 Effect on Inspector-Operator Comparisons

One principal aim of an Agency inspection is to make a quantitative verification of the facility MUF. This verification makes use of the $\hat{D}$ statistic, treated in detail in the next chapter, but which for present purposes is defined as simply the estimate of the difference between the facility MUF and the inspector's estimate of this quantity. It is based on paired comparisons between the operator's and inspector's measured values on an item-by-item basis.

As with MUF, the quantity $\hat{D}$ is affected by errors of measurement. Unlike MUF, which is affected only by facility measurements, the uncertainty in $\hat{D}$ is also affected by uncertainties in inspector measurements. Also, whereas the standard deviation of MUF is often limited in size by systematic errors, this may not necessarily be the case with $\hat{D}$ because much fewer measurements are made by the inspector than by the facility. Further, only those facility measurements that are involved in the inspector comparisons affect the standard deviation of $\hat{D}$, so that the contribution to the random error variance of $\hat{D}$ due to the facility measurements will be relatively much larger than their contribution to the variance of MUF.

As will be seen in Chapter 4, the random error variances will determine how many measurements the inspector will make and how he will allocate these among the various flow and inventory items. Thus, for purposes of inspection, it <u>is</u> necessary to have good estimates of the inspector's and the facility's random error variances.

It may be true in some applications that systematic error variances for the operator and/or the inspector are of such a size that the inspection sample sizes are limited in the sense that further measurements beyond a minimum number will have negligible effect on the variance of $\hat{D}$. In this limiting case, if the systematic error variances for the facility and for the inspector are roughly equivalent in size, then the variance of $\hat{D}$ is twice the variance of MUF.

Another statistic of importance in Agency inspections is the so-called (MUF-$\hat{D}$) statistic, which is interpreted as the facility MUF adjusted for biases on the basis of the inspection data. An attractive property of (MUF-$\hat{D}$) is that it is unaffected by the operator's systematic errors. In a sense, it is the facility MUF with the operator's systematic errors replaced by the inspector's. The advantage is obvious: the inspector should be better able to evaluate and control his systematic errors than he can the operator's. This (MUF-$\hat{D}$) statistic will also be studied in detail in later chapters.

2.6 ERROR ESTIMATION

It should be apparent from the preceding sections that it is important to have valid information about measurement error variances. This information can come from a potentially large number of sources. In the balance of this chapter, methods will be given for obtaining estimates of the various measurement parameters. The techniques given are not intended to include all possible means of estimating measurement error variances, but do include those most often applied.

There are five main sub-topics: Measurements of Standards, Calibration of Measurement Systems, Measurements of Non-standard Materials, Error Estimation in the Presence of Rounding Errors, and Interlaboratory Test Data.

2.6.1 Measurements of Standards

A physical standard is an item having an assigned value associated with it for the characteristic in question. The value may be known without error, or it may have an error associated with it. In Section 2.6.1.1, the case of a single standard is considered. In 2.6.1.2, several physical standards are involved. In 2.6.1.3, measurements are made on a standard distributed over time.

2.6.1.1 Single Standard

A given standard is measured n times under a given set of conditions. The data are to be used to estimate the measurement bias and make a decision whether or not to apply a bias correction to measurement data generated by the measurement system under the same set of conditions. Whether or not the bias correction is to be applied, the systematic error variance for the measured result is to be estimated. In the case of an uncorrected result, the mean square error is used to describe the systematic error variance. The mean square error is the expected value of the square of the difference between the measured and true values. An estimate of the random error variance for the measured result will also be given, but this estimate may be unrealistic in some instances. Better estimates of random error variances come from measurements on production items (see Section 2.6.3).

Method 2.1

Notation

μ_o = assigned value of standard

σ_o = standard deviation of assigned value

$\bar{x}$ = average of the n measurements on the standard

s^2 = sample variance of these n measurements

y_j = measured value for production item j

Results

The estimated bias is

$$\hat{\theta} = (\bar{x} - \mu_o) \quad \text{(eq. 2.6.1)}$$

The bias corrected result is

$$y_j' = y_j - \hat{\theta} \quad \text{(eq. 2.6.2)}$$

If the bias correction is applied, the estimate of the systematic error variance for the bias corrected result is

$$\sigma_o^2 + s^2/n \qquad \text{(eq. 2.6.3)}$$

If the bias correction is not applied, the estimate is

$$\hat{\theta}^2 = (\bar{x} - \mu_o)^2 \qquad \text{(eq. 2.6.4)}$$

which does not involve σ_o^2, the variance of the assigned standard value. (It is noted that neither estimate is of very high quality in a statistical sense in this problem situation since they are one-degree-of-freedom estimates).

Based on statistical considerations, one would tend to apply a bias correction if the expression (2.6.3) is smaller than (2.6.4). If applied, the correction should be made at the time the measurement is made and not after the fact because of the administrative problems occurring when correcting past data.

Whether the bias correction is applied or not, the estimated random error variance for the reported result is simply s^2.

Basis

The principle of maximum likelihood is applied [2.2]. For a full discussion of this principle as applied to this particular problem situation, see [2.3]. Reference [2.4] is also pertinent.

Examples

EXAMPLE 2.1 (a)

A plutonium standard has an assigned value of 22.12% Pu. Its uncertainty is described by the standard deviation, σ_o = 0.04% Pu. Twelve analyses are made on the standard. Analyze the data to see if a bias correction should be made, and find the systematic and random error variances for the reported result (bias corrected or not). The data are listed.

x_i = % Pu	x_i = % Pu	x_i = % Pu
22.12	22.16	22.06
22.06	22.09	22.08
22.16	22.13	22.05
22.07	22.08	22.06

The pertinent values are

μ_o = 22.12 $\qquad$ $\bar{x}$ = 22.0933

σ_o = 0.04 $\qquad$ s^2 = 0.001552

By (eq. 2.6.1), the estimated bias is

$$\hat{\theta} = -0.0267$$

If the bias correction is applied, the systematic error variance of a reported result is

$$(0.04)^2 + (0.001552)/12 = 0.001729$$

If the bias correction is not applied, then the systematic error variance is

$$(-0.0267)^2 = 0.000713$$

Since this is smaller than 0.001729, the appropriate action would be to not apply the bias correction. This is because the standard value is so poorly known; there is not conclusive evidence that the measurement system is biased. Whether or not the bias correction is applied, the random error variance is 0.001552.

EXAMPLE 2.1 (b)

A standard weight is weighed at periodic intervals on a scale used to weigh fuel columns as they are loaded into rods. The scale reads net weights directly to the nearest 0.5 gram. On n = 42 weighings of the standard, the following frequency of the observed minus standard weights was found. The standard weight is to be known without error.

$(x_i - \mu_o)$ grams	Frequency
1.0	1
0.5	5
0.0	14
-0.5	19
-1.0	2
-1.5	1
	42

The pertinent values are

$$(\bar{x} - \mu_o) = -0.226 \qquad \sigma_o = 0 \qquad s^2 = 0.2220$$

If the bias correction is applied, the systematic error variance is 0.2220/42 = 0.00529. If it is not applied, it is $(-0.226)^2 = 0.05108$, almost ten times as large. Based solely on statistical considerations, it would seem appropriate in this case to apply the bias correction.

In this particular application, however, it is difficult to justify correcting past data for the bias which, although statistically significant, is quite small. The problem in making the correction to past data is an administrative one, likely affecting also fuel rods already shipped. A more appropriate action in this instance would be to try and adjust the scale to eliminate or reduce the bias or, failing that, possibly to bias correct future data as they are generated.

2.6.1.2 Several Standards

There are two general types of situations in which more than one standard may be measured. In the first type of application, a given scale is to be used to measure gross and tare weights to determine the net weight for an item. The bias in the scale is to be evaluated at both the gross and tare weight ranges in order to estimate the bias for a reported net weight. In the second situation, more than one physical standard is measured but it is assumed that any bias is constant over the range covered by the standards. The standards may have different associated uncertainties, and they may be measured different numbers of times. The problem is to estimate the overall bias and to find the random and systematic error variances of the reported result on a production item, whether or not it is corrected for bias.

In what follows, Method 2.2 applies to the first situation and Method 2.3 to the second.

Method 2.2

Notation

μ_g = assigned value of gross weight standard

μ_t = assigned value of tare weight standard

σ_g = standard deviation of assigned gross weight standard value

σ_t = standard deviation of assigned tare weight standard value

$\bar{x}_g$ = average of n_g measurements on gross weight standard

$\bar{x}_t$ = average of n_t measurements on tare weight standard

s_g^2 = sample variance of the n_g measurements

s_t^2 = sample variance of the n_t measurements

y_j = measured net weight for item j

Results

The estimated bias in the net weight is

$$\hat{\theta} = (\bar{x}_g - \mu_g) - (\bar{x}_t - \mu_t) \qquad \text{(eq. 2.6.5)}$$

The bias- corrected result is

$$y_j' = y_j - \hat{\theta} \qquad \text{(eq. 2.6.6)}$$

If the bias correction is applied, the estimate of the systematic error variance for the bias corrected result is

$$\sigma_g^2 + \sigma_t^2 + s_g^2/n_g + s_t^2/n_t \qquad \text{(eq. 2.6.7)}$$

If the bias correction is not applied, the estimate is simply $\hat{\theta}^2$, from (eq. 2.6.5).

Based on statistical considerations, one would tend to apply a bias correction if the expression (2.6.7) is smaller than $\hat{\theta}^2$. If applied, the correction should be made at the time the measurement is made, and not after the fact.

Whether the bias correction is applied or not, the random error variance for the reported result is

$$s_g^2 + s_t^2 \qquad \text{(eq. 2.6.8)}$$

Basis

The basis is the same as for Method 2.1 with a simple extension to include both the gross and tare weight standards.

Examples

EXAMPLE 2.2 (a)

A case history dealing with the estimation of scale accuracy and precision is given in reference [2.5]. Suppose that standards S and P in that reference and weighed in combination correspond to a typical gross weight while standard B is the tare weight standard. From the reference, the following information is derived.

$$\mu_g = 8878.0 \text{ g} \quad ; \qquad \mu_t = 1591.7 \text{ g} \quad ; \qquad \sigma_g = \sigma_t = 0.97 \text{ g}$$

Assume that weighings of these standards yield the following data:

$$n_g = 30 \qquad\qquad n_t = 20$$

$$\bar{x}_g = 8881.3 \text{ g} \qquad\qquad \bar{x}_t = 1589.9 \text{ g}$$

$$s_g = 6.2 \text{ g} \qquad\qquad s_t = 4.9 \text{ g}$$

The estimated bias in the net weight is

$$\hat{\theta} = 3.3 + 1.8 = 5.1 \text{ g}$$

If the bias correction is applied, the systematic error variance of the bias corrected result is

$$2(0.97)^2 + (6.2)^2/30 + (4.9)^2/20 = 4.36 \text{ g}^2$$

If the bias correction is not applied, this variance is

$$(5.1)^2 = 26.01 \ g^2$$

In this example, since 4.36 <26.01, one would apply the bias correction on the basis of statistical considerations alone, assuming that scale adjustments could not reduce the bias to a more acceptable lower value.

Method 2.3

Notation

m = number of standards

n_k = number of measurements on standard k

μ_k = assigned value for standard k

σ_k = standard deviation of assigned value, standard k

$\bar{x}_k$ = average of the measurements on standard k

s_k^2 = sample variance of these measurements

Results

The estimated bias is the weighted average

$$\hat{\theta} = \sum_{k=1}^{m} w_k (\bar{x}_k - \mu_k) \Big/ \sum_{k=1}^{m} w_k \qquad \text{(eq. 2.6.9)}$$

where

$$w_k = (\sigma_k^2 + s^2/n_k)^{-1} \qquad \text{(eq. 2.6.10)}$$

and where s^2 is the estimated random error variance,

$$s^2 = \sum_{k=1}^{m} (n_k - 1) \ s_k^2/(n-m) \qquad \text{(eq. 2.6.11)}$$

n being the total number of observations,

$$n = \sum_{k=1}^{m} n_k \qquad \text{(eq. 2.6.12)}$$

If the bias correction is applied, the estimate of the systematic error variance for the bias corrected result is

$$\left(\sum_{k=1}^{m} w_k\right)^{-1} \qquad \text{(eq. 2.6.13)}$$

If the bias correction is not applied, the estimate is simply $\hat{\theta}^2$, from (eq. 2.6.9).

As with Methods 2.1 and 2.2, one would tend to correct for bias if expression (2.6.13) is smaller than $\hat{\theta}^2$.

Basis

The estimate of the bias is a weighted estimate where the weights are the inverses of the variances of the estimated biases for each condition. This is a standard weighting procedure and leads to the result that the variance of the weighted average is the reciprocal of the summed weights [2.6].

Examples

EXAMPLE 2.3 (a)

Three standards are used in controlling the measurement quality of a mass spectrometer. The standard values in percent U-235 are 2.013, 3.009, and 4.949 respectively. The observations on these standards are as follows:

Standard 1	Standard 2	Standard 3
2.013	3.009	4.953
2.018	3.013	4.957
2.015	3.010	4.949
2.015	3.017	4.946
2.013	3.009	
2.010	3.008	
	3.013	
	3.010	
	3.011	
	3.006	

Assume that errors are constant on a relative basis. From Section 2.3.2, then, it is appropriate to transform the data to natural logarithms. The standard values in this transformed scale become:

$$\mu_1 = \ell n\ 2.013 = 0.69963 \quad ; \quad \mu_2 = 1.10161 \quad ; \quad \mu_3 = 1.59919$$

The errors in the standards are each 0.05% relative at one standard deviation.

$$\sigma_1 = \sigma_2 = \sigma_3 = 0.0005$$

For the transformed data, the averages and variances are:

$$\bar{x}_1 = 0.70012 \qquad \bar{x}_2 = 1.10214 \qquad \bar{x}_3 = 1.59964$$

$$s_1^2 = 1.7798 \times 10^{-6} \quad s_2^2 = 1.0567 \times 10^{-6} \quad s_3^2 = 0.9330 \times 10^{-6}$$

Further parameter values are:

$$m = 3 \quad n_1 = 6 \quad n_2 = 10 \quad n_3 = 4 \quad n = 20$$

The first step is to calculate s^2 from (eq. 2.6.11)

$$s^2 = \Big[(5)\ (1.7798) + (9)\ (1.0567) + (3)\ (0.9300)\Big] \times 10^{-6}/17$$

$$= 1.2470 \times 10^{-6}$$

The weights are, from (eq. 2.6.10),

$$w_1 = \Big[(0.0005)^2 + 1.2470 \times 10^{-6}/6\Big]^{-1} = 2.1842 \times 10^6$$

$$w_2 = 2.6688 \times 10^6 \quad ; \qquad w_3 = 1.7802 \times 10^6$$

$$\sum_{k=1}^{3} w_k = 6.6332 \times 10^6$$

The estimated bias is, from (eq. 2.6.9),

$$\hat{\theta} = \Big[(2.1842)\ (0.00049) + \ .\ .\ .\ + (1.7802)\ (0.00045)\Big]/6.6332$$

$$= 0.0004954$$

If the bias correction is applied, the estimate of the systematic error variance for the bias corrected result is, by (2.6.13)

$$(6.6332 \times 10^6)^{-1} = 0.1508 \times 10^{-6}$$

The standard deviation is 0.00039 or 0.039% relative.

If the bias correction is not applied, then the estimated error variance is

$$(0.0004954)^2 = 0.2454 \times 10^{-6}$$

Since this is larger than 0.1508×10^{-6}, one would make the bias correction based on statistical considerations.

2.6.1.3 Measurements at Different Times

Thus far in the discussion, it has been assumed that the standard or standards being measured by the measurement system has been measured under a fixed set of conditions. The estimate of the bias or systematic error derived from the data apply to items measured under that same set of conditions.

In many measurement systems, the bias will not remain stable as the conditions change. Commonly, it is not possible to identify the reasons for shifting biases that may be observed in different time frames. In some cases, reasons for shifting biases may be apparent, e.g., changes in measuring instruments, operators, or environmental conditions. Whatever the explanations, in describing the total error of measurement, the effects of an overall average bias and of the degree to which it may shift from one set of conditions to another must be taken into account. In addition, the assigned value of the standard may have an associated uncertainty, and this effect must be included.

Method 2.4 to follow, then, deals with the problem in which a standard with assigned value μ_o is measured n_i times under condition i. The data are to be used to estimate an average bias, possibly apply a bias correction to future measurements on production items, and obtain estimates of the random and systematic measurement errors, whether or not the bias correction is applied.

Method 2.4

Notation

μ_o = assigned value for standard

x_{ij} = observed value for j-th measurement under condition i

n_i = number of measurements made under condition i

n = total number of measurements

m = number of conditions

σ_o = standard deviation of assigned value

σ^2_ε = random error variance

σ^2_θ = variance among condition means (short term systematic error variance

y_{kj} = measured value for production item j measured under condition k

Results

Calculate the following quantities:

$$T_i = \sum_{j=1}^{n_i} x_{ij} \quad ; \quad T = \sum_{i=1}^{m} T_i$$

$$S_0 = T^2/n \qquad S_1 = \sum_{i=1}^{m} T_i^{\,2}/n_i$$

$$S_2 = \sum_{i=1}^{m} \sum_{j=1}^{n_i} x_{ij}^2 \qquad P = \left(n^2 - \sum_{i=1}^{m} n_i^2\right) \Big/ n(m-1)$$

$$M_B = (S_1 - S_0)/(m-1) \qquad M_W = (S_2 - S_1)/(n-m)$$

Then, the estimated bias is

$$\hat{\theta} = (T/n - \mu_o) \qquad \text{(eq. 2.6.14)}$$

The bias corrected result is

$$y_{kj}' = y_{kj} - \hat{\theta} \qquad \text{(eq. 2.6.15)}$$

The estimates of σ_ε^2 and σ_θ^2 are

$$\hat{\sigma}_\varepsilon^{\,2} = M_W \qquad \text{(eq. 2.6.16)}$$

$$\hat{\sigma}_\theta^{\,2} = (M_B - M_W)/P \qquad \text{(eq. 2.6.17)}$$

If the bias correction is applied, the estimate of the systematic error variance for the bias corrected result, y_{kj}', is

$$\sigma_0^2 + \hat{\sigma}_\theta^2 \sum_{i=1}^{m} n_i^2/n^2 + \hat{\sigma}_\varepsilon^2/n \qquad \text{(eq. 2.6.18)}$$

If the bias correction is not applied, this estimate is simply $\hat{\theta}^2$, from (eq. 2.6.14)

Basis

The statistical technique that forms the basis for this method is the one way analysis of variance. The parameter estimates are derived from the analysis of variance table.

One Way ANOVA Table

Source of Variation	Degrees of Freedom	Mean Square	Expected Mean Square
Between conditions	m-1	M_B	$\sigma_\varepsilon^2 + P\ \sigma_\theta^2$
Within conditions	n-m	M_W	σ_ε^2

The analysis of variance is covered in many standard texts. See, for example, [2.7].

Examples

EXAMPLE 2.4 (a)

Mass spectrometer measurements are made at four time periods on a known standard of nominal 3.046% U-235. The standard deviation associated with this value is 0.0006% U-235. The data are as follows:

Time 1	Time 2	Time 3	Time 4
3.095	3.044	3.019	3.090
3.086	3.078	3.045	3.073
3.058	3.046	3.022	3.053
3.073	3.060		3.081
	3.023		
	3.072		

In this example, it is not necessary to transform the data to logarithms because only the one standard is used. One could, of course, perform the transformation, in which case the estimates of the standard deviations would be on a relative basis rather than an absolute basis.

The various quantities are calculated.

$T_1 = 12.312 \quad T_2 = 18.323 \quad T_3 = 9.086 \quad T_4 = 12.297$

$T = 52.018$

$S_0 = 159.168960 \qquad S_1 = 159.174242$

$S_2 = 159.178232 \qquad P = 4.157$

$M_B = 0.001761 \qquad M_W = 0.000307$

Then, by (eq. 2.6.14),

$$\hat{\theta} = 0.0139$$

By (eq. 2.6.16) and (eq. 2.6.17),

$$\hat{\sigma}_{\varepsilon}^{2} = 0.000307 \qquad \text{(random error)}$$

$$\hat{\sigma}_{\theta}^{2} = 0.000350 \qquad \text{(short term systematic error)}$$

If the bias correction is applied, the systematic error variance for the bias corrected result is, by (eq. 2.6.18),

$$(0.0006)^2 + (0.000350)\ (77)/289 + (0.000307)/17 = 0.000112$$

If the bias correction is not applied, this variance becomes

$$(0.0139)^2 = 0.000193$$

Since 0.000193 exceeds 0.000112, one would tend to correct for bias in this instance.

EXAMPLE 2.4 (b)

In the example just concluded, suppose that the time grouping were ignored. The data would then be analyzed by Method 2.1. The bias estimate, $\hat{\theta}$ is still 0.0139. However, the estimate of the systematic error variance would be different. For these data:

$$s^2 = 0.0005795$$

$$n = 17$$

Thus, from (eq. 2.6.3), if the bias correction is applied, the estimate of the systematic error variance is

$(0.0006)^2 + (0.0005795)/17 = 0.00003445$, compared with 0.000112 when the time grouping is taken into account as it should be. If the bias correction were not applied, the estimate of the systematic error variance would be the same as in the preceding example.

This example illustrates the importance of correctly specifying the model for the statistical analysis. The existence of the short-term systematic error in this set of data must be accounted for in the analysis.

2.6.2 Calibration of Measurement Systems

The problem of calibration is closely related to that of measuring standards (Section 2.6.1) in that physical standards are also used in the calibration problem. In calibration, the measured response (Y) is related to the standard value (X) in some functional way, depicted by Y = f(X). The quantities Y and X may not be in the same units, e.g., X may be in grams U-235 and Y may be in count rates for an NDA counter. The calibration problem involves estimating the parameters for the function f(X) so that the relationship may be used to relate an observed response for a production item, observed on the Y scale, to an estimated value on the X scale for that item. Various statements about error variances are also made on the basis of the calibration data.

Various calibration problems are treated. First the case in which the functional relationship is linear and the variance of the measured response is constant over the range of calibration is covered.

2.6.2.1 Linear Calibration; Constant Variance

The measured response, Y, is related to the assigned value, X, by a linear relationship. At any fixed value of X, Y is normally distributed with mean value given by the linear relationship and with unknown but constant variance. The calibration data consist of n pairs of observations.

The calibration process leads to obtaining estimates of the parameters (slope, intercept, variance). The estimated calibration equation is then used for measuring production items, and random and systematic error variances are derived for the production item characteristic value corresponding to this response.

Two cases are considered. In Method 2.5, it is assumed that the intercept parameter is known. When this known value is zero, a special case, then the calibration curve passes through the origin. Method 2.6 covers the case when the value for the intercept parameter is not known.

Method 2.5

Notation

(y_i, x_i) = i-th data point; $i = 1, 2, \ldots, n$

α = intercept parameter (known)

β = slope parameter (unknown)

$y_i = \alpha + \beta x_i$, calibration equation

σ^2 = variance of y_i at given x_i

Results

Calculate the following quantities:

$$S_1 = \sum_{i=1}^{n} x_i \qquad S_2 = \sum_{i=1}^{n} x_i y_i \qquad S_3 = \sum_{i=1}^{n} x_i^2$$

$$S_4 = \sum_{i=1}^{n} y_i \qquad S_5 = \sum_{i=1}^{n} y_i^2$$

The parameter estimates are

$$\hat{\beta} = (S_2 - \alpha S_1)/S_3 \qquad \text{(eq. 2.6.19)}$$

$$(n-1)\,\hat{\sigma}^2 = (S_5 - 2\alpha S_4 + n\alpha^2) - (S_2 - \alpha S_1)^2/S_3 \qquad \text{(eq. 2.6.20)}$$

For a production item, the measured response is y_0. The corresponding characteristic value for the item is x_0, estimated by

$$\hat{x}_0 = (y_0 - \alpha)/\hat{\beta} \qquad \text{(eq. 2.6.21)}$$

The quantity $\hat{\beta}$, once calculated, behaves as a constant. Thus, the uncertainty in $\hat{\beta}$ affects $\hat{x}_0$ as a systematic error. Denoting the systematic error variance of $\hat{x}_0$ by $V_s(\hat{x}_0)$, it is given by

$$V_s(\hat{x}_0) = (y_0 - \alpha)^2\, V(\hat{\beta})/\hat{\beta}^4$$

$$= \hat{x}_0^{\,2} V(\hat{\beta})/\hat{\beta}^2 \qquad \text{(eq. 2.6.22)}$$

where $V(\hat{\beta})$ is the variance of $\hat{\beta}$, given by

$$V(\hat{\beta}) = \hat{\sigma}^2/S_3 \qquad \text{(eq. 2.6.23)}$$

The random error variance of $\hat{x}_0$ is denoted by $V_r(\hat{x}_0)$

$$V_r(\hat{x}_0) = \hat{\sigma}^2/\hat{\beta}^2 \qquad \text{(eq. 2.6.24)}$$

Consider k measured responses: y_{01}, y_{02}, ..., y_{0k} and the corresponding values calculated by (eq. 2.6.21) and denoted by $\hat{x}_{01}$, $\hat{x}_{02}$, ..., $\hat{x}_{0k}$. Letting $\hat{x}_{ot}$ be the sum of these $\hat{x}_{oj}$ values, consider the random and systematic error variances of $\hat{x}_{ot}$. These are

$$V_r(\hat{x}_{ot}) = k\ \hat{\sigma}^2/\hat{\beta}^2 \qquad \text{(eq. 2.6.25)}$$

and
$$V_s(\hat{x}_{ot}) = \hat{x}_{ot}^2\ V(\hat{\beta})/\hat{\beta}^2 \qquad \text{(eq. 2.6.26)}$$

Equivalently, if $\bar{x}_0$ denotes the average value, i.e., $\hat{x}_{ot}$ divided by k, then

$$V_s(\hat{x}_{ot}) = k^2\bar{x}_0^2\ V(\hat{\beta})/\hat{\beta}^2 \qquad \text{(eq. 2.6.27)}$$

Equation 2.6.22 may be considered a special case of (eq. 2.6.27) with k=1.

Basis

The estimate given by (eq. 2.6.19) and (eq. 2.6.20) are maximum likelihood estimates [2.2]. Equivalently, $\hat{\beta}$ is derived from the principle of least squares, i.e., it is the value of β that minimizes the sum of squares:

$$Q = \sum_{i=1}^{n} (y_i - \alpha - \beta x_i)^2 \qquad \text{(eq. 2.6.28)}$$

The expressions for the variances of the quantities of interest are based on error propagation methods to be discussed in Chapter 3.

Examples

EXAMPLE 2.5 (a)

A SAM NDA instrument is calibrated for use in measuring non-fissile plutonium. Calibration data relating net count data in CPMx10^{-3} to grams of non-fissile plutonium are as follows:

x_i = g Pu	y_i = net count rate (CPMx10^{-3})
104.29	141.022
208.58	286.928
312.87	420.571
406.73	545.497
417.16	557.069

Assuming a zero intercept linear model with constant variance, estimate the calibration curve. Four coupons are then counted, the counts per minute being 549,172; 319,614; 277,328; and 401,616 respectively. Estimate the total amount of plutonium in these four coupons, and find the associated random and systematic error variances.

The quantities S_1-S_5 are calculated.

$S_1 = 1449.63$ $\quad S_2 = 660395.5742$ $\quad S_3 = 491721.4159$

$S_4 = 1951.087$ $\quad S_5 = 886987.6955$

By (eq. 2.6.19) and (eq. 2.6.20), with $\alpha = 0$

$$\hat{\beta} = 1.3430$$

$$\hat{\sigma}^2 = 14.5057$$

For the four coupons counted, from (eq. 2.6.21)

$\hat{x}_{01} = 408.914$ $\quad \hat{x}_{02} = 237.985$ $\quad \hat{x}_{03} = 206.499$

$\hat{x}_{04} = 299.044$ $\quad \hat{x}_{0t} = 1152.442$ g Pu

From (eq. 2.6.25), (eq. 2.6.23), and (eq. 2.6.26),

$$V_r(\hat{x}_{ot}) = (4)(14.5057)/(1.3430)^2 = 32.170 \text{ g}^2 \text{ Pu}$$

$$V(\hat{\beta}) = 14.5057/491721.4159 = 29.50 \times 10^{-6}$$

$$V_s(\hat{x}_{ot}) = (1152.442)^2(29.50) \times 10^{-6}/(1.3430)^2 = 21.722 \text{ g}^2 \text{ Pu}$$

EXAMPLE 2.5 (b)

In Example 2.5 (a), assume that the intercept is known to be $\alpha = 10$. Then,

$$\hat{\beta} = 1.3135$$

$$\hat{\sigma}^2 = 11.6980 \quad \text{(a better fit to the data)}$$

$$V_r(\hat{x}_{ot}) = 27.121 \text{ g}^2 \text{ Pu}$$

$$V(\hat{\beta}) = 23.79 \times 10^{-6}$$

$$V_s(\hat{x}_{ot}) = 18.314 \text{ g}^2 \text{ Pu}$$

Method 2.6

Notation

Same as for Method 2.5 except that the calibration curve is rewritten to be

$y_i = \gamma + \beta\ (x_i - \bar{x})$, where

$\bar{x}$ = average of the n x_i values

Results

Calculate the quantities S_1-S_5 as in Method 2.5. The parameter estimates are:

$$\hat{\gamma} = S_4/n \qquad \text{(eq. 2.6.28a)}$$

$$\hat{\beta} = (S_2-S_1S_4/n)/(S_3-S_1{}^2/n) \qquad \text{(eq. 2.6.29)}$$

$$(n-2)\ \hat{\sigma}^2 = S_5-S_4{}^2/n - (S_2-S_1S_4/n)^2/(S_3-S_1{}^2/n) \qquad \text{(eq. 2.6.30)}$$

For a measured response y_0, the corresponding value of x_0 is estimated by

$$\hat{x}_0 = [(y_0 - \hat{\gamma})/\hat{\beta}] + \bar{x} \qquad \text{(eq. 2.6.30a)}$$

The variances of $\hat{\gamma}$ and $\hat{\beta}$ are given below. They have zero covariance.

$$V(\hat{\gamma}) = \hat{\sigma}^2/n \qquad \text{(eq. 2.6.31)}$$

$$V(\hat{\beta}) = \hat{\sigma}^2/(S_3-S_1{}^2/n) \qquad \text{(eq. 2.6.32)}$$

For k measured responses, with $\bar{x}_0$ denoting the average value calculated from the calibration curve, the random and systematic error variances of the total, $\hat{x}_{ot}$, are

$$V_r(\hat{x}_{ot}) = k\ \hat{\sigma}^2/\hat{\beta}^2 \qquad \text{(eq. 2.6.33)}$$

$$V_s(\hat{x}_{ot}) = k^2[V(\hat{\gamma}) + (\bar{x}_0-S_1/n)^2V(\hat{\beta})]/\hat{\beta}^2 \qquad \text{(eq. 2.6.34)}$$

Basis

The basis is the same as for Method 2.5.

Examples

EXAMPLE 2.6 (a)

Example 2.5 (a) is reworked assuming that the intercept value is not known. From (eq. 2.6.28a)- (eq. 2.6.34),

$$\hat{\gamma} = 390.22 \quad , \quad \hat{\beta} = 1.3260 \quad , \quad \hat{\sigma}^2 = 11.2712$$

$$V(\hat{\gamma}) = 2.2542 \quad , \; V(\hat{\beta}) = 157.78 \times 10^{-6}$$

$$V_r(\hat{x}_{ot}) = 25.641 \; g^2Pu \quad , \quad V_s(\hat{x}_{ot}) = 1.282 \; g^2Pu$$

Note that for this particular set of $\hat{x}_{oj}$ values, the systematic error of the total, $\hat{x}_{ot}$, is much smaller for the unknown intercept than for the known intercept case. This is because (x_0-S_1/n) is very nearly zero, and is not a general result.

EXAMPLE 2.6 (b)

For the data of Example 2.2 (a), estimate the scale calibration curve assuming that the relationship is linear with unknown intercept. An observed weight on the scale is then 6616.4 grams. What is the weight corrected for bias? What are the random and systematic error variances for this corrected weight?

In the notation of Method 2.6, x_i = 8878.0 for the first 30 observations and x_i = 1591.7 for the last 20 observations. The quantities S_1-S_5 are calculated.

$$S_1 = (30)(8878.0)+(20)(1591.7) = 298{,}174$$

$$S_2 = (8878.0)(8881.3)(30) + (1591.7)(1589.9)(20) = 2{,}416{,}058{,}319$$

$$S_3 = (30)(8878.0)^2 + (20)(1591.7)^2 = 2{,}415{,}236{,}698$$

$$S_4 = (30)(8881.3) + (20)(1589.9) = 298{,}237$$

$$S_5 = (29)(6.2)^2 + (30)(8881.3)^2 + (19)(4.9)^2 + (20)(1589.9)^2$$

$$= 2{,}416{,}881{,}902$$

From (eq. 2.6.28a)-(eq. 2.6.32),

$$\hat{\gamma} = 5964.74$$

$$\hat{\beta} = 1.00070$$

$$\hat{\sigma}^2 = 32.7271$$

$$V(\hat{\gamma}) = 0.65454 \qquad V(\hat{\beta}) = 5.1370 \times 10^{-8}$$

At y_0 = 6616.4 g, the bias corrected weight is

$$\hat{x}_0 = (6616.4 - 5964.74)/1.00070 + 5963.48$$

$$= 6614.68 \text{ g}$$

From (eq. 2.6.33) and (eq. 2.6.34), with k = 1, the random and systematic errors for the bias corrected result are

$$V_r(\hat{x}_0) = 32.7271/(1.00070)^2 = 32.6813 \text{ g}^2$$

$$V_s(\hat{x}_0) = 0.6754 \text{ g}^2$$

Note: It was assumed for illustrative purposes in this rework of Example 2.2 (a) that the standards were known without error. This assumption is not valid, as was pointed out in Example 2.2 (a).

2.6.2.2 Linear Calibration; Non-Constant Variance

The calibration problem is identical to that discussed in Section 2.6.2.1 except that the variance in the response variable is not constant over the calibration range. This variance, denoted by σ_i^2 at the value x_i, is a known quantity.

As was the case in Section 2.6.2.1, two situations are covered. In Method 2.7, the intercept parameter is assumed to be known, while this parameter has an unknown value in Method 2.8.

Method 2.7

Notation

The notation is identical to that in Method 2.5 except that σ^2 is replaced by σ_i^2, a known quantity.

Results

Calculate the following quantities.

$$w_i = 1/\sigma_i^2 \text{ for each observation}$$

$$S_1 = \sum_{i=1}^{n} w_i x_i \qquad S_2 = \sum_{i=1}^{n} w_i x_i y_i \qquad S_3 = \sum_{i=1}^{n} w_i x_i^2$$

$$S_4 = \sum_{i=1}^{n} w_i y_i \qquad S_5 = \sum_{i=1}^{n} w_i y_i^2$$

The estimate of β is

$$\hat{\beta} = (S_2 - \alpha S_1)/S_3 \qquad \text{(eq. 2.6.35)}$$

Its variance is

$$V(\hat{\beta}) = 1/S_3 \qquad \text{(eq. 2.6.36)}$$

For $\hat{x}_0 = (y_0 - \alpha)/\hat{\beta}$, y_0 being a measured response for a production item, the random and systematic error variances are

$$V_r(\hat{x}_0) = \sigma_0^2/\hat{\beta}^2 \qquad \text{(eq. 2.6.37)}$$

and

$$V_s(\hat{x}_0) = \hat{x}_0^2 V(\hat{\beta})/\hat{\beta}^2 \qquad \text{(eq. 2.6.38)}$$

where σ_0^2 is the known variance of y_0.

For k measured responses, with $\hat{x}_{0t}$ being the total for the $\hat{x}_{01}, \hat{x}_{02}, \ldots, \hat{x}_{0k}$ values,

$$V_r(\hat{x}_{0t}) = \sum_{i=1}^{k} \sigma_{0i}^2/\hat{\beta}^2 \qquad \text{(eq. 2.6.39)}$$

$$V_s(\hat{x}_{0t}) = \hat{x}_{0t}^2\, V(\hat{\beta})/\hat{\beta}^2 \qquad \text{(eq. 2.6.40)}$$

Basis

The basis is the same as that for Method 2.5 except that a weighted least squares estimate of β is found (2.8). The quantity minimized is

$$Q = \sum_{i=1}^{n} (y_i - \alpha - \beta\, x_i)/\sigma_i^2$$

Examples

EXAMPLE 2.7 (a)

In calibrating a uranium solid waste barrel NDA system based on gamma counting, the model Y = βX is assumed, where Y is the net counts per 100 seconds and X is the grams U-235 in standard barrels. Given the following data for which the variance σ_i^2 is also known, estimate the calibration curve. Three production barrels are then counted, the count rates per 100 seconds being 20,192; 13,919; and 42,267 respectively. Estimate the total amount of U-235 in these barrels and find the random and systematic error variances for this estimate.

x_i	y_i	σ_i^2	w_i
1	2853	1.65×10^5	6.06×10^{-6}
4	11611	2.34×10^5	4.27×10^{-6}
7	18072	3.03×10^5	3.30×10^{-6}
10	27554	4.43×10^5	2.26×10^{-6}
15	38649	6.91×10^5	1.45×10^{-6}
20	53464	12.50×10^5	0.80×10^{-6}

The quantities S_1-S_5 are calculated

$$S_1 = 0.00010659 \qquad S_2 = 2.951828 \qquad S_3 = 0.00110833$$

$$S_4 = 0.28759004 \qquad S_5 = 7871.2519$$

For $\alpha = 0$, $\hat{\beta}$ is calculated from (eq. 2.6.35)

$$\hat{\beta} = 2663$$

Its variance is

$$V(\hat{\beta}) = 902.26$$

For the three production barrels,

$$\hat{x}_{ot} = (20192 + 13919 + 42267)/2663$$

$$= 28.68 \text{ g U-235}$$

By interpolation in the data table, (logarithmically on σ_i^2)

$$\sigma_{01}^2 = 3.30 \times 10^5 \qquad \sigma_{02}^2 = 2.66 \times 10^5 \qquad \sigma_{03}^2 = 7.99 \times 10^5$$

$$\sum_{i=1}^{k} \sigma_{0i}^2 = 13.95 \times 10^5$$

By (eq. 2.6.39) and (eq. 2.6.40),

$$V_r(\hat{x}_{ot}) = 13.95 \times 10^5/(2663)^2 = 0.1967$$

$$V_s(\hat{x}_{ot}) = 0.1047$$

Method 2.8

Notation

Same as for Method 2.7 except that the calibration curve is rewritten to be

$$y_i = \gamma + \beta\,(x_i - \bar{x}), \text{ where}$$

$$\bar{x} = \text{weighted average of the } n\ x_i \text{ values}$$

Results

Calculate w_i and the quantities S_1-S_5 as in Method 2.7. Also calculate S_6, the sum of the w_i's. The parameter estimates are:

$$\hat{\gamma} = S_4/S_6 \qquad \text{(eq. 2.6.41)}$$

$$\hat{\beta} = (S_2 - S_1S_4/S_6)/(S_3 - S_1^2/S_6) \qquad \text{(eq. 2.6.42)}$$

The variances are:

$$V(\hat{\gamma}) = {}^1/S_6 \qquad \text{(eq. 2.6.43)}$$

$$V(\hat{\beta}) = {}^1/(S_3 - S_1^2/S_6) \qquad \text{(eq. 2.6.44)}$$

For a measured response y_0, the corresponding value of x_0 is estimated by

$$\hat{x}_0 = [(\hat{y}_0 - \hat{\gamma})/\hat{\beta}] + \bar{x} \qquad \text{(eq. 2 6.44a)}$$

For k measured responses (k = 1 is a special case), with $\hat{x}_{ot}$ being the total amount and x_o being $\hat{x}_{ot}/k$,

$$V_r(\hat{x}_{ot}) = \sum_{i=1}^{k} \sigma_{oi}^2/\hat{\beta}^2 \qquad \text{(eq. 2.6.45)}$$

$$V_s(\hat{x}_{ot}) = k^2[V(\hat{\gamma}) + (\bar{x}_o - \bar{x})^2V(\hat{\beta})]/\hat{\beta}^2 \qquad \text{(eq. 2.6.46)}$$

where

$$\bar{x} = S_1/S_6 \qquad \text{(eq. 2.6.47)}$$

Basis

The basis is the same as for Method 2.7.

Examples

EXAMPLE 2.8 (a)

Rework Example 2.7 (a) given that the intercept is not zero, but is to be estimated.

The quantities S_1-S_5 were calculated in Example 2.7 (a) along with the w_i. Also,

$$S_6 = \sum_{i=1}^{k} w_i = 18.14 \times 10^{-6}$$

The parameter estimates are

$$\hat{\gamma} = 15854 \qquad \hat{\beta} = 2618$$

The variances are

$$V(\hat{\gamma}) = 55127 \qquad V(\hat{\beta}) = 2074.64$$

For the total, $\hat{x}_{ot}$,

$$V_r(\hat{x}_{ot}) = 13.95 \times 10^5/(2618)^2 = 0.2035$$

$$\hat{x}_{ot} = (76378 - 47562)/2618 + (3)(5.8760)$$

$$= 28.63 \text{ g U-235}$$

$$V_s(\hat{x}_{ot}) = 0.1090$$

2.6.2.3 Single Point Calibration

A problem commonly encountered in inspection may be called a single point calibration problem and is described as follows, in terms of a specific type of example. Suppose that the percent U-235 of items is to be measured by NDA, and further suppose that the expected range of percent U-235 values is quite narrow. Assuming that no physical calibration standards are available, common practice is to make NDA measurements on a small number of randomly selected items. These items are then sampled with the samples measured by destructive analysis (e.g., mass spectrometer). The calculated average result for these samples becomes the assigned standard value.

Mathematically, this problem reduces to a linear calibration with a zero intercept with the added feature that the uncertainty in the assigned standard value is taken into account. In the event the percent U-235 (or whatever other quantity is being measured) varies over a range such that a single point calibration is not desirable, then the Method 2.5 may be applied. For Method 2.5, it is assumed that the assigned standard values are known without error, which is not strictly valid. However, for the application under discussion the uncertainty in the assigned standard value based on the destructive analysis is quite small relative to that of the NDA measurement and can safely be ignored. (If there is concern on this point, one may use statistical methods that take into account errors in both variables.)

Method 2.9 treats the single point calibration problem.

Method 2.9

Notation

y_i = i-th NDA measurement on the standard

n = number of NDA measurements on the standard

$\bar{y}$ = average of the n measurements

s^2 = variance of the n measurements

μ_X = value assigned the standard, based on destructive analyses

σ_0^2 = variance of μ_X

Results

The calibration equation is written

$$Y = \beta X$$

The parameter, β, is estimated by

$$\hat{\beta} = \bar{y}/\mu_X \qquad \text{(eq. 2.6.48)}$$

At a given observed value, y_0, for a production item, the corresponding x_0 value is

$$\hat{x}_0 = y_0/\hat{\beta} \qquad \text{(eq. 2.6.49)}$$

The random and systematic error variances of $\hat{x}_0$ are, respectively,

$$V_r(\hat{x}_0) = s^2/\hat{\beta}^2 \qquad \text{(eq. 2.6.50)}$$

$$V_s(\hat{x}_0) = (y_0{}^2/\hat{\beta}^2)(\sigma_0^2/\mu_X^2 + s^2/n\bar{y}^2) \qquad \text{(eq. 2.6.51)}$$

Basis

The basis for estimating the parameter β is the same as for Method 2.5. The formulas for $V_r(\hat{x}_0)$ and $V_s(\hat{x}_0)$ are based on methods for propagating errors to be covered in Chapter 3.

Examples

EXAMPLE 2.9 (a)

NDA measurements of percent U-235 are to be made on Zr/U billets. No physical standards are available. A randomly selected billet is measured 10 times by NDA and then sectioned to provide samples for destructive analyses. Some 14 determinations are made of percent U-235 on these samples, and 21 determinations of percent uranium. The response of interest is the percent U-235 in the billet.

The average of the 14 percent U-235 measurements is 92.878% with a standard deviation on this average of 0.0091% absolute. For the 21 determinations of percent uranium, the average is 1.6882% and the associated absolute standard deviation is 0.0010%. Thus,

$$\mu_X = (0.92878)\ (1.6882) = 1.5680\%$$

The variance, σ_0^2, is calculated by propagation of error techniques to be discussed in Chapter 3.

$$\sigma_0^2 = (0.92878)^2(.0010)^2 + (1.6882)^2(0.000091)^2$$

$$= 88.62 \times 10^{-8}$$

The ten measurements by NDA are, in counts per unit time,

9149	9243	9219
9212	9245	9203
8923	9186	
9203	9208	

These give:

$$n = 10 \qquad \bar{y} = 9179.1 \qquad s^2 = 8846.54$$

Suppose a production billet is then counted to give 9031 counts. The percent U-235 for that billet is then estimated by:

$$\hat{\beta} = 9179.1/1.5680 = 5854$$

$$\hat{x}_0 = 9031/5854 = 1.5427\%$$

$$V_r(\hat{x}_0) = 8846.54/(5854)^2 = 0.0002581$$

$$V_s(\hat{x}_0) = (9031/5854)^2\left(88.62 \times 10^{-8}/(1.5680)^2 + 8846.54/(10)(9179.1)^2\right)$$

$$= 0.00002585$$

2.6.2.4 <u>SAM-2 Calibration for Percent U-235</u>

Another problem somewhat unique to inspection activities involves the calibration of the NDA SAM-2 instrument for the measurement of percent U-235. The problem differs from calibration problems discussed previously because there are two measured responses, one corresponding to a background correction. The fact that the background correction is now not a simple subtraction as was true for other NDA applications already treated makes the problem more complicated, involving the estimation of another parameter. Method 2.10 indicates how this problem may be treated.

<u>Method 2.10</u>

<u>Notation</u>

x_i = percent U-235 for standard, i-th measurement

y_{1i} = net count rate, source plus background, for i-th measurement

y_{2i} = net background count rate for i-th measurement

n = number of standard measurements

$Y_1-\beta_1 Y_2 = \beta_2 X$ (calibration curve)

Results

Calculate the following quantities. All summations run from 1 to n.

$$S_1 = \sum x_i\, y_{1i} \qquad S_2 = \sum x_i\, y_{2i} \qquad S_3 = \sum y_{1i}\, y_{2i}$$

$$S_4 = \sum y_{1i}^2 \qquad S_5 = \sum y_{2i}^2 \qquad S_0 = \sum x_i^2$$

The parameters β_1 and β_2 are estimated by:

$$\hat{\beta}_1 = (S_0S_3-S_1S_2)/S \qquad \text{(eq. 2.6.52)}$$

$$\hat{\beta}_2 = (S_1S_5-S_2S_3)/S \qquad \text{(eq. 2.6.53)}$$

where $$S = S_0S_5-S_2^2 \qquad \text{(eq. 2.6.54)}$$

The variances of $\hat{\beta}_1$ and of $\hat{\beta}_2$ and the covariance between them are given by

$$V(\hat{\beta}_1) = S_0\sigma_1{}^2/S + (S_0S_4-S_1^2)S_0\sigma_2^2/S^2 \qquad \text{(eq. 2.6.55)}$$

$$V(\hat{\beta}_2) = S_5\sigma_1{}^2/S + (S_0S_3^2 + S_2^2S_4-2S_1S_2S_3)\sigma_2^2/S^2 \qquad \text{(eq. 2.6.56)}$$

$$CV(\hat{\beta}_1,\hat{\beta}_2) = S_2V(\hat{\beta}_1)/S_0 \qquad \text{(eq. 2.6.56a)}$$

In these equation, $\sigma_1{}^2$ and $\sigma_2{}^2$ are the variances of y_{1i} and y_{2i} respectively and are estimated from replicated data as the sample variances (see example 2.10 a).

For a production item with measured responses y_{10} and y_{20}, the percent U-235 is $\hat{x}_0$. Its random and systematic error variances are respectively

$$V_r(\hat{x}_0) = \sigma_1{}^2/\beta_2{}^2 + (\hat{\beta}_1/\hat{\beta}_2)^2\, \sigma_2^2 \qquad \text{(eq. 2.6.56b)}$$

$$V_s(\hat{x}_0) = \left(y_{20}^2\, V(\hat{\beta}_1) + \hat{x}_0^2\, V(\hat{\beta}_2) + 2\, \hat{x}_0 y_{20} CV(\hat{\beta}_1,\hat{\beta}_2)\right)/\hat{\beta}_2^2 \qquad \text{(eq. 2.6.56c)}$$

where $\sigma_1{}^2$ and $\sigma_2{}^2$ are again replaced by their estimates.

Basis

The parameters β_1 and β_2 are estimated by assigning them values that minimize

$$Q = \sum_{i=1}^{n} (y_{1i} - \beta_1 y_{2i} - \beta_2 x_i)^2$$

Taking the partial derivatives of Q with respect to β_1 and β_2, and equating to 0, one obtains

$$S_3-\beta_1 S_5-\beta_2 S_2= 0$$

$$S_1-\beta_1 S_2-\beta_2 S_0= 0$$

from which the estimating equation (eq. 2.6.52) - (eq. 2.6.54) easily follow.

To derive the expressions for $V(\hat\beta_1)$, $V(\hat\beta_2)$, $CV(\hat\beta_1,\hat\beta_2)$, $V_r(\hat x_0)$, and $V_s(\hat x_0)$, one must apply Method 3.12 to follow. The details of the derivation are not included here. They are due to Neuilly (2.29).

Examples

EXAMPLE 2.10 (a)

Calibration data for a SAM-2 instrument are tabled. An inspected item produces the count rates: y_{10} = 90,000 and y_{20}= 49,200. Estimate the percent U-235 for that item and find the random and systematic error variances of the estimate.

x = % U-235	y_1(CPM)	y_2(CPM)
2.55	97950	49049
2.55	98008	49868
2.55	98130	49475
2.55	98218	49979
0.72	62268	46671
0.72	62569	46874
0.72	62345	47189
0.72	62802	47208

The quantities S_0-S_5 are calculated

S_0 = 28.0836 S_3 = 31,201,379,410

S_1 = 1,180,368.78 S_4 = 54,099,216,790

S_2 = 641,164.29 S_5 = 18,669,048,990

Then, by (eq. 2.6.54),

$$S = 1.1320246 \times 10^{11}$$

so the estimates are

$\hat\beta_1$ = 1.0551 $\hat\beta_2$ = 17942.7

Therefore, x_0 = 90,000/17942.7 - $\left[(1.0551)/(17942.7)\right]$49,200

= 2.123 % U-235

Next, $\sigma_1{}^2$ and $\sigma_2{}^2$ are estimated by

$$\hat{\sigma}_1{}^2 = \frac{1}{6}\left[\sum_{i=1}^{4}(y_{1i} - \bar{y}_1)^2 + \sum_{i=5}^{8}(y_{1i} - \bar{y}_2)^2\right] = 36,221$$

$$\hat{\sigma}_2{}^2 = 122,790$$

Assuming that $\sigma_1{}^2 = \sigma_2{}^2$, the estimate is the average of 36,221 and 122,790, or 79,506. This value is used in place of $\sigma_1{}^2$ and $\sigma_2{}^2$ in (eq. 2.6.55) and (eq. 2.6.56). The results are:

$$V(\hat{\beta}_1) = 4.168 \times 10^{-5} \qquad V(\hat{\beta}_2) = 27732$$

$$CV(\hat{\beta}_1,\hat{\beta}_2) = -0.9516$$

The random error variance of $\hat{x}_0$ is, from (eq. 2.6.56b),

$$V_r(\hat{x}_0) = 79,506\ (3.10616 + 3.45789) \times 10^{-9}$$

$$= 5.29 \times 10^{-4}\ ; \sqrt{V_r(\hat{x}_0)} = 0.0228$$

The systematic error variance is found from (eq. 2.6.56c)

$$V_s(\hat{x}_0) = (3.1339 + 3.8824 - 6.1748) \times 10^{-4}$$

$$= 0.845 \times 10^{-4}; \sqrt{V_s(x_0)} = 0.0092$$

The overall standard deviation of $\hat{x}_0$ is 0.0246% U-235. Since $\hat{x}_0$ = 2.123% U-235, this corresponds to a relative error standard deviation of 1.16%.

2.6.2.5 Several Calibration Data Sets; Linear Model

Measurement systems are recalibrated on a routine schedule because of difficulties in maintaining a stable measurement system. This is especially true for NDA measurement systems.

Assume that the calibration curve to be applied is the one based on the most recent set of data. Further assume that the calibration may shift from one time frame to another, the degree of shift being that described by prior calibrations. Then, the current set of calibration curve parameters may be regarded as being randomly selected from a population of parameters, just as prior sets of estimated parameters were also selected.

The linear model with unknown intercept and constant variance is assumed.

Method 2.11

Notation

Same as for Method 2.5. It is convenient to use this notation rather than the Method 2.6 notation because each data set may not have the same value for $\bar{x}$.

Results

For the most recent set of data, estimate the parameters of the calibration curve by following Method 2.6. Recalculate $\hat{\alpha}$ from the equation

$$\hat{\alpha} = \hat{\gamma} - \hat{\beta}\,\bar{x} \qquad \text{(eq. 2.6.57)}$$

List the corresponding paired $\hat{\alpha}$, $\hat{\beta}$ values from the previous calibrations. Let the total number of calibrations, including the most recent one, be m. Given the sets $(\hat{\alpha}_i, \hat{\beta}_i)$ for i=1, 2, ..., m, compute the sample variances among the $\hat{\alpha}_i$ values, and the $\hat{\beta}_i$ values, and the sample covariance between the $\hat{\alpha}_i$ and $\hat{\beta}_i$ values. Denote these quantities by

$$s^2_{\hat{\alpha}}\,, \quad s^2_{\hat{\beta}}\,, \quad s_{\hat{\alpha},\hat{\beta}} \qquad \text{respectively}$$

For k measured responses (k = 1 is a special case), with $\hat{x}_{ot}$ being the total amount and $\bar{x}_o$ being $\hat{x}_{ot}/k$, and with $\bar{y}_o$ being similarly defined, the random and systematic error variances are respectively

$$V_r(\hat{x}_{ot}) = k\hat{\sigma}^2/\hat{\beta}^2 \qquad \text{(eq. 2.6.58)}$$

$$V_s(\hat{x}_{ot}) = k^2\bar{x}_o^{\,2} \left\{ \frac{s^2_{\hat{\alpha}}}{(\bar{y}_o-\hat{\alpha})^2} + \frac{s^2_{\hat{\beta}}}{\hat{\beta}^2} + \frac{2s_{\hat{\alpha},\hat{\beta}}}{\hat{\beta}(\bar{y}_o-\hat{\alpha})} \right\} \qquad \text{(eq. 2.6.59)}$$

In (eq. 2.6.58), $\hat{\sigma}^2$ is calculated from (eq. 2.6.30) using the most recent data set.

Basis

The conservative assumption is made that the most recent calibration curve holds steady throughout the time frame in question. With this assumption, and assuming that the true calibration curve lies within the envelope defined by other curves for the same measurement system, the error variance describing this curve to curve variation becomes a systematic error variance. Under a less conservative assumption, one could use an average calibration curve in some sense, using the argument that the true calibration curve is fixed rather than variable, and the purpose of recalibrating is to obtain a better and better estimate of this average calibration. The true state of nature probably falls between these two extremes, but the conservative position is advocated.

As a final note, although Method 2.11 is based on the constant variance case, it may also be applied with little concern to the non-constant variance case. This is because the curve to curve variability is probably of dominant importance, and the random error for a given curve is of little consequence. In (eq. 2.6.58), simply replace $k\hat{\sigma}^2$ as calculated from (eq. 2.6.30) by

$$\sum_{i=1}^{k} \sigma_{0i}^2 \text{ , using (eq. 2.6.39)}$$

Examples

EXAMPLE 2.11 (a)

Suppose the data of Example 2.7 (a) represent the most recent set of calibration data. Further suppose that prior calibrations resulted in the following parameter estimates:

$\hat{\alpha}$	$\hat{\beta}$
355	2414
730	2564
2831	2484

From Example 2.8 (a), which uses the data of Example 2.7 (a), the parameter estimates are

$$\hat{\beta} = 2618$$

$$\hat{\gamma} = 15854$$

$$\bar{x} = 1.0659 \times 10^{-4}/18.14 \times 10^{-6} = 5.876$$

$$\hat{\alpha} = 15854 - (2618)(5.876) = 471 \quad \text{, from (eq. 2.6.57)}$$

The quantities $s_{\hat{\alpha}}^2$, $s_{\hat{\beta}}^2$, and $s_{\hat{\alpha},\hat{\beta}}$ are calculated, using all four calibration curve parameter estimates.

$$s_{\hat{\alpha}}^2 = 1{,}361{,}295 \qquad s_{\hat{\beta}}^2 = 8024 \qquad s_{\hat{\alpha},\hat{\beta}} = -20{,}423$$

For the three production barrels of Examples 2.7 (a) and 2.8 (a), the reported amount of U-235 was 28.63g, as calculated in the latter example.

From (eq. 2.6.58), with $k\hat{\sigma}^2$ replaced by

$$\sum_{i=1}^{k} \sigma_{0i}^2 = 13.95 \times 10^5 \quad ,$$

$V_r(\hat{x}_{ot}) = 0.2035$, as in Example 2.8 (a). From (eq. 3.6.59),

$$V_s(\hat{x}_{ot}) = (28.63)^2 \left\{ \frac{1,361,295}{(25459 - 471)^2} + \frac{8024}{(2618)^2} - \frac{40846}{(2618)\ (24988)} \right\}$$

$$= 2.2349\ g^2\ \text{U-235}$$

Note that this is much larger than as previously calculated because the curve to curve variation is now taken into account.

2.6.2.6 Linear Calibration; Cumulative Model

The so-called linear calibration with cumulative model is encountered in the calibration of process vessels. The vessel in question must be a straight wall tank with a minimum amount of internal piping in order for the linear model to apply. For more complex situations, a statistical expert should be consulted.

In the calibration process, the response y is the liquid level measured by some technique. Observations of the liquid level are made at values of the measured volume, x. The variable x denotes the sum of measured increments, x_i. The calibration equation is of the form

$$y = \alpha' + \beta' x$$

In general terms, the error structure may be written symbolically as

$$y = \alpha' + \beta'\Sigma(x + e_x) + e_y \qquad \text{(eq. 2.6.60)}$$

where e_x represents the error in the measured increment x and e_y is the error in determining the liquid level y. The statistical procedure to use depends upon the relative sizes of e_x and e_y.

If there is little error in determining the weights or volumes of the liquid increments relative to the error in determining the liquid level, then the statistical procedure is that covered in 2.6.2.1. If the reverse is true, then the cumulative model applies. This terminology is properly descriptive because the error in a given x is the cumulative sum of errors in the increments of volume comprising x.

For the cumulative model, it is permissible to write the relationship in the form

$$x = \alpha + \beta y \qquad \text{(eq. 2.6.61)}$$

and obtain estimates of α and β directly. Recall that in 2.6.2.1, it was first necessary to estimate the parameters of the equation in which y was expressed as a function of x and then use the equation in its inverse form.

For the cumulative model discussed in Method 2.12, it will be noted that only the initial and final points are used in estimating the calibration parameters. The intermediate points are used to verify that the relationship is in fact linear and to provide an estimate of the variance of a measured increment.

Method 2.12

Notation

(x_1, y_1) = initial point

(x_n, y_n) = final point

y_o = measured response (after calibration)

x_o = estimated liquid level corresponding to y_o

α, β are defined by (eq. 2.6.61)

Results

The estimates of β and α are

$$\hat{\beta} = (x_n - x_1)/(y_n - y_1) \qquad \text{(eq. 2.6.62)}$$

$$\hat{\alpha} = x_1 - \hat{\beta}\, y_1 \qquad \text{(eq. 2.6.63)}$$

To calculate the variance of $\hat{\beta}$ and of $\hat{\alpha}$, and the systematic error variance of x_o, first compute

$$S = \sum_{i=2}^{n} (x_i - x_{i-1})^2/(y_i - y_{i-1}) \qquad \text{(eq. 2.6.64)}$$

Then,

$$V(\hat{\beta}) = \frac{(y_n - y_1)S - (x_n - x_1)^2}{(n-1)\ (y_n - y_1)^2} \qquad \text{(eq. 2.6.65)}$$

$$V(\hat{\alpha}) = y_1 y_n V(\hat{\beta}) \qquad \text{(eq. 2.6.66)}$$

$$V_s(x_o) = (y_1 y_n + y_o^2 - 2y_o y_1)V(\hat{\beta}) \qquad \text{(eq. 2.6.67)}$$

The random error variance of x_o is zero since y_o is assumed to be measured without error in the cumulative model.

In tank calibration applications, one is often interested in transfer amounts as determined by noting the difference between two measured responses, $(y_{o2} - y_{o1})$. The transfer amount, $\hat{\beta}(y_{o2} - y_{o1})$ has a systematic error variance given by

$$V_s(x_{o2} - x_{o1}) = (y_{o2} - y_{o1})^2\ V(\hat{\beta}) \qquad \text{(eq. 2.6.68)}$$

For the total of m transfers,

$$V_s\left[\sum_{i=1}^{m}(x_{02i}-x_{01i})\right]=\left[\sum_{i=1}^{m}(y_{02i}-y_{01i})\right]^2 V(\hat{\beta}) \qquad \text{(eq. 2.6.69)}$$

Basis

As is seen from (eq. 2.6.60), with e_y assumed to be zero for the cumulative model, the random errors for any two observations are not independent. Thus, it is inappropriate to use the methodology of 2.6.2.1 or 2.6.2.2 which require independence of observations.

As Mandel points out [2.10], one can remove this dependence by working with successive differences, or incremental additions of liquid in this application. The resulting estimates given in Method 2.12 are then weighted least squares estimates. Equivalently, one can use the original data and apply Aitkin's method of generalized least squares to obtain the estimates [2.11], as was demonstrated by Jaech [2.12].

Examples

EXAMPLE 2.12 (a)

Tank calibration data are given as follows:

y (in)	x (lb)	y	x	y	x
33.48	5025.25	64.48	11036.05	95.37	17042.26
41.20	6525.75	72.20	12537.42	99.28	17795.36
48.95	8027.40	79.93	14038.39	104.47	18797.41
56.72	9534.40	87.62	15540.81	109.64	19798.46

Assuming that the cumulative model applies, Method 2.12 is applied. From (eq. 2.6.62) and (eq. 2.6.63),

$$\hat{\beta} = 14773.21/76.16 = 193.976$$

$$\hat{\alpha} = 5025.25 - (193.976)(33.48) = -1469.07$$

From (eq. 2.6.64),

$$S = (1500.50)^2/7.72 + (1501.65)^2/7.75 + \ldots + (1001.05)^2/5.17$$

$$= 2{,}865{,}680.853$$

The variances of the calibration parameter estimates are given by (eq. 2.6.65) and (eq. 2.6.66).

$$V(\hat{\beta}) = \frac{(76.16)(2,865,680.853) - (14773.21)^2}{(11)(76.16)^2} = 0.0395$$

$$V(\hat{\alpha}) = (33.48)(109.64)(0.0395) = 144.99$$

Suppose that an observed manometer reading is 98.74 inches. The estimated weight of the liquid in the tank is

$$x_0 = -1469.07 + (193.976)(98.74) = 17684.12 \text{ lbs.}$$

From (eq. 2.6.67), its variance is

$$V_s(x_0) = \Big[(33.48)(109.64) + (98.74)^2 - 2(98.74)(33.48)\Big] (0.0395)$$
$$= 268.94 \text{ lbs}^2$$

Suppose that 5 transfers are made as follows:

	Manometer Reading		
Transfer	Before	After	Difference
1	101.62	34.91	66.71
2	104.29	36.18	68.11
3	97.01	33.99	63.02
4	88.15	35.07	53.08
5	100.72	37.66	63.06
		Total	313.98

The estimated total weight of the transferred liquid is

$$(313.98)(193.976) = 60904.58 \text{ lbs.}$$

Its variance is given by (eq. 2.6.69)

$$V_s \text{ (transferred amount)} = (313.98)^2(0.0395) = 3894.06 \text{ lbs}^2$$

2.6.2.7 Several Calibration Data Sets; Cumulative Model

The discussion of 2.6.2.5 is applicable to this section also, the difference being that now the underlying model for any given calibration curve is the cumulative model rather than the independent model.

Method 2.13

Notation

Same as for Method 2.12.

Results

For the most recent set of data, estimate the parameters of the calibration curve by following Method 2.12. From the previous calibrations and including the most recent one, list the $\hat{\alpha}$ and $\hat{\beta}$ values. Let the total number of calibrations be m. Given the sets $(\hat{\alpha}_i, \hat{\beta}_i)$ for i = 1, 2, ..., m, compute the sample variances among the $\hat{\alpha}_i$ and the $\hat{\beta}_i$ values, and the sample covariance between them. Denote these quantities respectively by

$$s^2_{\hat{\alpha}},\ s^2_{\hat{\beta}},\ s_{\hat{\alpha},\hat{\beta}}$$

Then, at an observed y_0, the variance of the corresponding x_0 (which is a systematic error variance, the random error variance being zero) is

$$V_s(x_0) = s^2_{\hat{\alpha}} + y_0^2 s^2_{\hat{\beta}} + 2y_0\, s_{\hat{\alpha},\hat{\beta}} \qquad \text{(eq. 2.6.70)}$$

For a transfer volume,

$$V_s(x_{02}-x_{01}) = (y_{02}-y_{01})^2 s^2_{\hat{\beta}} \qquad \text{(eq. 2.6.71)}$$

For the total of m transfers,

$$V_s\left[\sum_{i=1}^{m} (x_{02i} - x_{01i})\right] = \left[\sum_{i=1}^{m} (y_{02i} - y_{01i})\right]^2 s^2_{\hat{\beta}} \qquad \text{(eq. 2.6.72)}$$

Basis

The basis discussion for Method 2.11 is also applicable here.

Examples

EXAMPLE 2.13 (a)

Suppose that the data of Example 2.12 (a) represent the most recent set of calibration data. Further suppose that there were two prior calibrations with the following parameter estimates:

$\hat{\alpha}$	$\hat{\beta}$
-1408.77	192.792
-1505.74	194.250

Then,

$$s^2_{\hat{\alpha}} = 2397.33$$

$$s^2_{\hat{\beta}} = 0.60045$$

$$s_{\hat{\alpha},\hat{\beta}} = -37.1376$$

Suppose that $y_0 = 98.74$. The corresponding x_0 value is 17684.12 lbs., from Example 2.12 (a). Its systematic error variance is given by (eq. 2.6.70)

$$V_s(x_0) = 2397.33 + (98.74)^2(0.60045) + (197.48)(-37.1376)$$

$$= 917.54 \text{ lbs}^2$$

2.6.2.8 Nonlinear Calibration

Thus far in the discussion of calibration equations, it has been assumed that the calibration curve is linear. In some applications, this assumption may not be valid. In NDA applications, for example, depending on the range of the calibration, the curve may depart from linearity.

Assume that, at worst, a nonlinear calibration curve may be represented by a quadratic model. The adequacy of this assumption has been demonstrated in a number of NDA applications. Further assume a constant variance, zero intercept model. For applications that do not satisfy these assumptions, qualified statistical advice should be sought. (An examination of the residuals is helpful in checking on the validity of the assumptions.)

Method 2.14

Notation

(y_i, x_i) = i-th data point; i=1, 2, ..., n

α, β = parameters of the model

$$y_i = \alpha x_i + \beta x_i^2$$

σ^2 = variance of y_i at given x_i

Results

Calculate the following quantities (all summations run from 1 to n)

$S_1 = \Sigma x_i^2$ $\qquad S_2 = \Sigma x_i^3$ $\qquad S_3 = \Sigma x_i^4$

$S_4 = \Sigma x_i y_i$ $\qquad S_5 = \Sigma x_i^2 y_i$ $\qquad S_6 = S_1 S_3 - S_2^2$

Then, the parameter estimates are:

$$\hat{\alpha} = (S_3S_4 - S_2S_5)/S_6 \qquad \text{(eq. 2.6.73)}$$

$$\hat{\beta} = (S_1S_5 - S_2S_4)/S_6 \qquad \text{(eq. 2.6.74)}$$

$$(n-2)\hat{\sigma}^2 = \sum_{i=1}^{n} (y_i - \hat{\alpha}x_i - \hat{\beta}x_i^2)^2 \qquad \text{(eq. 2.6.75)}$$

For a production item, the measured response is y_0. The corresponding x value is the solution of the equation

$$\hat{\beta}x_0^2 + \hat{\alpha}x_0 - y_0 = 0$$

The solution is

$$x_0 = -\hat{\alpha}\ (1- \sqrt{1+4\hat{\beta}y_0/\hat{\alpha}^2})/2\hat{\beta} \qquad \text{(eq. 2.6.76)}$$

The estimated variances of the estimated parameters, and the estimated covariance between them are:

$$V(\hat{\beta}) = S_1\hat{\sigma}^2/S_6 \qquad \text{(eq. 2.6.77)}$$

$$V(\hat{\alpha}) = S_3\hat{\sigma}^2/S_6 \qquad \text{(eq. 2.6.78)}$$

$$CV(\hat{\alpha},\hat{\beta}) = -S_2\hat{\sigma}^2/S_6 \qquad \text{(eq. 2.6.79)}$$

The systematic error variance of x_0 is denoted by $V_s(x_0)$ and is given by

$$V_s(x_0) = x_0^2 \left[V(\hat{\alpha}) + x_0^2\ V(\hat{\beta}) + 2x_0CV(\hat{\alpha},\hat{\beta})\right]/R^2 \qquad \text{(eq. 2.6.80)}$$

where

$$R = \hat{\alpha} + 2\hat{\beta}x_0 \qquad \text{(eq. 2.6.81)}$$

The random error variance of x_0 is

$$V_r(x_0) = \hat{\sigma}^2/R^2 \qquad \text{(eq. 2.6.82)}$$

Consider k measured responses: $y_{01}, y_{02}, \ldots, y_{0k}$ and the corresponding x values: $x_{01}, x_{02}, \ldots, x_{0k}$. Letting x_{0t} be the sum of these x_{0j} values, consider the random and systematic error variances of x_{0t}. To calculate these quantities, first compute

$$S_7 = \sum_{i=1}^{k} x_{0i}/R_i \qquad S_8 = \sum_{i=1}^{k} x_{0i}{}^2/R_i \qquad S_9 = \sum_{i=1}^{k} 1/R_i{}^2$$

with R_i defined in (eq. 2.6.81) for x_{0i}. Then,

$$V_s(x_{0t}) = S_7^2\ V(\hat{\alpha}) + S_8^2\ V(\hat{\beta}) + 2S_7S_8CV(\hat{\alpha},\hat{\beta}) \qquad \text{(eq. 2.6.83)}$$

$$V_r(x_{0t}) = S_9\hat{\sigma}^2 \qquad \text{(eq. 2.6.84)}$$

Basis

The parameter estimates are least squares estimates, i.e., are found by minimizing

$$Q = \sum_{i=1}^{n} (y_i - \alpha x_i - \beta x_i^2)^2$$

The quantity σ^2 is estimated by replacing α and β in this expression by their estimates and dividing Q by (n-2), the degrees of freedom.

The expressions for the variances of the quantities of interest are based on error propagation methods to be discussed in Chapter 3.

Examples

EXAMPLE 2.14 (a)

An NDA instrument is calibrated for use in measuring the amount of U-235 in containers of solid waste. Calibration data are as follows:

x_i = g U-235	y_i = net count rate/100 sec.
1.13	2629
4.52	11455
8.03	19512
11.03	27365
16.03	38701
21.61	50136
27.33	62111
32.88	71647

The quantities are calculated.

S_1 = 2759.8254 S_2 = 72,124.22288 S_3 = 2,030,156.602

S_4 = 6,270,327.66 S_5 = 162,032,220.2 S_6 = 400,974,230.0

By (eq. 2.6.73) and (eq. 2.6.74),

$\hat{\alpha}$ = 2602 $\hat{\beta}$ = -12.624

For x_i = 1.13, 4.52, ..., 32.88, the quantities $P_i = \hat{\alpha}x_i + \hat{\beta}x_i^2$ are calculated. These are used in computing $\hat{\sigma}^2$ from (eq. 2.6.75)

x_i	$P_i = 2602\, x_i - 12.624 x_i^2$	$y_i - P_i$
1.13	2924	-295
4.52	11503	-48
8.03	20080	-568
11.03	27164	201
16.03	38466	235
21.61	50334	-198
27.33	61683	428
32.88	71906	-259

$\hat{\sigma}^2 = \Sigma(y_i - P_i)^2/6 = 132{,}841$

The quantities $V(\hat{\beta})$, $V(\hat{\alpha})$, and $CV(\hat{\alpha},\hat{\beta})$ are calculated from (eq. 2.6.77), (eq. 2.6.78), and (eq. 2.6.79).

$$V(\hat{\beta}) = 0.91432$$

$$V(\hat{\alpha}) = 672.58$$

$$CV(\hat{\alpha},\hat{\beta}) = -23.894$$

A production item is now counted, the net count rate being 32,145 counts/100 sec. From (eq. 2.6.76), the corresponding amount of U-235 for that item is

$$x_o = -2602\ (1-\sqrt{1-0.23975})/(-25.248)$$

$$= 13.20 \text{ grams}$$

Its random and systematic error variances are calculated. First, from (eq. 2.6.81),

$$R = 2269$$

Then, from (eq. 2.6.80) and (eq. 2.6.82),

$$V_s(x_o) = (13.20)^2\ (672.58 + 159.31 - 630.80)/(2269)^2$$

$$= 0.00681$$

$$V_r(x_o) = 132{,}841/(2269)^2 = 0.02580$$

Three additional production items are now counted, the estimated amounts of U-235 being 4.11, 7.69, and 19.83 g respectively. The total U-235 for the four items is x_{ot} = 44.83 g. To find the systematic and random error variances for x_{ot}, apply (eq. 2.6.83) and (eq. 2.6.84). First compute S_7, S_8, and S_9.

i	x_i	R_i	x_i/R_i	x_i^2/R_i	$1/R_i^2$
1	13.20	2269	0.00582	0.07679	1.9424×10^{-7}
2	4.11	2498	0.00165	0.00676	1.6026×10^{-7}
3	7.69	2408	0.00319	0.02456	1.7246×10^{-7}
4	19.83	2101	0.00944	0.18716	2.2654×10^{-7}
			$S_7 = 0.02010$	$S_8 = 0.28852$	$S_9 = 7.5350 \times 10^{-7}$

$$V_s(x_{ot}) = 0.27173 + 0.07611 - 0.27713 = 0.07071$$

$$V_r(x_{ot}) = 0.10010$$

2.6.2.9 Nonlinear Calibration; Several Calibration Data Sets

The decision of 2.6.2.5 is applicable to this section also, the difference being that now the underlying model for any given calibration curve is the quadratic model with zero intercept, discussed in 2.6.2.8.

Method 2.15

Notation

Same as for Method 2.14.

Results

For the most recent set of data, estimate the parameters of the calibration curve by following Method 2.14. From the previous calibrations, and including the most recent one, list the $\hat{\alpha}$ and $\hat{\beta}$ values. Compute the sample variances of the m $\hat{\alpha}$'s and $\hat{\beta}$'s, and the sample covariance between them, calling the results $s^2_{\hat{\alpha}}$, $s^2_{\hat{\beta}}$, and $s_{\hat{\alpha},\hat{\beta}}$ respectively. Then, at an observed y_0, the systematic error variance of the corresponding x_0 is given by (eq. 2.6.80) with:

$$V(\hat{\alpha}) \text{ replaced by } s^2_{\hat{\alpha}}$$

$$V(\hat{\beta}) \text{ replaced by } s^2_{\hat{\beta}}$$

$$CV(\hat{\alpha},\hat{\beta}) \text{ replaced by } s_{\hat{\alpha},\hat{\beta}}$$

The random error variance of x_0 is given by (eq. 2.6.82), using the value for $\hat{\sigma}^2$ from the most recent calibration.

For k measured responses, x_{0t} has systematic error variance given by (eq. 2.6.83), making the same replacements for $V(\hat{\alpha})$, $V(\hat{\beta})$, and $CV(\hat{\alpha},\hat{\beta})$ as just indicated. For the random error variance of x_{0t}, use (eq. 2.6.84) with the value of $\hat{\sigma}^2$ based on the most recent calibration.

Basis

The basis discussion for Method 2.11 is also applicable here.

Examples

EXAMPLE 2.15 (a)

Suppose that the data of Example 2.14 (a) represent the most recent set of calibration data. Further suppose that there were four prior calibrations with the following parameter estimates:

$\hat{\alpha}$	$\hat{\beta}$
2944	-22.761
2656	-12.648
3214	-34.792
2573	-11.219

Then, based on the 5 sets of values:

$$s_{\hat{\alpha}} = 7\ 894$$

$$s_{\hat{\beta}} = 101.224$$

$$s_{\hat{\alpha},\hat{\beta}} = -2755.20$$

From Example 2.14 (a), at y_o = 32,145 counts/100 sec., x_o = 13.20 g and R = 2269. Thus, using the data from all five calibrations,

$$V_s(x_o) = (13.20)^2(75894 + 17637 - 72737)/(2269)^2$$

$$= 0.70375\ g^2$$

Again using the data of Example 2.14 (a), for the four production items, x_{ot} = 44.83 g, and

$$V_s(x_{ot}) = 30.6619 + 8.4263 - 31.9562 = 7.1320\ g^2$$

2.6.3 Measurements of Non-Standard Materials

Thus far, estimation of measurement error parameters has been based on data resulting from the measurement of physical standards. Techniques have been given for estimating error variances based on such data, and for giving guidance with respect to the need for bias correcting the data.

Although, as was repeatedly shown, analyses of standards data provide estimates of random error variances as well as systematic, the emphasis with such data is on the systematic errors. Random error variances often tend to be underestimated for physical standards for a number of reasons; it is difficult to include in the preparation of standards all the factors that might affect measurement repeatability. Usually, it is better to base estimation of random error variances on data that result from repeat measurements of production items. Techniques for doing this are covered in this section, with each technique discussed keyed to a given situation.

It is also noted that if one is willing to make certain assumptions about data structure, then systematic error variances may also be estimated from measurements made on production items. Instances in which this may be done, and techniques for doing so, are indicated as the various situations are covered.

The first topic in this section deals with the analysis of variance as applied to replicate measurement data.

2.6.3.1 Replicate Measurements; Analysis of Variance

Replicate or repeat measurements are made on the same or similar production items. The scatter in the repeat measurements is used to estimate the random error variance using a statistical technique known as the one way analysis of variance.

It is important to keep in mind just which measurement error parameter is being estimated in a given case. Depending on how the measurements are performed, for example, the random error variance being estimated may be the sum of sampling and analytical error variances, say, or it may be just the analytical error variance. This will be pointed out in the examples.

In Method 2.16, the one way analysis of variance as applied to this problem situation is discussed.

Method 2.16

Notation

x_{ij} = observed value for j-th measurement on production item i

n_i = number of measurements made on production item i

n = total number of measurements (sum of n_i values)

m = number of production items measured

σ^2 = random error variance

Results

Following Method 2.4, σ^2 is estimated by (eq. 2.6.16). That is, $\hat{\sigma}^2$ is the same calculated quantity as was $\hat{\sigma}_\varepsilon^2$ in Method 2.4.

Basis

The basis is the same as for Method 2.4.

Examples

EXAMPLE 2.16 (a)

Five cans of UO_2 powder are weighed at random and at routine intervals on a given scale. Assume that the actual contents do not change in weight, and estimate σ^2 from the following data (weights in kg).

Can 1	Can 2	Can 3	Can 4	Can 5
22.038	22.616	21.418	19.811	24.095
22.041	22.615	21.425	19.825	24.120
22.033	22.617	21.414	19.808	24.096
22.048	22.608		19.802	24.105
	22.603		19.795	24.105
	22.610		19.799	24.118
				24.113

Here, $n_1 = 4$ $n_2 = 6$ $n_3 = 3$ $n_4 = 6$ $n_5 = 7$ $n = 26$ $m = 5$

$T_1 = 88.160$ $T_2 = 135.669$ $T_3 = 64.257$

$T_4 = 118.840$ $T_5 = 168.752$

$$S_1 = (88.160)^2/4 + \ldots + (168.752)^2/7$$

$$= 12809.04773$$

$$S_2 = (22.038)^2 + (22.041)^2 + \ldots + (24.113)^2$$

$$= 12809.04923$$

Then,

$$\hat{\sigma}^2 = (12809.04923 - 12809.04773)/(26-5)$$

$$= 0.00007143 \text{ kg}^2$$

$$= 71.43 \text{ g}^2$$

2.6.3.2 Duplicate Measurements; Paired Differences

If the replicate measurements of 2.6.3.1 are, for all items, duplicate measurements (i.e., two measurements per item), then the analysis of the data is simplified. Of course, the one way analysis of variance technique of Method 2.16 may also be applied to duplicate data to give identical results, but there is no need to perform this more difficult analysis.

Method 2.17

Notation

x_i, y_i = the two measurement results for item i

k = number of items measured in duplicate

d_i = the difference, $x_i - y_i$

σ^2 = the random error variance for the measurement method in question

Results

There are two estimators of σ^2 that may be used. Denote these by $\hat{\sigma}^2$ and $\tilde{\sigma}^2$ respectively.

$$\hat{\sigma}^2 = \sum_{i=1}^{k} d_i^2/2k \qquad \text{(eq. 2.6.85)}$$

$$\tilde{\sigma}^2 = \sum_{i=1}^{k} \frac{d_i^2 - (\sum_{i=1}^{k} d_i)^2/k}{2(k-1)} \qquad \text{(eq. 2.6.86)}$$

The estimator $\hat{\sigma}^2$ is preferred if there is assurance that the x_i and y_i values are not relatively biased. The estimator $\tilde{\sigma}^2$ is preferred if there is some reason to believe that such a bias might exist, e.g., if all of the initial measurements were made in one time frame and all of the repeat measurements in another.

Basis

The bases for the two estimators are very simple. The expected values of $\hat{\sigma}^2$ and $\tilde{\sigma}^2$ are both σ^2 if x_i and y_i have the same expected value for all i. If they have different expected values, then the expected value of $\tilde{\sigma}^2$ is σ^2.

Examples

EXAMPLE 2.17 (a)

Sampled UO_2 sintered pellets were split into two parts with one sample analyzed for percent uranium by the gravimetric method and the other retained for later analysis by the same method. Over a given time period, 42 samples were thus analyzed. The difference data (d_i values) are tabled. Values are in percent uranium.

-.003	.003	.002	.028	.000
-.021	.021	.006	.008	.000
-.018	.045	.011	.009	-.013
.034	.040	.015	-.005	-.002
.025	-.007	.031	.005	-.004
-.001	.003	.020	.008	-.015
.014	-.002	.009	.002	
-.005	.018	-.013	-.001	
.011	-.012	.019	.004	

For these data,

$$\sum_{i=1}^{42} d_i = 0.269 \qquad \sum_{i=1}^{42} d_i^2 = 0.011339$$

Thus, by (eq. 2.6.85) and (eq. 2.6.86), the two estimates are:

$$\hat{\sigma}^2 = 0.0001350 \qquad \tilde{\sigma}^2 = 0.0001173$$

The fact that $\tilde{\sigma}^2$ is somewhat less than $\hat{\sigma}^2$ is an indication that there may be a relative bias between the two sets of measurements.

EXAMPLE 2.17 (b)

Containers of grinder sludge are resampled after having been in storage for varying lengths of time. The results in percents uranium for 15 containers are as follows.

x_i	y_i	d_i	x_i	y_i	d_i
77.72	80.03	-2.31	77.30	78.12	-0.82
77.64	79.24	-1.60	80.00	78.76	1.24
77.88	77.68	0.20	73.95	73.92	0.03
74.98	76.82	-1.84	75.69	72.87	2.82
76.67	70.41	6.26	75.82	77.81	-1.99
76.54	78.60	-2.06	81.48	77.23	4.25
70.43	68.01	2.42	74.35	78.67	-4.32
78.41	75.95	2.46			

For these data,

$$\sum_{i=1}^{15} d_i = 4.74 \qquad \sum_{i=1}^{15} d_i^2 = 117.5092$$

By (eq. 2.6.85) and (eq. 2.6.86), the two estimates are:

$$\hat{\sigma}^2 = 3.9170 \qquad \tilde{\sigma}^2 = 4.1433$$

In this example, σ^2 measures the combined effects of sampling and analytical errors. One could obtain separate estimates of these two errors by making replicate analyses of at least some of the samples. In this particular instance, however, sampling error is clearly dominant and so σ^2 may be interpreted as being the sampling error variance. In the previous example, the reverse was true, i.e., the sampling error may be assumed to be negligible so that σ^2 measured the analytical error.

2.6.3.3 Grubbs' Analysis; Two Measurement Methods

In the paired data analysis method just treated, the assumption is made that both measurements are made by the same measurement method or, if the methods are different, their random error variances are the same. In many safeguards applications, this assumption is not valid. Paired data occur naturally, for example, in shipper-receiver comparisons and for inspection data in which operator and inspector measurements are compared on a paired basis. In both instances, there may be little basis for assuming that the measurement error parameter values are the same for both parties; their measurement methods may in fact be quite different.

By appropriately treating the data, it is possible to obtain estimates of each of the two random error variances. The caution is made, however, that the resulting estimates may be of disappointing quality. In order for the estimates derived from this approach to be useful, it is necessary that the measurement errors be large relative to the scatter among the items being measured. There are other requirements imposed on the data set, as discussed by Jaech [2.13].

A number of statistical methods are given below. Method 2.18 gives the Grubbs' estimators and provides alternate estimators when one of the estimates is negative. In Method 2.19, an indication is given of what estimates to use when the random error of measurement is known for one of the two parties. This occurs in inspection situations, for example, when the inspector's error variance may be known based on a large body of past experience. Method 2.20 indicates how and under what conditions one can estimate systematic error variances from paired data.

Method 2.18

Notation

x_i = measured value for item i, one measurement method

y_i = measured value for item i, second method

n = number of pairs of values

s_x^2 = sample variance for the x_i values

s_y^2 = sample variance for the y_i values

s_{xy} = sample covariance for the (x_i,y_i) values

σ_1^2 = random error variance, method 1

σ_2^2 = random error variance, method 2

Results

The parameter estimates are

$$\hat{\sigma}_1^2 = s_x^2 - s_{xy} \quad \text{(eq. 2.6.87)}$$

$$\hat{\sigma}_2^2 = s_y^2 - s_{xy} \quad \text{(eq. 2.6.88)}$$

In the event $\hat{\sigma}_1^2$ is negative, set the estimate equal to zero. The parameter σ_2^2 is then estimated by

$$s_x^2 + s_y^2 - 2s_{xy} \qquad \text{(eq. 2.6.89)}$$

Equation (eq. 2.6.89) also provides the estimate of σ_1^2 should $\hat{\sigma}_2^2$ of (eq. 2.6.88) be negative.

If s_{xy} is negative, replace it by zero in estimating σ_1^2 and σ_2^2.

Basis

The method of estimation is due to Grubbs [2.14]. When one of the estimates is negative, the constrained maximum likelihood estimates apply [2.15].

Examples

EXAMPLE 2.18 (a)

Samples are taken from containers of grinder sludge and split. One subsample is analyzed by a titration method and the second by a spectrophotometric method. The analytical results are displayed below, in percent uranium.

Container	Titration (x_i)	Spectrophotometric (y_i)
1	75.44	73.96
2	77.46	75.98
3	72.22	74.15
4	75.85	75.98
5	74.28	77.44
6	76.82	77.61
7	74.24	70.30
8	77.87	80.27
9	75.32	78.75
10	76.17	73.70
11	73.21	77.93
12	75.65	74.70
13	76.93	73.38
14	72.36	73.67
15	79.15	76.01
16	75.90	77.01
17	77.03	76.71
18	75.90	73.71

The sample variances and covariance are

$$s_x^2 = 3.433035 \qquad s_y^2 = 5.751694 \qquad s_{xy} = 1.317212$$

By (eq. 2.6.87) and (eq. 2.6.88)

$\hat{\sigma}_1^2$ = 2.115823 (titration)

$\hat{\sigma}_2^2$ = 4.434482 (spectrophotometric)

Suppose that one of the two random error variances (say σ_2^2) may be assumed known. One might decide to use this information in estimating σ_1^2. This may not be a good decision. Method 2.19 provides a way to determine which estimate of $\sigma_1{}^2$ to use when $\sigma_2{}^2$ is known.

Method 2.19

Notation

Same as for Method 2.18. Also,

σ_μ^2 = variance of characteristic from one item to the next, excluding effect of measurement errors

(σ_μ^2 is estimated by s_{xy})

$R = \sigma_1{}^2/\sigma_2{}^2$

Results

If knowledge about the value of $\sigma_2{}^2$ is ignored, then $\sigma_1{}^2$ is estimated by (eq. 2.6.87).

If the value of $\sigma_2{}^2$ is taken into account, then $\sigma_1{}^2$ is estimated by

$$\tilde{\sigma}_1{}^2 = s_x^2 + s_y^2 - 2s_{xy} - \sigma_2{}^2 \qquad \text{(eq. 2.6.90)}$$

The expression in (eq. 2.6.90) is identically the same as provided by finding each paired difference, calculating the variance of the differences, and subtracting $\sigma_2{}^2$.

The choice of which of the two estimators is preferred depends on two quantities; A and T, where

$$A = (3R + 2)/(R + 1) \qquad \text{(eq. 2.6.91)}$$

$$T = \sigma_\mu{}^2/\sigma_2{}^2 \qquad \text{(eq. 2.6.92)}$$

The following rule applies:

(1) If $T<2$, use $\hat{\sigma}_1{}^2$

(2) If $T>3$, use $\tilde{\sigma}_1{}^2$

(3) If $2 \le T \le 3$, use $\hat{\sigma}_1{}^2$ if $A<T$; otherwise, use $\tilde{\sigma}_1{}^2$

Basis

The criterion for selecting one estimator over the other is the sampling variance of each estimator. The estimator with the smaller variance is regarded as the better one. For details, see [2.16].

Examples

EXAMPLE 2.19 (a)

In Example 2.18 (a), suppose that $\sigma_2{}^2 = 4.00$ is a known quantity. The quantity σ_μ^2 is not known, but is replaced by its estimate, $s_{xy} = 1.317$. Then, by (eq. 2.6.92),

$$T = 1.317/4.00 = 0.33$$

Since this is less than 2, use the estimator $\hat{\sigma}_1{}^2$, i.e., ignore the knowledge about the value of $\sigma_2{}^2$.

Normally, one obtains information about systematic errors by measuring standards, and not production items. However, by making certain assumptions, it is possible to obtain estimates of systematic error variances from paired data also. In a sense, such estimates may be more realistic than those based on standards data.

Two key assumptions are made:

(1) Both measurement methods are indeed applied to the same items, i.e., the item being measured in no way changes in true value after the first measurement is made.

(2) The systematic error or bias associated with each of the two measurement methods is a random variable with zero mean and variance that is called the systematic error variance for that method.

With these assumptions kept in mind, Method 2.20 is now considered.

Method 2.20

Notation

Same as for Method 2.18. Also

$\bar{x}$, $\bar{y}$ = average of the x_i and y_i measurements respectively

σ_{s1}^2 = systematic error variance for method 1

σ_{s2}^2 = systematic error variance for method 2

Results

The sum of the two systematic error variances, σ_{S1}^2 and σ_{S2}^2, is estimated by

$$\hat{\sigma}_{S1}^{\,2} + \hat{\sigma}_{S2}^{\,2} = (\bar{x} - \bar{y})^2 - (s_x^2 + s_y^2 - 2s_{xy})/n \qquad \text{(eq. 2.6.93)}$$

One cannot assign separate values to the two parameters unless: (1) one value is known; or (2) the two values are the same (or one is a known function of the other).

In passing, it is noted that the expression $(s_x^2 + s_y^2 - 2s_{xy})$ is the same as the variance of the $d_i = (x_i - y_i)$ values. This latter value is, of course, easier to compute.

Basis

Under the stated assumptions, the expected value of $(\bar{x} - \bar{y})^2$ is

$$\sigma_{S1}^{\,2} + \sigma_{S2}^{\,2} + (\sigma_1^2 + \sigma_2^2)/n$$

so that $(\hat{\sigma}_{S1}^2 + \hat{\sigma}_{S2}^2)$ is found by equating the observed value $(\bar{x}-\bar{y})^2$ to its expected value and replacing $(\sigma_1^{\,2} + \sigma_2^2)$ by its estimate.

If the bias for either method shifts over some time frame, and if data are collected over several time frames, then a more complex statistical analysis is performed. For an illustrative example, see [2.17].

Examples

EXAMPLE 2.20 (a)

When nitrate solution is being loaded into a recovery plant for further purification, samples are drawn from each container and analyzed for percent plutonium using two different analytical techniques. Sample data from 20 containers are given below.

Sample	x_i	y_i	d_i	Sample	x_i	y_i	d_i
1	13.11	13.00	0.11	11	13.26	13.01	0.25
2	15.14	14.90	0.24	12	11.00	11.06	-0.06
3	13.22	13.01	0.21	13	12.74	12.75	-0.01
4	13.67	13.65	0.02	14	13.69	13.69	0.00
5	10.48	10.61	-0.13	15	10.43	10.40	0.03
6	15.37	15.11	0.26	16	11.38	11.30	0.08
7	12.37	12.40	-0.03	17	12.26	12.27	-0.01
8	12.50	12.63	-0.13	18	12.89	12.70	0.19
9	11.46	11.71	-0.25	19	13.33	13.30	0.03
10	14.28	14.21	0.07	20	11.88	11.90	-0.02

In applying (eq. 2.6.93), replace $(\bar{x} - \bar{y})^2$ by its equivalent, $\bar{d}^2$, and replace $(s_x^2 + s_y^2 - 2s_{xy})$ by its equivalent, s_d^2. The values are

$$\bar{d} = 0.0425 \qquad \bar{d}^2 = 0.001806 \qquad s_d^2 = 0.018883$$

Then,

$$\hat{\sigma}_{s1}^2 + \hat{\sigma}_{s2}^2 = 0.001806 - 0.018883/20$$

$$= 0.000862$$

Suppose σ_{s2} were known to be 0.02% Pu. Then,

$$\hat{\sigma}_{s1}^2 = 0.000862 - (.02)^2 = 0.000462$$

$$\hat{\sigma}_{s1} = 0.021\% \text{ Pu}$$

If there were no prior knowledge about either parameters, then it might be reasonable to make them equal so that each σ_{si}^2 would be estimated by 0.000862/2 = 0.000431.

2.6.3.4 Grubbs' Analysis; More Than Two Measurement Methods With Constant Relative Bias

In the discussion of Section 2.6.3.3, it was pointed out that the estimates of the two measurement error random variances were not useful unless the measurement errors were large relative to the item to item variation. When more than two measurement methods are involved, this difficulty disappears; the quality of the estimates is then not a function of the relative sizes of the random errors of measurement.

Method 2.21 gives the technique for estimating random error variances for each of N measurement methods with N ≥ 3. Each item is measured no more than once by each of the N methods. It is assumed that any relative biases among measurement methods are constant over all items measured. This assumption is relaxed in 2.6.3.5.

Method 2.21

Notation

x_{ik} = measured value for item k, method i

N = number of measurement methods

n = number of items measured

$d_{ijk} = x_{ik} - x_{jk}$ for all pairs i, j

v_{ij} = sample variance among the n d_{ijk} values for each i,j. There are N(N-1)/2 v_{ij} values

σ_i^2 = random error variance, method i

Results

For each method i, calculate

$$S_i = \sum_{j \neq i} v_{ij} \qquad \text{(eq. 2.6.94)}$$

Also sum all the v_{ij} values and call this sum V. Then, σ_i^2 is estimated from

$$\hat{\sigma}_i^2 = S_i/(N-2) - V/(N-1)(N-2) \qquad \text{(eq. 2.6.95)}$$

Basis

The method of estimation is given in [2.14] and in [2.18]. The latter reference contains an example for N = 6 measurement methods.

Examples

EXAMPLE 2.21 (a)

Turnings from a zirconium-uranium billet are distributed among three laboratories as indicated below for analysis of percent uranium. The results are tabulated.

x_{ik} (% U)

Turn Number (k)	Lab 1	Lab 2	Lab 3
1	77	81	78
2	85	--	90
3	80	--	84
4	85	81	--
5	90	79	88
6	--	82	90
7	--	75	88
8	92	84	89
9	81	84	--

The d_{ijk} values are calculated

k	d_{12k}	d_{13k}	d_{23k}
1	-4	-1	3
2	--	-5	--
3	--	-4	--
4	4	--	--
5	11	2	-9
6	--	--	-8
7	--	--	-13
8	8	3	-5
9	-3	--	--

The variances, v_{ij}, are

$v_{12} = 43.70$ $\qquad v_{13} = 12.50$ $\qquad v_{23} = 35.80$

Then, $V = 43.70 + 12.50 + 35.80 = 92.00$

$S_1 = 43.70 + 12.50 = 56.20$

$S_2 = 43.70 + 35.80 = 79.50$

$S_3 = 12.50 + 35.80 = 48.30$

By (eq. 2.6.95),

$\hat{\sigma}_1{}^2 = 56.20 - 92.00/2 = 10.20$; $\hat{\sigma}_1 = 3.19\%$ U

$\hat{\sigma}_2^2 = 33.50$; $\hat{\sigma}_2 = 5.79\%$ U

$\hat{\sigma}_3^2 = 2.30$; $\hat{\sigma}_3 = 1.52\%$ U

If one can make the same assumptions as were made prior to Method 2.20, then estimates of systematic error variances may also be derived from data of this type. The technique for obtaining these estimates is given by Method 2.22.

Method 2.22

Notation

Same as Method 2.21. Also,

$\bar{x}_i$ = average of the x_{ik} values

$= \sum_{k=1}^{n} x_{ik}/n$, where

n = number of measurements for each method

σ_{si}^2 = systematic error variance for method i

Results

First, perform the calculations of Method 2.21 to obtain estimates of the random error variances, $\hat{\sigma}_i^2$. Then, the sum of the systematic error variances, σ_{si}^2, is estimated by

$$\sum_{i=1}^{N} \hat{\sigma}_{si}^{2} = Ns_{\bar{x}i}^{2} - \sum_{i=1}^{N} \hat{\sigma}_i^2/n \qquad \text{(eq. 2.6.96)}$$

where $s_{\bar{x}i}^2$ is the sample variance among the N $\bar{x}_i$ values.

One cannot obtain separate estimates of the systematic error variances without additional information. For example, one might be willing to assume that all systematic error variances are the same, or are related to one another in some known way.

Basis

The result in (eq. 2.6.96) is based on the fact that the expected value of $Ns_{\bar{x}i}^2$ is

$$\sum_{i=1}^{N} \sigma_{si}^2 + \sum_{i=1}^{N} \sigma_i^2/n$$

so that

$$\sum_{i=1}^{N} \hat{\sigma}_{si}^2$$

is found by equating the expected value of $Ns_{\bar{x}i}^2$ to its observed value and replacing σ_i^2 by its estimate for all i.

Examples

EXAMPLE 2.22 (a)

One cannot perform this analysis on the data of Example 2.21 (a) because it is necessary that all measurement methods measure all items in order to obtain estimates of the systematic error variance. This assumption was not necessary when using Grubbs' method to estimate random error variances.

EXAMPLE 2.22 (b)

Percent plutonium analyses on sintered pellet samples are reported for 13 samples by four laboratories. Analyze the following data to find estimates of the random error variances. Assume that all four laboratories have the same systematic error variance and obtain its estimate.

Sample	Lab 1	Lab 2	Lab 3	Lab 4
1	4.62	4.57	4.64	4.56
2	4.61	4.67	4.61	4.55
3	4.72	4.71	4.80	4.79
4	4.66	4.67	4.65	4.66
5	4.68	4.61	4.72	4.52
6	4.67	4.67	4.77	4.64
7	4.69	4.67	4.71	4.67
8	4.68	4.67	4.75	4.65
9	4.72	4.69	4.71	4.69
10	4.70	4.70	4.73	4.66
11	4.59	4.60	4.66	4.58
12	4.59	4.55	4.63	4.59
13	4.61	4.63	4.66	4.64

Method 2.21 is followed. The columns of differences, d_{ijk}, are found.

d_{12k}	d_{13k}	d_{14k}	d_{23k}	d_{24k}	d_{34k}
.05	-.02	.06	-.07	.01	.08
-.06	.00	.06	.06	.12	.06
.01	-.08	-.07	-.09	-.08	.01
-.01	.01	.00	.02	.01	-.01
.07	-.04	.16	-.11	.09	.20
.00	-.10	.03	-.10	.03	.13
.02	-.02	.02	-.04	.00	.04
.01	-.07	.03	-.08	.02	.10
.03	.01	.03	-.02	.00	.02
.00	-.03	.04	-.03	.04	.07
-.01	-.07	.01	-.06	.02	.08
.04	-.04	.00	-.08	-.04	.04
-.02	-.05	-.03	-.03	-.01	.02

The variances, v_{ij} are:

$v_{12} = 0.001117$ $\quad$ $v_{23} = 0.002397$

$v_{13} = 0.001214$ $\quad$ $v_{24} = 0.002559$

$v_{14} = 0.002876$ $\quad$ $v_{34} = 0.003177$

Then

$$V = 0.00117 + \ldots + 0.003177 = 0.013340$$

$$S_1 = 0.001117 + 0.001214 + 0.002876 = 0.005207$$

$$S_2 = 0.006073$$

$$S_3 = 0.006788$$

$$S_4 = 0.008612$$

From (eq. 2.6.95),

$$\hat{\sigma}_1^2 = 0.005207/2 - 0.013340/6 = 0.000380$$

$$\hat{\sigma}_2^2 = 0.000813$$

$$\hat{\sigma}_3^2 = 0.001171$$

$$\hat{\sigma}_4^2 = 0.002083$$

Method 2.22 is now followed. The means of the four columns of data are found

$\bar{x}_1 = 4.6569$ $\bar{x}_2 = 4.6469$ $\bar{x}_3 = 4.6954$ $\bar{x}_4 = 4.6308$

The sample variance among these four mean values is

$$s_{\bar{x}i}^2 = 0.000754$$

From (eq. 2.6.96),

$$\sum_{i=1}^{4} \hat{\sigma}_{si}^2 = 4\ \hat{\sigma}_s^2 = (4)(0.000754) - (0.004447)/13$$

from which

$$\hat{\sigma}_s^2 = 0.000668$$

2.6.3.5 More Than Two Measurement Methods With Non-Constant Relative Bias

The problem under discussion is the same as that in 2.6.3.4 except for an important distinction. In the methodology of 2.6.3.4, it was assumed that any relative biases among measurement methods are constant over all the items measured. In Method 2.23 to follow, this assumption is no longer made.

Method 2.23

Notation

x_{ik}, N, and n as in Method 2.21

s_i^2 = sample variance among the x_{ik} values, method i

s_{ij} = sample covariance among the x_{ik} and x_{jk} values, methods i and j

μ_k = true (but unknown) value for item k

σ_μ^2 = variance among the μ_k values

α_i, β_i = parameters describing the relative bias among measurement methods, defined by the model

$$E(x_{ik}) = \alpha_i + \beta_i \mu_k \qquad \text{(eq. 2.6.97)}$$

σ_i^2 = random error variance, method i

Arbitrarily, set $\beta_1 = 1$. This does not affect the estimate of the σ_i^2 parameters.

Results

For i = 2, 3, ..., N, calculate

$$\hat{\beta}_i = \left(\prod_{j \neq 1, i}^{N} s_{ij} \Big/ s_{1j} \right)^{1/(N-2)} \qquad \text{(eq. 2.6.98)}$$

Calculate

$$\hat{\sigma}_\mu^2 = \left(\prod_{\substack{i=3 \\ j<i, \neq 1}}^{N} s_{1i} s_{1j} \Big/ s_{ij} \right)^{2/(N-1)(N-2)} \qquad \text{(eq. 2.6.99)}$$

The parameters are estimated by

$$\hat{\sigma}_i^2 = s_i^2 - \hat{\beta}_i^2 \, \hat{\sigma}_\mu^2 \quad \text{, where } \hat{\beta}_1 = 1 \qquad \text{(eq. 2.6.100)}$$

Basis

Two methods of estimation have been suggested for this model [2.19], [2.20]. The latter reference forms the basis for the estimation Method 2.23. See also [2.21] and [2.22].

Examples

EXAMPLES 2.23 (a)

In Table III of [2.17], data are given corresponding to 5 NDA measurements of plutonium-bearing solid wastes. After transforming the data to natural logarithms, Method 2.21, based on the constant relative bias model, was applied, giving the following 5 estimates as reported in [2.17].

$\hat{\sigma}_1^2 = 0.07162$ $\qquad$ $\hat{\sigma}_4^2 = 0.00413$

$\hat{\sigma}_2^2 = 0.00100$ $\qquad$ $\hat{\sigma}_5^2 = 0.04076$

$\hat{\sigma}_3^2 = 0.00805$

The data are re-analyzed after removing the assumption that relative biases are constant. The transformed data are given below as natural logarithms of grams of plutonium.

Container	Measurement Method (See [2.17]) 1	2	3	4	5
1	3.906	3.906	3.661	2.944	2.923
2	1.740	1.792	1.649	1.030	1.131
3	1.131	1.482	0.956	0.693	0.693
4	2.128	2.128	2.015	1.308	1.386
5	3.144	2.760	2.542	1.946	1.705

The appropriate variances and covariances are calculated.

$s_1^2 = 1.235259$ $\quad s_{12} = 1.043854$ $\quad s_{24} = 0.852077$

$s_2^2 = 0.920771$ $\quad s_{13} = 1.112266$ $\quad s_{25} = 0.802107$

$s_3^2 = 1.032645$ $\quad s_{14} = 0.972217$ $\quad s_{34} = 0.893879$

$s_4^2 = 0.789404$ $\quad s_{15} = 0.897719$ $\quad s_{35} = 0.847349$

$s_5^2 = 0.711128$ $\quad s_{23} = 0.963278$ $\quad s_{45} = 0.741087$

From (eq. 2.6.98),

$$\hat{\beta}_2 = (s_{23}s_{24}s_{25}/s_{13}s_{14}s_{15})^{1/3} = 0.878584$$

$$\hat{\beta}_3 = (s_{23}s_{34}s_{35}/s_{12}s_{14}s_{15})^{1/3} = 0.928645$$

$$\hat{\beta}_4 = (s_{24}s_{34}s_{45}/s_{12}s_{13}s_{15})^{1/3} = 0.815103$$

$$\hat{\beta}_5 = (s_{25}s_{35}s_{45}/s_{12}s_{13}s_{14})^{1/3} = 0.764160$$

From (eq. 2.6.99),

$$\hat{\sigma}_\mu^2 = \left(\frac{s_{13}s_{12}}{s_{23}} \cdot \frac{s_{14}s_{12}}{s_{24}} \cdot \frac{s_{14}s_{13}}{s_{34}} \cdot \frac{s_{15}s_{12}}{s_{25}} \cdot \frac{s_{15}s_{13}}{s_{35}} \cdot \frac{s_{15}s_{14}}{s_{45}}\right)^{1/6}$$

$$= 1.188312$$

The parameter estimates are then given by (eq. 2.6.100).

$$\hat{\sigma}_1{}^2 = 1.235259 - (1)\ (1.188312) = 0.04695$$

$$\hat{\sigma}_2{}^2 = 0.920771 - (0.878584)^2\ (1.188312) = 0.00350$$

$$\hat{\sigma}_3{}^2 = 0.00787$$

$$\hat{\sigma}_4{}^2 = -0.00010 \quad (=0)$$

$$\hat{\sigma}_5{}^2 = 0.01722$$

In comparing these estimates with those based on the constant relative bias model, it is noted that the results are quite different. The model assumption is obviously quite important. With so few data points, it is difficult to determine with any confidence which model is the more appropriate one in this instance. In [2.20], expressions are given for the sampling variance of $\hat{\beta}_i$ which will permit testing whether or not the slopes differ significantly from unity, i.e., whether or not the constant relative bias model applies.

2.6.3.6 Combining Parameter Estimates from Different Experiments

Estimates of measurement error parameters will often come from a number of data sources or experiments. In application, one requires a single estimate of each parameter, one that is the "best" estimate in some sense. The problem of how to obtain such an estimate is covered by Method 2.24.

Method 2.24

Notation

There are n observed variances, where

v_j = j-th observed variance

The expected value of v_j is of the form

$$\sum_{i=1}^{k} c_{ij}\ \theta_i$$

where θ_i = i-th measurement error parameter, i = 1, 2, ..., k

c_{ij} = known constant

For example, θ_1 may be the random error variance due to sampling and θ_2 may be the random error variance due to analytical for a given analytical method. If 12 containers are sampled with the samples all analyzed in duplicate, and if the observed variance among the 12 results is, say, 100 units^2, then, assuming that the containers have the same nominal value for the characteristic of interest (e.g. percent uranium), one estimating equation would be

$$\theta_1 + 0.5\ \theta_2 = 100$$

Assume that the v_j have different sampling variances, and hence, should be weighted differently. Further assume (for the moment) that each weight is known:

w_j = weight associated with v_j

Results

A k by k matrix, A, is formed. This is a symmetric matrix whose element in row i and column h is

$$a_{ih} = \sum_{j=1}^{n} w_j c_{ij} c_{hj} \qquad \text{(eq. 2.6.101)}$$

A k by 1 column matrix, V, is formed. In row h, the element is

$$V_h = \sum_{j=1}^{n} w_j c_{hj} v_j \qquad \text{(eq. 2.6.102)}$$

Invert A to give A^{-1} and perform the matrix multiplication

$$A^{-1}V$$

The element in row h of $A^{-1}V$ is the estimate of the parameter θ_h.

Basis

The weighted least squares estimation method is used [2.23]. If an unweighted analysis is performed (i.e., if all the v_j values have about the same sampling uncertainty), let $w_j = 1$ for all i.

The method assumes that the w_j values are known. In practice, w_j will be a function of the θ's which are, of course, not known, and hence, w_j would not be known either. An iterative estimation procedure is available in this instance [2.24]. Method 2.24 is in essence one step in this iterative procedure, for the procedure calculates weights at each stage of the estimation process based on the estimates of the parameters from the prior stage.

Examples

EXAMPLE 2.24 (a)

Several data sources provide estimates of sampling error, (θ_1), of analytical error by titration, (θ_2), and of analytical error by spectrophotometry, (θ_3), in the measurement of percent uranium in UO_2 dirty powder scrap. The following equations are derived

$$\theta_1 = 2.89$$

$$\theta_2 = 6.28$$

$$\theta_3 = 4.98$$

$$\theta_1 + \theta_2 = 8.13$$

$$\theta_1 + 0.5\theta_3 = 6.70$$

In the notation of this method, the c_{ij} and v_j values are tabulated

j	c_{1j}	c_{2j}	c_{3j}	v_j
1	1	0	0	2.89
2	0	1	0	6.28
3	0	0	1	4.98
4	1	1	0	8.13
5	1	0	0.5	6.70

Assume that the v_j are based on the following numbers of replicate measurements:

$$m_1 = 29 \quad m_2 = 10 \quad m_3 = 43 \quad m_4 = 18 \quad m_5 = 7$$

It is well known that when sampling from a normal distribution (as will be assumed here), the sampling variance of an estimated variance is twice the square of the true variance divided by the degrees of freedom, $m_j - 1$ [2.25]. Also, since the weights are the reciprocals of the sampling variances, they are

$$w_1 = 28/2\theta_1^2 \qquad w_2 = 9/2\theta_2^2 \qquad w_3 = 42/2\theta_3^2$$

$$w_4 = 17/2(\theta_1+\theta_2)^2 \qquad w_5 = 6/2(\theta_1+0.5\theta_3)^2$$

These are functions of the unknown parameter values. An iterative procedure is needed, using an arbitrary starting point. A reasonable starting point could be the estimates based on the first three equations:

$$\hat{\theta}_1 = 2.89 \qquad \hat{\theta}_2 = 6.28 \qquad \hat{\theta}_3 = 4.98$$

The weights for the first iteration are then,

$$w_1 = 1.676 \qquad w_2 = 0.114 \qquad w_3 = 0.847$$

$$w_4 = 0.101 \qquad w_5 = 0.104$$

Then, from (eq. 2.6.101),

$$a_{11} = 1.676 + 0.101 + 0.104 = 1.881$$

$$a_{22} = 0.114 + 0.101 = 0.215$$

$$a_{33} = 0.847 + 0.25\,(0.104) = 0.873$$

$$a_{12} = 0.101$$

$$a_{13} = 0.5\,(0.104) = 0.052$$

$$a_{23} = 0$$

The A matrix is

$$\begin{pmatrix} 1.881 & 0.101 & 0.052 \\ 0.101 & 0.215 & 0 \\ 0.052 & 0 & 0.873 \end{pmatrix}$$

By (eq. 2.6.102),

$$V_1 = (1.676)\ (2.89) + (.101)(8.13) + (.104)(6.70) = 6.362$$

$$V_2 = (.114)(6.28) + (.101)(8.13) = 1.537$$

$$V_3 = (.847)(4.98) + (.5)(.104)(6.70) = 4.566$$

The V matrix is

$$V = \begin{pmatrix} 6.362 \\ 1.537 \\ 4.566 \end{pmatrix}$$

The inverse matrix, A^{-1}, is found. Procedures for finding the inverse of a matrix are found in many texts. Most computer program packages contain matrix inversion routines. Certain pocket calculators with program cards or tabs also permit rapid inversion of matrices of order 4 by 4 or smaller. The inverse of A is found to be

$$A^{-1} = \begin{pmatrix} 0.5463 & -0.2566 & -0.0325 \\ -0.2566 & 4.7717 & 0.0153 \\ -0.0325 & 0.0153 & 1.1474 \end{pmatrix}$$

The estimates are the elements of A^{-1} V.

$$\hat{\theta}_1 = (.5463)(6.362) - (.2566)(1.537) - (.0325)(4.566) = 2.933$$

$$\hat{\theta}_2 = 5.771$$

$$\hat{\theta}_3 = 5.056$$

Note how these estimates compare with the inputs for this iteration. They could then be used to determine the weights for the next iteration, and the process could continue until convergence.

2.6.4 Error Estimation in the Presence of Rounding Errors

In some safeguards applications, most notably, with weighing data, the effect of rounding errors on the total error of measurement cannot be ignored. If a scale rounds to the nearest 10 grams, say, and if replicate measurements are made on the same item, it is quite likely that all recorded weights would agree exactly. If one were then to follow the procedures in 2.6.3.1 to estimate the random error variance due to weighing, the estimate would be zero, for any variability in the weighing process would be obscured by the rounding error. Clearly, it would be misleading to assert that there is zero random error. By similar reasoning, if one weighed known standards when rounding error is relatively large, the conclusion in following the procedures of 2.6.1 would be that the systematic error variance is also zero, for all recorded measurements would be in perfect agreement with the assigned standard value. Such a conclusion is also invalid.

By statistical theory based on the uniform or rectangular distribution, for a measurement process in which data are rounded to the nearest u units, the systematic error standard deviation can be no smaller than $u/\sqrt{12}$ units. The same is true of the random error standard deviation for a single weighing [2.26].

If rounding error is not the sole error, but is nevertheless too large to be ignored, e.g., if the measurement data fall into 2 or 3 groups, then an iterative method of obtaining the estimates of the parameters has been developed. The methodology is beyond the scope of this Volume, but the model and estimation procedure are referenced [2.27]. The computer program based on this estimation method is in the Agency library. It is recommended that the method be used whenever measurement data are grouped in 2 or 3 cells because of rounding.

2.6.5 Interlaboratory Test Data

In many Agency safeguards applications, more than one analytical laboratory is involved in some way. Most notably, this occurs whenever interlaboratory sample exchange or "round robin" programs are instigated in order to obtain information on measurement error parameters. Another instance in which such data are generated is whenever inspection samples are sent to more than one laboratory for analysis.

In the sections to follow, the analyses of interlaboratory data are considered. In 2.6.5.1, samples of a single standard reference material are sent to a number of laboratories in a round robin exercise. In 2.6.5.2, non-standard materials are distributed to a number of laboratories for analysis. Section 2.6.5.3 considers the same problem, but it is now assumed that random errors of measurement may be laboratory dependent. The distribution of inspection samples to a number of laboratories with different specified patterns of distribution is discussed in 2.6.5.4.

2.6.5.1 Single Standard Reference Material

The purpose of an exercise in which samples of single standard reference material are sent to participating laboratories is to identify factors that contribute to the uncertainty of a measured value, and assess their importance. To accomplish this, it is necessary that detailed instructions be sent the laboratories relative to how the samples should be measured. That is, one might be interested in obtaining estimates of differences due to replicate measurements of aliquots, differences among aliquots, etc. To obtain such estimates, it is necessary that the analyses be performed according to some plan.

The statistical analysis of the data resulting from such an exercise is called a nested or hierarchal analysis of variance with the plan or experimental design called a general unbalanced nested design. To permit a simpler presentation of the analysis, it will be assumed in Method 2.25 that there are 4 identified variance components to be estimated, and the example will specify the 4 components in question. It should be readily apparent from the analysis of Method 2.25 how the procedures should be altered to account for fewer or more than 4 components. Further, the 4 variance components identified in the example can, of course, be replaced by other components.

Method 2.25

Notation

μ = true value of item being measured

A_i = deviation from μ for level i of factor A; i = 1, 2, ..., a

$B_{j(i)}$ = deviation from μ for level j of factor B within level i of factor A; j = 1, 2, ..., b_i

$C_{k(i,j)}$ = deviation from μ for level k of factor C within level j of factor B within level i of factor A; (k = 1, 2, ..., c_{ij})

$D_{\ell(i,j,k)}$ = deviation from μ for level ℓ of factor D within level k of factor C within level j of factor B within level i of factor A; (ℓ = 1, 2, ..., d_{ijk})

The extension to additional components is obvious.

$Y_{ijk\ell}$ = given observed value related to A_i, $B_{j(i)}$, etc. by the model

$$Y_{ijk\ell} = \mu + A_i + B_{j(i)} + C_{k(i,j)} + D_{\ell(i,j,k)} \qquad \text{(eq. 2.6.103)}$$

A_i is selected at random from a population with zero mean and variance σ_A^2

$B_{j(i)}$ is selected at random from a population with zero mean and variance σ_B^2

σ_C^2 and σ_D^2 are similarly defined

Results

The problem is to obtain estimates of σ_A^2, σ_B^2, σ_C^2, and σ_D^2. Note the implicit assumption that σ_B^2, say, is the same for all levels of factor A, i.e., there are not a values of $\sigma_{B(i)}^2$ to estimate, but only the one. Similar statements apply to σ_C^2 and σ_D^2.

The statistical analysis proceeds as follows. The parameters will be estimated by solving the following four equations, starting from the last and working upward.

$$\left.\begin{aligned} M_A &= \hat{\sigma}_D^2 + Q_1\hat{\sigma}_C^2 + P_1\hat{\sigma}_B^2 + R_1\hat{\sigma}_A^2 \\ M_B &= \hat{\sigma}_D^2 + Q_2\hat{\sigma}_C^2 + P_2\hat{\sigma}_B^2 \\ M_C &= \hat{\sigma}_D^2 + Q_3\hat{\sigma}_C^2 \\ M_D &= \hat{\sigma}_D^2 \end{aligned}\right\} \qquad \text{(eq. 2.6.104)}$$

The procedures for obtaining the M's, Q's, P's, and R_1 are given below. Calculate the following quantities:

$$n_{ij} = \sum_k d_{ijk} \qquad n_i = \sum_j n_{ij} \qquad n = \sum_i n_i$$

$$F_A = a-1$$

$$F_B = \sum_i b_i - a$$

$$F_C = \sum_i \sum_j c_{ij} - \sum_i b_i$$

$$F_D = n - \sum_i \sum_j c_{ij}$$

$$T_{ijk} = \sum_\ell y_{ijk\ell} \qquad T_{ij} = \sum_k T_{ijk}$$

$$T_i = \sum_j T_j \qquad T = \sum_i T_i$$

$$S_0 = T^2/n \qquad S_1 = \sum_i T_i^2/n_i$$

$$S_2 = \sum_i \sum_j T_{ij}^2/n_{ij} \qquad S_3 = \sum_i \sum_j \sum_k T_{ijk}^2/d_{ijk}$$

$$S_4 = \sum_i \sum_j \sum_k \sum_\ell y_{ijk\ell}^2$$

$$S_A = S_1 - S_0 \qquad S_B = S_2 - S_1$$

$$S_C = S_3 - S_2 \qquad S_D = S_4 - S_3$$

The M's are:

$$\left.\begin{array}{ll} M_A = S_A/F_A & M_B = S_B/F_B \\ M_C = S_C/F_C & M_D = S_D/F_D \end{array}\right\} \qquad \text{(eq. 2.6.105)}$$

$$G_i = (^1/n_i - {}^1/n)/F_A$$

$$G_{ij} = (^1/n_{ij} - {}^1/n_i)/F_B$$

$$G_{ijk} = (^1/d_{ijk} - {}^1/n_{ij})/F_C$$

The Q's are:

$$\left.\begin{aligned} Q_1 &= \sum_i \sum_j \sum_k d_{ijk}^2 G_i \\ Q_2 &= \sum_i \sum_j \sum_k d_{ijk}^2 G_{ij} \\ Q_3 &= \sum_i \sum_j \sum_k d_{ijk}^2 G_{ijk} \end{aligned}\right\} \qquad \text{(eq. 2.6.106)}$$

The P's are:

$$\left.\begin{aligned} P_1 &= \sum_i \sum_j n_{ij}^2 G_i \\ P_2 &= \sum_i \sum_j n_{ij}^2 G_{ij} \end{aligned}\right\} \qquad \text{(eq. 2.6.107)}$$

Finally,

$$R_1 = \sum_i n_i^2 G_i \qquad \text{(eq. 2.6.108)}$$

In solving (eq. 2.6.104), $\underset{\cdot}{M}_D$ cannot be a negative quantity, but it may be that one or more of the estimates of σ_A^2, σ_B^2, and σ_C^2 may be negative. In this event, equate each such estimate to zero and obtain pooled estimates of the other parameters by appropriately combining the equations.

Basis

The distinctive feature of a nested experimental design is noted. Consider factor $B_{j(i)}$, for example. The j-th level of this factor only has meaning with respect to the i-th level of factor A. For example, if A and B represent laboratory and instrument effects respectively, then instrument 1 for laboratory 1 is not the same as instrument 1 for laboratory 2, say. If one were speaking of instrument types (or analytical methods), then the model would not be nested. It would then be called a crossed-classification and a different statistical analysis would be required.

Many textbooks treat the nested or heirarchal design, but most restrict their treatment to the case where there are equal numbers of observations at each level. For a good treatment of the general unbalanced case treated here, which is the case most likely to be encountered in practice due to "missing" observations even if the plan itself is balanced, see [2.28].

Examples

EXAMPLE 2.25 (a)

Samples of the NBS standard reference material 949/d were distributed to a number of laboratories for analysis of plutonium concentration. For those laboratories that used a potentiomitry method, the data are tabled. Tabular entries are in percent recovery minus 100. The identified factors are:

A : laboratories

B : times within laboratories

C : aliquots within times within laboratories

D : determinations within aliquots within times within laboratories

Lab	1	2	2	3	3	4
Time	1	1	2	1	2	1
Aliquot	1	1	1	1	1	1
	.03	.09	.45	-.10	-.08	.02
	.10	.21	.23	-.05	-.11	-.10
	-.10	.09	-.01	-.12	-.07	-.04
	-.08			-.01	-.13	
				-.07	-.09	

Lab	4	4	4	5	5	6
Time	1	2	2	1	1	1
Aliquot	2	1	2	1	2	1
	.00	-.08	-.09	.06	.26	.06
	-.03	-.10	-.07	.53	.25	.30
	-.04	.01	.01	.11	.29	.57
				.24	.63	
				.42	.35	
				.53	.08	

Lab	6	6	7	7	7
Time	1	2	1	1	2
Aliquot	2	1	1	2	1
	.23	.26	.09	.02	-.042
	.07	.14	-.06	.04	.013
	.48	.41	-.10		.018
					.044
					-.017

The various quantities are calculated

$n_{11} = 4$	$n_{42} = 6$	$n_1 = 4$	$n = 63$
$n_{21} = 3$	$n_{51} = 12$	$n_2 = 6$	
$n_{22} = 3$	$n_{61} = 6$	$n_3 = 10$	
$n_{31} = 5$	$n_{62} = 3$	$n_4 = 12$	
$n_{32} = 5$	$n_{71} = 5$	$n_5 = 12$	
$n_{41} = 6$	$n_{72} = 5$	$n_6 = 9$	
		$n_7 = 10$	

$F_A = 6$ $\qquad$ $F_B = 12 - 7 = 5$

$F_C = 17 - 12 = 5$ $\qquad$ $F_D = 63 - 17 = 46$

$T_{111} = -.05$	$T_{412} = -.07$	$T_{612} = .78$
$T_{211} = .39$	$T_{421} = -.17$	$T_{621} = .81$
$T_{221} = .67$	$T_{422} = -.15$	$T_{711} = -.07$
$T_{311} = -.35$	$T_{511} = 1.89$	$T_{712} = .06$
$T_{321} = -.48$	$T_{512} = 1.86$	$T_{721} = .016$
$T_{411} = -.12$	$T_{611} = .93$	

$T_{11} = -.05$	$T_{32} = -.48$	$T_{61} = 1.71$
$T_{21} = .39$	$T_{41} = -.19$	$T_{62} = .81$
$T_{22} = .67$	$T_{42} = -.32$	$T_{71} = -.01$
$T_{31} = -.35$	$T_{51} = 3.75$	$T_{72} = .016$

$T_1 = -.05$	$T_4 = -.51$	$T_7 = .006$
$T_2 = 1.06$	$T_5 = 3.75$	
$T_3 = -.83$	$T_6 = 2.52$	$T = 5.946$

$S_0 = 0.561189$	$S_1 = 2.155935$	$S_2 = 2.172618$
$S_3 = 2.180340$	$S_4 = 3.009182$	

$S_A = 1.594746$ $\qquad$ $S_B = 0.016683$

$S_C = 0.007722$ $\qquad$ $S_D = 0.828842$

The M's are; from (eq. 2.6.105)

$M_A = 0.265791$ $\qquad$ $M_B = 0.003337$

$M_C = 0.001544$ $\qquad$ $M_D = 0.018018$

Then,

$G_1 = .039021$	$G_{11} = 0$	$G_{42} = .016667$
$G_2 = .025132$	$G_{21} = .033333$	$G_{51} = 0$
$G_3 = .014021$	$G_{22} = .033333$	$G_{61} = .011111$
$G_4 = .011243$	$G_{31} = .02$	$G_{62} = .044444$
$G_5 = .011243$	$G_{32} = .02$	$G_{71} = .02$
$G_6 = .015873$	$G_{41} = .016667$	$G_{72} = .02$
$G_7 = .014021$		

$G_{111} = 0$	$G_{412} = .033333$	$G_{612} = .033333$
$G_{211} = 0$	$G_{421} = .033333$	$G_{621} = 0$
$G_{221} = 0$	$G_{422} = .033333$	$G_{711} = .026667$
$G_{311} = 0$	$G_{511} = .016667$	$G_{712} = .06$
$G_{321} = 0$	$G_{512} = .016667$	$G_{721} = 0$
$G_{411} = .033333$	$G_{611} = .033333$	

The Q's are; from (eq. 2.6.106),

$Q_1 = 3.9534$ $\quad Q_2 = 3.5600$ $\quad Q_3 = 3.4800$

From (eq. 2.6.107),

$P_1 = 5.6216$ $\quad P_2 = 4.6000$

From (eq. 2.6.108),

$R_1 = 8.8570$

The estimating equations (eq. 2.6.104) are

$$0.265791 = \hat{\sigma}_D^2 + 3.9534\hat{\sigma}_C^2 + 5.6216\hat{\sigma}_B^2 + 8.8570\hat{\sigma}_A^2$$

$$0.003337 = \hat{\sigma}_D^2 + 3.5600\hat{\sigma}_C^2 + 4.6000\hat{\sigma}_B^2$$

$$0.001544 = \hat{\sigma}_D^2 + 3.4800\hat{\sigma}_C^2$$

$$0.018018 = \hat{\sigma}_D^2$$

Since $M_C<M_D$, the estimate of σ_C^2 is zero. The revised estimate of σ_D^2 is found by taking a weighted average using the last two equations. The weights are F_C and F_D respectively. (These are called degrees of freedom.)

$$(5)(0.001544) = 5\hat{\sigma}_D^2$$

$$(46)(0.018018) = 46\hat{\sigma}_D^2$$

$$0.836548 = 51\hat{\sigma}_D^2 \quad \Longrightarrow \quad \hat{\sigma}_D^2 = 0.016403$$

From the second equation, it is also evident that the estimate of σ_B^2 is zero. Again weighting,

$$(5)(0.003337) = 5\hat{\sigma}_D^2$$

$$0.836548 = 51\hat{\sigma}_D^2$$

$$0.853233 = 56\hat{\sigma}_D^2 \quad \Longrightarrow \quad \hat{\sigma}_D^2 = 0.015236$$

From the first equation, then,

$$0.265791 = \hat{\sigma}_D^2 + 8.8570\hat{\sigma}_A^2$$

so that

$$\hat{\sigma}_A^2 = (0.265791 - 0.015236)/8.8570 = 0.028289$$

Thus, the estimates are:

$$\hat{\sigma}_A^2 = 0.028289 \quad ; \quad \hat{\sigma}_B^2 = \hat{\sigma}_C^2 = 0 \quad ; \quad \hat{\sigma}_D^2 = 0.015236$$

The differences among laboratories is the dominant effect. This component, σ_A^2, is a systematic error variance, while σ_D^2 is a random error variance.

2.6.5.2 Several Samples; Non-Standard Materials

The discussion in the previous section is now extended to accommodate the following changes:

(1) The samples distributed for analysis are not reference standards, but are samples of production materials.

(2) The samples are identified, i.e., sample 1 sent to laboratory 1, say, is sampled from the same population as sample 1 sent to laboratory 2.

The second change in assumptions requires a different method of statistical analysis. If the samples were not so identified, the model would still be a nested model and the Method 2.25 could be applied. The fact that the samples are identified results in a model that is partly crossed and partly nested.

To handle this case, follow Method 2.26.

Method 2.26

Notation

Same as for Method 2.25 except that the S's, F's, Q's, etc. have an additional subscript, h, to indicate the sample number. Thus, S_{hA} is the same as S_A in Method 2.25, except that it applies to sample h.

Results

Perform the analysis of Method 2.25 for each sample. Replace the key equations, (eq. 2.6.104), by the following equivalent equations for sample h.

$$\left.\begin{aligned} S_{hA} &= F_{hA}\,(\hat{\sigma}_D^2 + Q_{h1}\hat{\sigma}_C^2 + P_{h1}\hat{\sigma}_B^2 + R_{h1}\hat{\sigma}_A^2) \\ S_{hB} &= F_{hB}\,(\hat{\sigma}_D^2 + Q_{h2}\hat{\sigma}_C^2 + P_{h2}\hat{\sigma}_B^2) \\ S_{hC} &= F_{hC}\,(\hat{\sigma}_D^2 + Q_{h3}\hat{\sigma}_C^2) \\ S_{hD} &= F_{hD}\,\hat{\sigma}_D^2 \end{aligned}\right\} \qquad \text{(eq. 2.6.109)}$$

Sum these over the h samples to obtain the four equations which provide the estimates of σ_A^2, σ_B^2, σ_C^2, and σ_D^2. (Note: Method 2.26 also assumes that there are the 4 variance components for simplicity in discussion; there may be more or fewer.)

Next, letting $\bar{y}_h$ be the average of all the observations for sample h, compute the variance among these sample means, calling the variance $s_{\bar{y}}^2$. This is equated to its expected value, given using (eq. 2.6.110), and solved for σ_s^2, the variance due to sampling of the material in question. In (eq. 2.6.110), V_h is the variance of the sample mean for sample h, and the sums indicated are for that sample.

$$V_h = \sigma_s^2 + (\sigma_A^2 \sum_i n_i^2 + \sigma_B^2 \sum_{i,j} n_{ij}^2 + \sigma_C^2 \sum_{i,j,k} d_{ijk}^2 + \sigma_D^2 n_h)/n_h^2 \qquad \text{(eq. 2.6.110)}$$

where n_h is the total number of observations for sample h. Then σ_s^2 is estimated from

$$\hat{\sigma}_s^2 = s_{\bar{y}}^2 - \sum_{h=1}^{H} (V_h - \sigma_s^2)/H \qquad \text{(eq. 2.6.111)}$$

where σ_A^2, σ_B^2, σ_C^2, and σ_D^2 are replaced by their estimates.

Basis

For this partly crossed and partly nested model, because of the unbalanced nature it is most straightforward to perform the analysis of Method 2.25 separately for each sample and then appropriately combine results to give the overall estimates of σ_A^2, σ_B^2, ... suggested by (eq. 2.6.109).

To estimate σ_S^2, the error variance due to sampling of the material in question, the variance among the sample means is calculated, equated to its expected value, and the equation solved for σ_S^2.

Examples

EXAMPLE 2.26 (a)

Two distinct and identified samples of Pu oxide are distributed to a number of laboratories for analysis of percent plutonium by a given analytical method. The data, in percent Pu minus 86 are tabled. Note, by comparison with the Example 2.25 (a), that there is now no "time" effect. As a result, there are only three levels in the nested model.

Sample 1

Lab	1	1	1	2	2	2
Aliquot	1	2	3	1	2	3
	.188	.348	.388	.157	1.007	.837
	.178	.298	.338	.847	.837	.657
	.158	.368	.398			
	.198	.298	.388			

Lab	3	3	4	4	4	5
Aliquot	1	2	1	2	3	1
	.651	.581	.451	.545	.586	.888
	.611	.641	.530	.608	.702	.720
	.591	.581	.625	.685	.730	.814

Lab	5	5	6	6	6	7
Aliquot	2	3	1	2	3	1
	.377	.661	.885	-.063	.645	.515
	.383	.651	.645	.625	.615	.635
	.388	.394	.366	1.094	.475	.555

Lab	7	7	8	8	8
Aliquot	2	3	1	2	3
	.645	.825	.677	.698	.346
	.645	.715	.492	.532	.456
	.615	.725	.640	.580	.466
			.636	.526	.617
			.645	.608	.656

Sample 2

Lab	1	1	1	2	2	2
Aliquot	1	2	3	1	2	3
	.318	.048	.128	.580	.730	.680
	.138	.228	.398	.660	.890	.110
	.388	.098	.198	.610		.890
	.158	.218	.168			

Lab	3	3	3	4	4	5
Aliquot	1	2	3	1	2	1
	.587	.507	.587	.690	.662	.570
	.567	.457	.587	.683	.546	.542
	.547	.467	.487	.793	.596	.828

Lab	5	5	6	6	6	7
Aliquot	2	3	1	2	3	1
	.867	.503	.380	.400	.440	.478
	.390	.242	.170	.610	.380	.698
	.782	.622	.510	.370	.450	.328

Lab	7	7
Aliquot	2	3
	.418	.158
	.548	.318
	.488	.458

Since the analysis for each of the two samples is so similar to that demonstrated in Example 2.25 (a), the only difference being that now there are three levels rather than four, the calculations are not all indicated. Rather, (eq. 2.6.109) is shown for each of the two samples, and for their sum.

Sample 1:

$$1.127201 = 7\ (\hat{\sigma}_C^2+3.217119\hat{\sigma}_B^2+9.257067\hat{\sigma}_A^2)$$

$$0.659001 = 15\ (\hat{\sigma}_C^2+3.266685\hat{\sigma}_B^2)$$

$$1.325515 = 52\ \hat{\sigma}_C^2$$

Sample 2:

$$1.506556 = 6\ (\hat{\sigma}_C^2+3.098098\hat{\sigma}_B^2+8.806384\hat{\sigma}_A^2)$$

$$0.308824 = 13\ (\hat{\sigma}_C^2+3.096190\hat{\sigma}_B^2)$$

$$0.950346 = 42\ \hat{\sigma}_C^2$$

Summed over both samples:

$$2.633757 = 13\hat{\sigma}_C^2 + 41.108421\hat{\sigma}_B^2 + 117.637773\hat{\sigma}_A^2$$

$$0.967825 = 28\hat{\sigma}_C^2 + 89.250745\hat{\sigma}_B^2$$

$$2.275861 = 94\hat{\sigma}_C^2$$

Solving these:

$\hat{\sigma}_C^2 = 0.024211$ (Replicates)

$\hat{\sigma}_B^2 = 0.003248$ (Aliquots)

$\hat{\sigma}_A^2 = 0.018578$ (Laboratories)

Proceeding to find the estimate of σ_S^2, the two sample means are

$\bar{y}_1 = 0.5603$ $\qquad$ $\bar{y}_2 = 0.4733$

from which $\quad s_{\bar{y}}^2 = 0.003785$

From (eq. 2.6.110),

$$V_1 = \sigma_S^2 + 0.003000$$

$$V_2 = \sigma_S^2 + 0.003301$$

Thus, from (eq. 2.6.111),

$$\hat{\sigma}_S^2 = 0.003785 - 0.006301/2$$

$= 0.000635$ (Sampling)

2.6.5.3 Laboratory-Dependent Random Error

For the case in which several samples of a given material are distributed to a number of laboratories, it may be appropriate to apply a Grubbs' analysis as described in 2.6.3.4, or the analysis described in 2.6.3.5 if the assumptions underlying the Grubbs' analysis are not valid. The main advantage of the Grubbs' analysis is that it provides estimates of the random error variance for each laboratory. It is not required that all have the same random error variances, an assumption implicit in Methods 2.25 and 2.26. Further, the estimate of the error may be more realistic than one based on observing the scatter among reported results.

2.6.5.4 Distribution of Inspection Samples to Several Laboratories

A problem of special interest to Agency safeguards is that concerned with the distribution of inspection samples to more than one laboratory. The effect of this action on statistical inference is discussed in the next chapter. In this section, the problem of analyzing the inspection data to make inferences about errors of measurement is considered.

There are, of course, any number of ways in which samples may be distributed. Three representative distribution plans are treated here. For variations on these plans, competent statistical advice should be sought.

For each plan, n items of a given material are selected for analysis, and there are L laboratories to perform the analysis. There are two reported results per sampled item so that the total number of observations is 2n for each plan. The plans under consideration are:

Plan 1: n/L samples are sent to each of the L laboratories. Each laboratory performs duplicate analyses on each sampled item.

Plan 2: The n samples are each split into two parts or subsamples. There are only L=2 laboratories, each of which makes single analyses on each of the n paired subsamples.

Plan 3: Same as Plan 2 except that there are L(L-1)/2 pairs of laboratories, each pair of which are sent 2n/L(L-1) pairs of subsamples. Each laboratory makes a single analysis on each subsample.

Plan 1 will be covered by Method 2.27, Plan 2 by Method 2.28, and Plan 3 by Method 2.29. First, however, the mathematical model underlying all three plans is presented. The model is written to include components that may not be estimable with each plan, except perhaps in combination. The assumed model is:

$$x_{ijk} = \mu+\theta+\beta_i+\eta_j+\omega_{ijk} \qquad \text{(eq. 2.6.112)}$$

where

x_{ijk} = measured result for lab i, sample j, analytical determination k

μ = true average result for population being sampled

θ = overall deviation, or bias, common to all labs

β_i = deviation due to lab i

η_j = deviation due to sample j

ω_{ijk} = deviation for determination k, lab i, sample j

Some important points are noted from this model.

(1) It is not possible to obtain separate information about μ and θ from the inspection samples alone; only the net effect of both parameters can be studied.

(For example, the labs in total could all be biased high by, say, 0.1%, but this could not be detected unless known standards were used.)

(2) β_i, called the lab effect, should be thought of as the sum of two effects. One is the deviation for lab i averaged over all time frames, and the other is the deviation due to the particular time frame existing for the data in question. It is assumed that all inspection samples in question are analyzed in the same time frame. Thus, β_i is really the algebraic sum of the lab effect and the time-within-lab effect. It is assumed that β_i is a random variable with zero mean and variance σ_β^2.

(3) η_j is a random variable with zero mean and variance σ_η^2. This describes the variance between sample values, or the item to item variance.

(4) ω_{ijk} is a random variable with zero mean and variance σ_ω^2, called the analytical error.

(5) As a variation on the model that is not generally identified, it is recognized that a correlation can exist between replicate analytical results. The quantity σ_ω^2 is intended to represent the net effect of all factors normally operating within a laboratory in a given time frame. If, for a set of observations, some conditions that are normally permitted to vary are held constant, then the resulting estimate of σ_ω^2 will be biased low. To accommodate this possibility in the model, write the covariance between two results within lab i as

$$E(\omega_{ijk},\omega_{ij\ell}) = \rho_i\sigma_\omega^2 \qquad \text{(eq. 2.6.113)}$$

This representation is not wholly realistic or satisfactory, but it will serve to keep in mind at least in a semi-quantitative way what quantities are estimable with the different sample distribution plans.

Method 2.27

Under Plan 1, n/L samples are sent to each of the L laboratories with duplicate analyses performed on each sampled item.

Notation

The notation is given by (eq. 2.6.112) and following.

Results

The data are analyzed by a nested analysis of variance, but the analysis is simpler than by Method 2.25 because of the balance in the design. The estimating equations are

$$\left.\begin{aligned} M_1 &= 2n\hat{\sigma}_\beta^2/L + 2\hat{\sigma}_\eta^2 + \hat{\sigma}_\omega^2\,(L+\sum_i \rho_i)/L \\ M_2 &= 2\hat{\sigma}_\eta^2 + \hat{\sigma}_\omega^2\,(L+\sum_i \rho_i)/L \\ M_3 &= \hat{\sigma}_\omega^2\,(L-\sum_i \rho_i)/L \end{aligned}\right\} \qquad \text{(eq. 2.6.114)}$$

The M's are calculated from the data as follows. Calculate

$$Q_{ij} = \sum_k x_{ijk} \qquad T_i = \sum_j Q_{ij}$$

$$T = \sum_i T_i$$

$$S_0 = T^2/2n \qquad S_1 = L \sum_i T_i^2/2n$$

$$S_2 = \sum_i \sum_j Q_{ij}^2/2 \qquad S_3 = \sum_i \sum_j \sum_k x_{ijk}^2$$

The M's of (eq. 2.6.114) are calculated from

$$M_1 = (S_1-S_0)/(L-1)$$

$$M_2 = (S_2-S_1)/(n-L)$$

$$M_3 = (S_3-S_2)/n$$

From the first two equations of (eq. 2.6.114), it is evident that σ_β^2 is estimated by

$$\hat{\sigma}_\beta^2 = (M_1-M_2)L/2n \qquad \text{(eq. 2.6.115)}$$

From the equations for M_2 and M_3, it is also evident that σ_ω^2 and σ_η^2 can only be estimated if

$$\sum_i \rho_i = 0 \quad .$$

If $\sum_i \rho_i$

is positive, M_3 will tend to overestimate σ_ω^2, while σ_η^2 will tend to be underestimated. However, no matter what the size of

$$\sum_i \rho_i \quad ,$$

the sum of σ_η^2 and σ_ω^2 is estimated by $(M_2+M_3)/2$.

On the other hand, if the sampling error, σ_η^2, were known to be negligibly small relative to σ_ω^2, then

$$\hat{\sigma}_\omega^2 = (M_2+M_3)/2 \qquad \text{(eq. 2.6.116)}$$

and

$$\sum_i \rho_i = L(M_2-M_3)/(M_2+M_3) \qquad \text{(eq. 2.6.117)}$$

If ρ_i is to be estimated separately for each laboratory, then the data may be analyzed separately for each laboratory, as the example will illustrate.

The (eq. 2.6.114) are very instructive in pointing out just what combinations of parameters are being estimated by each statistic.

Basis

Given the model of (eq. 2.6.112) and the expressions for M_1, M_2, and M_3, the estimating equations (eq. 2.6.114) are found by equating the expected values of M_1, M_2, and M_3 to their respective observed values.

Examples

EXAMPLE 2.27 (a)

In an inspection, 24 sintered UO_2 pellets are sampled. Eight pellets are sent to each of labs 1, 2, and 3, and each lab performs duplicate analyses. The data, in percent U minus 80, are given below.

Lab 1		Lab 2		Lab 3	
Pellet	%U-80	Pellet	%U-80	Pellet	%U-80
1	8.056, 7.992	9	7.939, 8.107	17	8.144, 8.122
2	8.088, 7.999	10	7.883, 7.970	18	8.240, 8.279
3	8.044, 8.026	11	8.005, 7.923	19	8.132, 8.054
4	8.015, 8.117	12	8.064, 8.119	20	8.233, 8.266
5	7.897, 7.825	13	8.001, 7.922	21	8.079, 8.127
6	8.039, 8.099	14	7.977, 7.982	22	8.102, 8.130
7	7.950, 7.881	15	7.881, 7.921	23	7.977, 7.837
8	8.113, 8.068	16	7.946, 8.023	24	8.105, 8.048

The design parameter values are

$$n = 24 \qquad L = 3$$

The calculated quantities are:

$Q_{11} = 16.048$	$Q_{29} = 16.046$	$Q_{3,17} = 16.266$
$Q_{12} = 16.087$	$Q_{2,10} = 15.853$	$Q_{3,18} = 16.519$
$Q_{13} = 16.070$	$Q_{2,11} = 15.928$	$Q_{3,19} = 16.186$
$Q_{14} = 16.132$	$Q_{2,12} = 16.183$	$Q_{3,20} = 16.499$
$Q_{15} = 15.722$	$Q_{2,13} = 15.923$	$Q_{3,21} = 16.206$
$Q_{16} = 16.138$	$Q_{2,14} = 15.959$	$Q_{3,22} = 16.232$
$Q_{17} = 15.831$	$Q_{2,15} = 15.802$	$Q_{3,23} = 15.814$
$Q_{18} = 16.181$	$Q_{2,16} = 15.969$	$Q_{3,24} = 16.153$

$T_1 = 128.209$ $T_2 = 127.663$ $T_3 = 129.875$

$T = 385.747$

$S_0 = 3100.015583$ $S_1 = 3100.181554$

$S_2 = 3100.491001$ $S_3 = 3100.557383$

The M's are calculated from these S values, and the estimating equations become:

$$0.082986 = 16\sigma_\beta^2 + 2\sigma_\eta^2 + \sigma_\omega^2 (1+\bar{\rho})$$

$$0.014736 = 2\sigma_\eta^2 + \sigma_\omega^2 (1+\bar{\rho})$$

$$0.002766 = \sigma_\omega^2 (1-\bar{\rho})$$

where $\bar{\rho} = (\rho_1+\rho_2+\rho_3)/3$

From the first two equations,

$$\hat{\sigma}_\beta^2 = (.082986 - .014736)/16 = 0.004266$$

If one can assume that $\bar{\rho} = 0$, then

$$\hat{\sigma}_\omega^2 = 0.002766$$

$$\hat{\sigma}_\eta^2 = (.014736 - .002766)/2 = 0.005985$$

Regardless of the value of $\bar{\rho}$, the sum of the two variance components is

$$\hat{\sigma}_\omega^2+\hat{\sigma}_\eta^2 = (.014736 + .002766)/2 = 0.008751$$

The sampling error for percent U in uranium pellets should be very small. If $\sigma_\eta = 0$, then from the last two of the estimating equations,

$$.014736 = \sigma_\omega^2(1+\bar{\rho})$$
$$.002766 = \sigma_\omega^2(1-\bar{\rho})$$

which gives $\bar{\rho} = 0.684$ (large positive correlation)

$$\hat{\sigma}_\omega^2 = 0.008751$$

The correlation coefficient is not likely to be the same for all three laboratories. The parameters ρ_1, ρ_2, and ρ_3 can be estimated separately for each lab by setting L=1 in Method 2.27 and using only the last two of the estimating equations (eq. 2.6.114). The summary results are, again assuming $\sigma_2^2=0$:

Lab 1 $\quad .013171 = \sigma_\omega^2(1+\rho_1) \qquad \hat{\sigma}_\omega^2 = 0.007783$

$\qquad\quad .002395 = \sigma_\omega^2(1-\rho_1) \qquad \hat{\rho}_1 = 0.692$

Lab 2 $\quad .006856 = \sigma_\omega^2(1+\rho_2) \qquad \hat{\sigma}_\omega^2 = 0.005282$

$\qquad\quad .003708 = \sigma_\omega^2(1-\rho_2) \qquad \hat{\rho}_2 = 0.298$

Lab 3 $\quad \hat{\sigma}_\omega^2 = 0.013188 \quad ; \quad \hat{\rho}_3 = 0.834$

Method 2.28

Under Plan 2, the n samples are each split into two subsamples. L=2, and each lot makes a single analysis on each of the n subsamples.

Notation

The notation is given by (eq. 2.6.112) and following.

Results

The data are analyzed by the method of Grubbs (see 2.6.3.3). In order to apply this, it is necessary that the sampling error variance, σ_η^2, be small relative to the analytical error variance, σ_ω^2. However, for large σ_η^2, the estimate of σ_ω^2 is meaningful if, in fact, σ_ω^2 is the same for both laboratories. The correlation coefficient, ρ_i, does not enter into the analysis since replicate measurements are not made.

Compute V_1 and V_2, the sample variances among the n values for labs 1 and 2 respectively. Also compute V_{12}, the sample covariance for the n pairs of values. Then, the estimating equations are:

$$\hat{\sigma}_\eta^2 = V_{12} \qquad \text{(eq. 2.6.118)}$$

$$\left.\begin{aligned} \hat{\sigma}_{\omega 1}^2 &= V_1 - V_{12} \\ \hat{\sigma}_{\omega 2}^2 &= V_2 - V_{12} \end{aligned}\right\} \qquad \text{(eq. 2.6.119)}$$

(Eq. 2.6.119) provides separate estimates of σ_ω^2 for each lab. If both labs have the same value of σ_ω^2, then this is estimated by

$$\hat{\sigma}_\omega^2 = (V_1+V_2-2V_{12})/2 \qquad \text{(eq. 2.6.120)}$$

The variance between labs, σ_β^2, is estimated by

$$\hat{\sigma}_\beta^2 = [(T_1-T_2)^2/n - (\hat{\sigma}_{\omega 1}^2 + \hat{\sigma}_{\omega 2}^2)]/2n \qquad \text{(eq. 2.6.121)}$$

where T_i is the total of all observations for Lab i.

Basis

The basis is the same as for Methods 2.18 and 2.20.

Examples

EXAMPLE 2.28 (a)

The 24 sintered pellets of Example 2.27 (a), were each split into two parts with one part of each pellet sent to Lab 1 and the other to Lab 2. The data are tabled below, all values being in (% U-80).

Pellet	Lab 1	Lab 2	Pellet	Lab 1	Lab 2	Pellet	Lab 1	Lab 2
1	7.985	8.025	9	8.139	7.992	17	8.042	7.868
2	7.862	7.975	10	8.051	7.893	18	8.034	8.057
3	8.013	7.938	11	7.940	8.019	19	7.981	8.016
4	8.061	8.071	12	7.918	7.859	20	7.917	7.947
5	8.016	7.862	13	8.204	7.959	21	7.913	7.933
6	7.857	7.866	14	8.059	7.842	22	7.999	7.992
7	8.036	8.010	15	8.044	7.992	23	7.923	8.194
8	8.170	7.974	16	7.948	8.022	24	8.033	8.047

Then,

$$V_1 = 0.007823 \qquad V_2 = 0.006850 \qquad V_{12} = 0.000044$$

From (eq. 2.6.118),

$$\hat{\sigma}_\eta^2 = 0.000044$$

From (eq. 2.6.119),

$$\hat{\sigma}_{\omega 1}^2 = 0.007779$$

$$\hat{\sigma}_{\omega 2}^2 = 0.006806$$

From (eq. 2.6.121), where $T_1 = 192.145$ and $T_2 = 191.353$

$$\hat{\sigma}_\beta^2 = (.026136 - .014585)/48 = 0.000241$$

Plan 2 involves only the two laboratories. When sampling errors are not small relative to analytical errors, and when separate estimates of the analytical error variance for the laboratories are desired, the paired subsamples should be distributed to more than two labs according to the Plan 3 distribution plan.

Method 2.29

Notation

The notation is given by (eq. 2.6.112) and following.

Results

To estimate $\sigma^2_{\omega i}$, follow Method 2.21, where N of 2.21 is the same as L, and where σ^2_i is the same as $\sigma^2_{\omega i}$.

The sampling variance, σ^2_η, is estimated by

$$\hat{\sigma}^2_\eta = \sum_{i=1}^{L} (s^2_i - \hat{\sigma}^2_{\omega i})/L \qquad \text{(eq. 2.6.122)}$$

where s^2_i is the variance among the measured values of Lab i.

To estimate the between lab variance, σ^2_β, calculate the means for each of the L(L-1)/2 columns of differences. Calling this mean $\bar{x}_{ij}$ for Labs i and j, then σ^2_β is estimated by

$$\hat{\sigma}^2_\beta = \sum_i \sum_j \bar{x}^2_{ij}/L(L-1) - (L-1)\sum_i \hat{\sigma}^2_{\omega i}/2n \qquad \text{(eq. 2.6.123)}$$

Basis

The basis for Method 2.21 which provides the estimates of $\sigma^2_{\omega i}$ was given earlier, under that method.

For (eq. 2.6.122) and (eq. 2.6.123), the expected values of Σs^2_i and of $\Sigma\Sigma\bar{x}^2_{ij}$ respectively are found and equated to the observed values of these statistics. The equations are then solved for the parameters to be estimated, σ^2_η and σ^2_β respectively. The quantity $\sigma^2_{\omega i}$ is replaced by its estimate for each lab i.

Examples

EXAMPLE 2.29 (a)

The 24 sintered pellets of Example 2.27 (a) were each split into two pairs. The 24 pairs were distributed among 4 labs, with each pair of labs receiving 4 pairs. The data, expressed as (% U-80) are tabled.

Pellet	Lab 1	Lab 2	Pellet	Lab 1	Lab 4	Pellet	Lab 2	Lab 4
1	8.030	8.048	9	7.839	8.088	17	7.986	8.140
2	8.062	7.954	10	7.932	8.217	18	7.785	8.165
3	7.994	7.981	11	7.970	8.064	19	8.009	8.061
4	8.036	7.988	12	8.020	8.130	20	8.015	8.048
	Lab 1	Lab 3		Lab 2	Lab 3		Lab 3	Lab 4
5	7.931	8.096	13	7.909	8.046	21	8.137	7.981
6	8.032	7.990	14	7.934	8.069	22	7.998	8.038
7	8.187	8.033	15	7.927	8.092	23	8.108	8.120
8	8.009	8.196	16	8.009	7.999	24	8.022	8.058

Here, $n = 24$ and $L = 4$

First, follow Method 2.21. The calculated differences are not shown, but the summarizing statistics are:

$v_{12} = 0.002920$ $\quad v_{23} = 0.006245$

$v_{13} = 0.027197$ $\quad v_{24} = 0.025373$

$v_{14} = 0.009334$ $\quad v_{34} = 0.008740$

Then, by (eq. 2.6.94),

$S_1 = 0.039451$ $\quad S_3 = 0.042182$ $\quad V = 0.079809$

$S_2 = 0.034538$ $\quad S_4 = 0.043447$

By (eq. 2.6.95),

$$\hat{\sigma}^2_{\omega 1} = (.039451)/2 - 0.079809/6 = 0.006424$$

$$\hat{\sigma}^2_{\omega 2} = 0.003968$$

$$\hat{\sigma}^2_{\omega 3} = 0.007790$$

$$\hat{\sigma}^2_{\omega 4} = 0.008422$$

To estimate σ^2_η by (eq. 2.6.122), first calculate s^2_i for each lab. The results are

$s_1{}^2 = 0.007146$ $\quad s_3{}^2 = 0.003964$

$s_2{}^2 = 0.004768$ $\quad s_4{}^2 = 0.004143$

Thus,

$$\hat{\sigma}_\eta^2 = (.000722 + .000800 - .003826 - .004279)/4$$

$$= -0.001646 \qquad \text{(call it 0)}$$

To estimate σ_β^2 by (eq. 2.6.123), first calculate the 6 mean differences.

$$\bar{x}_{12} = 0.03775 \qquad \bar{x}_{23} = -0.10675$$

$$\bar{x}_{13} = -0.03900 \qquad \bar{x}_{24} = -0.15475$$

$$\bar{x}_{14} = -0.18450 \qquad \bar{x}_{34} = 0.01700$$

Then,

$$\hat{\sigma}_\beta^2 = (.073618)/12 - 3\ (.026604)/48$$

$$= 0.004389$$

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Chapter 3

ERROR PROPAGATION

3.1 DEFINITION OF ERROR PROPAGATION

In the Safeguards Dictionary [3.1] prepared by the Brookhaven National Laboratory for the then United States Atomic Energy Commission, error propagation is defined as follows: "The determination of the value to be assigned as the uncertainty of a given quantity using mathematical formulae for the combination of measurement errors. Error propagation involves many considerations and the choice of formulae for computing the uncertainty depends upon the functional relations of the measurement parameters involved."

This definition is made with respect to safeguards applications since it speaks of combining measurement errors. Nevertheless, the definition accurately identifies error propagation as a procedure based on using mathematical formulae. It is consistent with another quote taken from an expository paper on error propagation by Birge [3.2], "The subject of the propagation of errors . . . is a purely mathematical matter, with very definite and easily ascertained conclusions." Although the practitioner might quarrel with the words "easily ascertained" there can be no quarrel with the principal point in both quotations, namely, error propagation is a mathematical exercise and hence leads to precise and well-defined procedures.

Although the error propagation procedures may be exact, this does not mean that the net effect of the propagated errors is exactly determined. The error propagation formulas draw a mathematically precise line from some model to the conclusions; any inexactness in an answer derived from error propagation comes not from drawing this mathematically precise line, (assuming that this line is drawn correctly), but rather from the starting point, that is, from the mathematical model. This underscores the importance of the model insofar as it corresponds to a valid description of reality.

The general error propagation problem may be formulated as follows. Let a random variable of interest (such as the MUF or material unaccounted for) be written as a specific function of a number of other random variables as in (eq. 3.1.1).

$$y = f(x_1, x_2, \ldots, x_k) \qquad \text{(eq. 3.1.1)}$$

The precise form of this function is known in the error propagation problem, that is, the problem is not to estimate the parameters of the function (which is a statistical problem), but is rather to find the uncertainty in y as a function of the uncertainties in the x's (a mathematical problem) given that the function is well defined. In the section to follow, error propagation is considered when f is a linear function, and the more general case for the nonlinear function is treated in Section 3.3.

3.2 ERROR PROPAGATION; ADDITIVE MODEL

Error propagation for the additive or linear model is covered by Method 3.1.

Method 3.1

Notation

Let x_i = value for the i-th random variable

a_i = i-th constant; i=1, 2, ..., k

$\mu_i = E(x_i)$, mean of x_i

$\sigma_i^2 = E(x_i-\mu_i)^2$, variance of x_i

$\sigma_{ij} = E[(x_i-\mu_i)(x_j-\mu_j)]$, covariance between x_i and x_j

y = value for response or dependent variable

$\mu_y = E(y)$, mean of y

$\sigma_y^2 = E(y-\mu_y)^2$, variance of y

Model

$$y = \sum_{i=1}^{k} a_i x_i \qquad \text{(eq. 3.2.1)}$$

Results

$$\mu_y = \sum_{i=1}^{k} a_i \mu_i \qquad \text{(eq. 3.2.2)}$$

$$\sigma_y^2 = \sum_{i=1}^{k} a_i^2 \sigma_i^2 + 2 \sum_{i=1}^{k-1} \sum_{j>i} a_i a_j \sigma_{ij} \qquad \text{(eq. 3.2.3)}$$

Basis

The formulas given by (eq. 3.2.2) and (eq. 3.2.3) are basic results found in most texts. See, for example, reference [3.3].

Examples

EXAMPLE 3.1 (a)

Consider a number of observations x_i, x_2, ..., x_k and calculate the sample mean or average, denoted by y

$$y = (x_1+x_2+\ldots+x_k)/k$$

If $E(x_i) = \mu$ and $E(x_i-\mu)^2 = \sigma^2$ for all i, with $E(x_i-\mu)(x_j-\mu) = 0$ for $i \neq j$, then application of (eq. 3.2.2) and (eq. 3.2.3), with $a_i = {}^1/k$ for all i, gives

$$E(y) = k\mu/k = \mu$$
$$\sigma_y^2 = k\sigma^2/k^2 = \sigma^2/k$$

Note: y, the sample average or mean, is usually denoted by $\bar{x}$. This example shows that the mean of $\bar{x}$ is μ and its variance is σ^2/k for the simple model considered here.

EXAMPLE 3.1 (b)

Some 19 UF_6 cylinders in a shipment are weighed over a period of four days. The model for a single weighing is written:

$$x_{i(j)} = \mu_{i(j)} + \delta + \theta_i + \varepsilon_{i(j)}$$

where

$x_{i(j)}$ = observed gross weight for cylinder j weighed on day i

$\mu_{i(j)}$ = corresponding true gross weight

δ = scale bias or systematic error

θ_i = error affecting all cylinders weighed on day i

$\varepsilon_{i(j)}$ = random error in weighing for cylinder j, day i

Assume that

$$E(\delta) = E(\theta_i) = E(\varepsilon_{i(j)}) = 0$$
$$E(\delta^2) = \sigma_\delta^2 \;;\; E(\theta_i^2) = \sigma_\theta^2 \;;\; E(\varepsilon_{i(j)}^2) = \sigma_\varepsilon^2$$
$$\sigma_\delta = 0.6 \text{ lbs} \;;\; \sigma_\theta = 1.2 \text{ lbs.} \;;\; \sigma_\varepsilon = 1.6 \text{ lbs.}$$

All covariances are zero.

Of the 19 cylinders, 4 are weighed on day 1, 3 on day 2, 7 on day 3, and 5 on day 4. Find the variance of the total observed weight.

Here,

$$\begin{aligned} y &= \Sigma x_{i(j)} \\ &= \Sigma\mu_{i(j)} + 19\delta + 4\theta_1 + 3\theta_2 + 7\theta_3 + 5\theta_4 + \Sigma\varepsilon_{i(j)} \end{aligned}$$

There are 19 terms in each indicated sum. Then, by (eq. 3.2.3), the variance of the total observed weight is

$$\sigma_y^2 = 361\sigma_\delta^2 + (16+9+49+25)\sigma_\theta^2 + 19\sigma_\varepsilon^2$$

$$= (361)(0.36) + (99)(1.44) + (19)(2.56)$$

$$= 321.16 \text{ lbs}^2$$

$$\sigma_y = 17.9 \text{ lbs}$$

3.3 GENERAL ERROR PROPAGATION; TAYLOR'S SERIES

The key equations, (eq. 3.2.2) and (eq. 3.2.3), apply only if the function indicated by (eq. 3.1.1) is linear. Although linear models are adequate approximations to reality in many applications, they cannot, of course, be expected to apply universally. In the area of safeguards applications, in fact, nonlinear models are frequently encountered, as was discussed in Section 2.3.2.

For nonlinear models, errors are propagated using an approximation based on Taylor's series. The method follows.

Method 3.2

Notation

The notation is the same as for Method 3.1, except that a_i is not defined, except in particular applications.

Model

In general terms, the model is schematically indicated by (eq. 3.1.1). A specific model must be written for each application.

Results

The mean of y is given approximately by

$$\mu_y \approx f(\mu_1, \mu_2, \ldots, \mu_k) \qquad \text{(eq. 3.3.1)}$$

The approximation to the variance of y is

$$\sigma_y^2 \approx \sum_{i=1}^{k} b_i^2\sigma_i^2 + 2\sum_{i=1}^{k-1}\sum_{j>i} b_i b_j \sigma_{ij} \qquad \text{(eq. 3.3.2)}$$

where $b_i = \frac{\partial b}{\partial x_i}$, evaluated at μ_i for all i

Note the similarity between the forms of (eq. 3.3.2) and (eq. 3.2.3).

Basis

The approximations given by (eq. 3.3.1) and (eq. 3.3.2) are based on approximating the function (eq. 3.1.1) by the linear terms of a Taylor's series approximation [3.4]. This approximation is

$$f(x_1,x_2,\ldots,x_k) \approx f(\mu_1,\mu_2,\ldots,\mu_k) + \sum_{i=1}^{k} \left(\frac{\partial f}{\partial x_i}\right) (x_i-\mu_i) \qquad \text{(eq. 3.3.3)}$$

Since $f(\mu_1,\mu_2,\ldots,\mu_k)$ is a constant, (eq. 3.2.3) can be applied immediately to (eq. 3.3.3), giving the result in (eq. 3.3.2). The transition from (eq. 3.3.3) to (eq. 3.3.2) is an exact one. The approximation comes about by (eq. 3.3.3) in which only the linear terms of Taylor's series are included. For most safeguards applications that are routinely encountered, this is an adequate approximation. Its adequacy depends on errors being "small" in a relative sense, e.g., smaller than 5%-10% relative. If there is concern about the adequacy of a result based on Taylor's series approximation in a given instance, statistical guidance should be sought.

Examples

EXAMPLE 3.2 (a)

In connection with Method 2.5 dealing with linear calibration, it was indicated that certain results were based on error propagations methods to be discussed in Chapter 3. With the tools now in hand, the results of interest can now be derived.

From eq. (2.6.21),

$$x_0 = (y_0-\alpha)/\hat{\beta}$$

where α is a constant, y_0 has variance estimated by $\hat{\sigma}^2$, and $\hat{\beta}$ has variance denoted by $V(\hat{\beta})$. Then, to apply (eq. 3.3.2), find the appropriate partial derivatives:

$$b_1 = \frac{\partial x_0}{\partial y_0} = \frac{1}{\hat{\beta}}$$

$$b_2 = \frac{\partial x_0}{\partial \hat{\beta}} = \frac{-(y_0-\alpha)}{\hat{\beta}^2}$$

Therefore, from (eq. 3.3.2), and keeping in mind that the covariance between y_0 and $\hat{\beta}$ is zero, the approximate variance of x_0 is estimated by

$$V(x_0) \approx \hat{\sigma}^2/\hat{\beta}^2 + (y_0-\alpha)^2V(\hat{\beta})/\hat{\beta}^4$$

In Method 2.5, the first term in this expression was regarded as the random error variance, and the second term, the systematic error variance.

EXAMPLE 3.2 (b)

In Method 2.14 dealing with nonlinear calibration, (eq. 2.6.76) gave the following expression for x_0 as a function of the random variables $\hat{\alpha}$ and $\hat{\beta}$. (Treat y_0 as a constant when focussing attention on the systematic error variance.)

$$x_0 = -\hat{\alpha}(1- \sqrt{1+4\hat{\beta}y_0/\hat{\alpha}^2})/2\hat{\beta}$$

$$= [-\hat{\alpha}+(\hat{\alpha}^2+4\hat{\beta}y_0)^{1/2}]/2\hat{\beta}$$

To apply (eq. 3.3.2),

$$b_1 = \frac{\partial x_0}{\partial\hat{\alpha}} = [-1+\hat{\alpha}(\hat{\alpha}^2+4\hat{\beta}y_0)^{-1/2}]/2\hat{\beta}$$

$$= (\hat{\alpha}-R)/2\hat{\beta}R$$

where $\quad R = (\hat{\alpha}^2+4\hat{\beta}y_0)^{1/2}$

Also,

$$b_2 = \frac{\partial x_0}{\partial\hat{\beta}} = \frac{2\hat{\beta}(2y_0R^{-1}) - (-\hat{\alpha}+R)2}{4\hat{\beta}^2}$$

$$= [2\hat{\beta}y_0+R(\hat{\alpha}-R)]/2\hat{\beta}^2R$$

It is noted that $\hat{\alpha}-R = -2\hat{\beta}x_0$ so the expressions for b_1 and b_2 reduce to

$$b_1 = -x_0/R$$

$$b_2 = (y_0-Rx_0)/\hat{\beta}R$$

where $\quad R = (\hat{\alpha}^2+4\hat{\beta}y_0)^{1/2} = \hat{\alpha}+2\hat{\beta}x_0$

Then, applying (eq. 3.3.2), and letting $V(\hat{\alpha})$ and $V(\hat{\beta})$ denote the variances of $\hat{\alpha}$ and of $\hat{\beta}$ respectively, with $CV(\hat{\alpha},\hat{\beta})$ the covariance between them, the result is

$$V_s(x_0) = (x_0^2/R^2)V(\hat{\alpha})+[(y_0-Rx_0)^2/\hat{\beta}^2R^2]V(\hat{\beta})$$

$$- 2x_0(y_0-Rx_0)/\hat{\beta}R^2\ CV(\hat{\alpha},\hat{\beta})$$

But since $y_0-Rx_0 = \beta x_0^2+\hat{\alpha}x_0-\hat{\alpha}x_0-2\hat{\beta}x_0^2$

$$= -\hat{\beta}x_0^2 \quad ,$$

the expression for the systematic error variance of x_0, $v_s(x_0)$ reduces to

$$V_s(x_0) = (x_0^2/R^2)[V(\hat{\alpha})+x_0^2V(\hat{\beta}) + 2x_0CV(\hat{\alpha},\hat{\beta})]$$

which is the result, (eq. 2.6.80).

Other applications of error propagation will be encountered later in this volume.

3.4 CALCULATION OF VARIANCE OF MUF

3.4.1 Definition of MUF

The word, MUF, is an acronym for material unaccounted for. It is defined as the difference between the book inventory and the physical inventory. This definition may be with respect to either the element or isotope weight.

The facility MUF for a given material balance (or accounting) period is a measure of the performance of the facility with respect to its control of the nuclear materials involved. The MUF, as verified by inspection, or alternately, as adjusted on the basis of inspection results, is the key index of performance used by the Agency in its quantitative assessment of facility performance.

The MUF calculation is represented schematically by the following equation:

$$MUF = I - O + B - E \qquad \text{(eq. 3.4.1)}$$

where I designates inputs, O designates outputs (which are sometimes subdivided into product and waste streams), B refers to beginning inventory, and E to ending inventory. The three terms in (eq. 3.4.1), I, O, and B, collectively represent the book inventory, while E represents the physical inventory. Note that the physical inventory for one accounting period becomes a part of the book inventory for the subsequent period.

The definition of MUF implicity assumes that the material balance is based completely on measured data. The use of by-difference accounting results in a meaningless MUF. For example, if the contents of waste streams are calculated as the differences between the measured amounts entering a process step and those exiting the step, it is clear that the calculated MUF would be zero over that particular material balance area, i.e., it would be meaningless as a performance index.

In order to judge the significance of a given MUF, either as verified or as adjusted by the inspection results, it is necessary to calculate its variance. A MUF is affected by many factors. For example, in taking inventory, if a measured value is improperly recorded, i.e., if a mistake is made in recording the value in question, this will affect the MUF. Such a mistake will not, however, affect the variance of MUF as will be defined here, for the variance of MUF is calculated to include only the uncertainties arising from the measurement process under the assumption that this process is functioning properly.

In this connection, it may be helpful to make a distinction between an observed MUF and a true MUF. The true MUF is the actual amount of material unaccounted for, excluding the effects of errors of measurement. It includes the effects of unmeasured inventories, process losses, recording mistakes, plus any diverted material. The observed MUF is a random variable whose expected value is the true MUF. In the absence of any errors of measurement, the observed MUF is identical to the true MUF. Equivalently stated, the variance of the (observed) MUF is zero in this case.

Clearly, an observed MUF together with its calculated variance are not sufficient information on which to make a judgement as to whether or not material has been diverted. Other information must be brought to bear to make such a judgement. However, it should also be clear that if the calculated variance of MUF is excessively large due to a poor measurement system, then one can never hope to carry the MUF evaluation beyond this first stage; any possible diversion would be quite obscurred by large errors of measurement and no reasonable judgement about diverted material could be made.

In the next section, it will be indicated how the variance of a given MUF can be calculated exactly by application of the error propagation formulas already given. Following that, a general approach to calculating the variance of MUF under specified rather non-restrictive assumptions will be considered.

3.4.2 Direct Application of Error Propagation Formulas

If a given calculated MUF is based on a simple model, then either Method 3.1 or Method 3.2 already given may be applied directly to calculate its variance. This is illustrated by the following example in which the variance of MUF is calculated by application of Method 3.1.

EXAMPLE 3.1 (c)

Consider the plutonium MUF in a somewhat simplified chemical reprocessing facility. The components of the material balance are identified as follows:

Inputs : A batch consists of a volume of material in the input accountability tank. Each volume contains nominally 10 kg Pu. Over the material balance period, there are 40 batches. The model for batch i is

$$x_{Ii} = T_{Ii} + \delta + \varepsilon_i$$

where x_{Ii} = measured plutonium in batch i

T_{Ii} = true plutonium in batch i

δ = systematic error for batch i

ε_i = random error for batch i

All quantities are expressed in kilograms. The model has been simplified in that δ and ε_i each represent the combined effects of several measurement errors (volume, sampling, analytical). Assume that δ and ε_i (and other measurement errors identified in what follows) have zero means with:

$$\sigma_\delta = 0.044 \text{ kg} \qquad \sigma_\varepsilon = 0.109 \text{ kg}$$

Product : Product is outputted as plutonium nitrate. Each batch contains nominally 22 kg Pu, and there are 18 batches over the material balance period. For batch i, the model is

$$x_{pi} = T_{pi} + \Delta + \eta_i$$

where X_{pi}, T_{pi}, Δ, and η_i are defined in a manner analagous to X_{Ii}, T_{Ii}, δ, and ε_i. Assume

$$\sigma_\Delta = 0.082 \text{ kg} \qquad \sigma_\eta = 0.191 \text{ kg}$$

Waste : Waste is batched with a nominal batch size of 0.25 kg Pu. There are 16 batches over the material balance period. The model is

$$x_{wi} = T_{wi} + \alpha + \omega_i \quad \text{, with}$$

$$\sigma_\alpha = 0.030 \text{ kg} \qquad \sigma_\omega = 0.052 \text{ kg}$$

Inventory: There are 10 batches (process vessels) in inventory. The inventory level is very low because physical inventories are taken after clean out. For beginning inventory:

$$x_{Bi} = T_{Bi} + \beta + \theta_i$$

and for ending inventory,

$$x_{Ei} = T_{Ei} + \beta + \gamma_i$$

Note that β appears in the model for both beginning and ending inventories and therefore will cancel in the MUF equation. The value of σ_β is hence immaterial. For θ_i and γ_i, assume

$$\sigma_\theta = \sigma_\gamma = 0.002 \text{ kg}$$

The model for MUF can then be written and Method 3.1 applied since the MUF equation is of the form (eq. 3.2.1).

$$\begin{aligned} \text{MUF} = & \sum_{i=1}^{40} T_{Ii} + 40\delta + \sum_{i=1}^{40} \varepsilon_i - \sum_{i=1}^{18} T_{pi} - 18\Delta - \sum_{i=1}^{18} \eta_i \\ & - \sum_{i=1}^{16} T_{Wi} - 16\alpha - \sum_{i=1}^{16} \omega_i + \sum_{i=1}^{10} T_{Bi} + 10\beta + \sum_{i=1}^{10} \theta i \end{aligned}$$

$$-\sum_{i=1}^{10} T_{Ei} - 10\beta - \sum_{i=1}^{10} \gamma_i$$

Then, assuming all covariances are zero, (eq. 3.2.3) is applied.

$$\text{Var (MUF)} = 1600\,(.044)^2 + 40(.109)^2 + 324(.082)^2 + 18(.191)^2$$
$$+256(.030)^2 + 16(.052)^2 + 10(.002)^2 + 10(.002)^2$$
$$= 6.6818\ \text{kg}^2\ \text{Pu}$$

Standard deviation (MUF) = 2.585 kg Pu

In this example, the standard deviations were expressed on an absolute basis, i.e., in kilograms plutonium. If they were expressed on a relative basis, multiplicative rather than additive models would have been written, and Method 3.2 applied. The answer, of course, would be the same.

3.4.3 Variance of Element MUF by General Approach

Although conceptually one can apply Method 3.1 or Method 3.2 in calculating the variance of any MUF, in practice it will often be difficult to write explicitly the model that is required to apply these methods. Even when it is possible to do so with moderate effort, as was done in the example just concluded, it may be much simpler to apply a general solution to the problem based on some simplifying assumptions set forth in the section to follow. The assumptions will rarely be 100% satisfied in practice, but (1) moderate departures from them will often have negligible effect; and (2) one can make slight alterations in the calculations to account for departures in assumptions if deemed necessary. This will be illustrated in the examples to follow.

The general approach to calculating the variance of element (and of isotope) MUF is an extension of methods documented in [3.5] and later expanded upon [3.6]. A computerized version of the calculations has been developed and is described in [3.7]. The computer code, identified as NUMSAS, is available. The program order form is reproduced as Annex 3.1.

3.4.3.1 Assumptions

The variance of element MUF is considered first. Isotope MUF will be treated in a later section.

In calculating the variance of element MUF, it is convenient to develop a hierarchy of classifications consisting of items, batches, strata, and components.

An item is a primary unit which has a weight, volume and destructive analysis or NDA measurement associated with it. A number of item collectively form a batch, where a batch consists of all items that are related because they have a common element concentration factor. In the event the element factor is uniquely determined for each item, then an item and a batch are identical, i.e., there is one item in that batch. Note that this definition of batch may be different than used in accounting reports to the Agency. A number of batches collectively form a stratum, which consists of all batches of like material. As will be illustrated in the examples, one has a certain amount of freedom in defining a stratum in a given application; strata of similar materials may be combined into a single stratum in order to reduce the amount of calculation at the expense of bending the assumptions somewhat. Finally, strata are combined to form a component of the MUF equation. There are four MUF components, identified in the schematic (eq. 3.4.1).

With the classifications in mind, the following assumptions are made:

(1) All random, short-term systematic, and long-term systematic error standard deviations are known and are expressed on a relative basis. For example, a 0.4% relative standard deviation is expressed as 0.004. (However, see Section 3.4.3.5.)

(2) Within a given batch, the number of samples drawn and the number of analyses per sample are both constants.

(3) Within a given stratum, the number of items per batch is constant.

(4) No more than one scale or analytical method is used in a given stratum.

(5) A given element concentration factor cannot apply to more than one stratum.

Some comments on these assumptions are helpful. First, with respect to the distinction between short-term and long-term systematic errors, it is not always evident just how a given error should be classified. Whenever a given measurement system is recalibrated, this signals a change in the error structure and introduces a new short-term systematic error. However, such error shifts may also occur in a measurement system even when the system is not recalibrated. Each error source should be evaluated using the methods of Chapter 2 to properly characterize it from point of view of how often the error may shift in value. When calculating the variance of MUF, whether an error is a short-term or long-term systematic error can have significant impact on the calculated variance of MUF, and hence, it is worth the effort to properly characterize each error.

With respect to assumption (4), this assumption can be relaxed if the measurement methods in question are of the same design (e.g., same type scale). In this event, the use of several scales, say, is equivalent to the use of one scale with several shifts in the systematic error (i.e., with a short-term systematic error). See examples 3.3 (a) and 3.4 (a).

As regards assumption (5), this assumption is not generally satisfied when dealing with isotope MUF as opposed to element MUF. Should the assumption be violated for element MUF, then the methods of Section 3.4.5 for isotope MUF may be applied.

Before continuing with the general methodology, the notation to be used in the following sections is summarized.

3.4.3.2 Notation

The following notation is used.

x_{kqpt} = total element weight in stratum k, where the element weight is found using bulk measurement method q, sampling is from material type p, and analytical technique t is used. If measurement is by NDA, regard the NDA instrument as an analytical method. "Dummy" methods can be used for the bulk and sampling measurements.

Note: It may be that within a stratum, the same systematic error does not affect all items, i.e., there is a short-term systematic error. Use parentheses to indicate the total element weight associated with "condition i" for a given measurement. For example:

$x_{kqpt(3)}$ = total element weight identified with condition 3 for analytical method t in stratum k

$x_{kq(2)pt}$ = total element weight identified with condition 2 for bulk method (e.g., scale) q in stratum k.

To continue,

δ = a relative standard deviation; subscripts identify a specific one.

s,g,r = first subscript on δ: s refers to a long term systematic error; g to a short term systematic error; r to a random error.

q,p,t = second, third, and fourth subscripts on δ; defined as for the subscripts on x; if the measurement method in question is a bulk method, replace p and t by dots; for example

$\delta_{r \cdot p \cdot}$ = random error standard deviation in sampling of material type p.

n_k = number of items per batch in stratum k.

m_k = number of batches in stratum k.

r_k = number of samples drawn per batch in stratum k to estimate the batch element concentration factor.

c_k = number of analyses per sample in stratum k

k = total number of strata

$V(\cdots)$ = variance of quantity within parentheses, for example,

$V(x_{kqpt})$ = variance of element weight in stratum k;

V(MUF) = variance of MUF

Note: s, g, and r subscripted on V is defined as when subscripted on δ; if V has no subscript, this refers to a total variance.

With this notation in mind, Method 3.3 will now provide formulas needed to compute the random error variance of MUF.

3.4.3.3 Random Error Variance of MUF

Method 3.3

Notation

The notation is given in 3.4.3.2.

Model

The schematic model for MUF is given by (eq. 3.4.1). For the individual measurements, the multiplicative model is used. This is discussed in 2.3.2.

Results

For stratum k, the random error variance of the total element weight is

$$V_r(x_{kqpt}) = x^2_{kqpt}(\delta^2_{rq..}/n_k m_k + \delta^2_{r.p.}/r_k m_k + \delta^2_{r..t}/c_k r_k m_k) \qquad \text{(eq. 3.4.2)}$$

To find $V_r(\text{MUF})$, $V_r(x_{kqpt})$ is summed over all the strata.

$$V_r(\text{MUF}) = \sum_{k=1}^{K} V_r(x_{kqpt}) \qquad \text{(eq. 3.4.3)}$$

If, in a given situation, a nominal (historical average, stoichiometric) element factor is used, then $\delta_{r.p.}$ and $\delta_{r..t}$ are both zero for that stratum. To avoid division by zero in (eq. 3.4.2), r_k may be set equal to one.

Basis

The formula for $V_r(\text{MUF})$ is derived by application of Method 3.2.

Examples

EXAMPLE 3.3 (a)

In this first example, consider a simplified material balance for a fuel fabrication facility. There are seven strata identified. There are 4 bulk measurement methods (q=1-4), 5 material types (p=1-5), and 4 analytical methods (t=1-4).

The error standard deviations are tabled.

$\delta_{r1..} = 0.000658$	$\delta_{r.1.} = 0.000531$	$\delta_{r..1} = 0.000433$
$\delta_{r2..} = 0.000877$	$\delta_{r.2.} = 0$	$\delta_{r..2} = 0.000568$
$\delta_{r3..} = 0$*	$\delta_{r.3.} = 0$*	$\delta_{r..3} = 0.0577$

$\delta_{r4..} = 0.00250$ $\delta_{r.4.} = 0.0181$ $\delta_{r..4} = 0.0274$

$\delta_{r.5.} = 0.0418$

*"dummy" methods

Stratum 1 is an input stratum consisting of containers of UO_2 powder. Stratum 2 is an output product stratum consisting of containers of sintered UO_2 pellets. Stratum 3 is an output waste stream stratum consisting of containers of solid waste measured by NDA. Stratum 4 is a beginning inventory stratum consisting of containers of dirty scrap. Stratum 5 is a beginning inventory stratum consisting of containers of grinder sludge. Stratum 6 is an ending inventory stratum containing the same kinds of material as Stratum 4. Stratum 7 is an ending inventory stratum containing the same kinds of material as Stratum 5.

Before continuing with this example, it is important to emphasize an important point. Before any calculations of the variance of MUF are performed, data for any items that are identical in both a plus and minus component of the MUF equation must be deleted. For example, if an item in beginning inventory were remeasured in ending inventory, then depending as the use made of the remeasured value, this item may or may not be included in the MUF variance calculation. If the remeasured value were booked, then the item would be included; if it were not booked, but were only used for verification of the previously booked value, then it would not be included--it would neither affect the MUF nor its variance.

To continue with the example, it is convenient to organize the parameter values in tabular form before performing the calculations of (eq. 3.4.2). This is done below.

Stratum (k)

	1	2	3	4	5	6	7
n_k	150	47,760	1	300	200	300	200
m_k	80	1	2770	6	4	6	4
r_k	5	240	1	10	12	10	12
c_k	1	1	1	1	1	1	1
q	1	2	3	4	4	4	4
p	1	2	3	4	5	4	5
t	1	2	3	4	4	4	4
x_{kqpt}(1)	240,000	238,800	1200	7200	4000	7200	4000

(1) entries are in kg uranium

Equation (3.4.2) is now applied for each stratum.

$V_r(x_{1111}) = 69.68\ kg^2\ U$

$V_r(x_{2222}) = 77.58\ kg^2\ U$

$V_r(x_{3333}) = 1.73\ kg^2\ U$

$V_r(x_{4444}) = 931.89\ kg^2\ U$

$V_r(x_{5454}) = 832.79\ kg^2\ U$

$V_r(x_{6444}) = 931.89\ kg^2\ U$

$V_r(x_{7454}) = 832.79\ kg^2\ U$

$V_r(MUF)$ is then computed using (eq. 3.4.3)

$$V_r(MUF) = 69.68 + 77.58 + \cdots + 832.79 = 3678.35\ kg^2\ U$$

EXAMPLE 3.3 (b)

This example deals with the plutonium MUF in a mixed oxide fuel fabrication plant. Except for the fact that the numbers of containers in beginning and ending inventories are not the same, the example contains no features not found in the previous example. It is included, however, because it will later serve to illustrate calculation of the short term systematic error variance.

In this facility, there are 10 strata identified (K=10). There are 5 bulk measurement methods, (q=1-5); 6 material types, (p=1-6); and 4 analytical methods, (t=1-4). The error standard deviations are listed.

$\delta_{r1\cdot\cdot} = 0.00025$	$\delta_{r\cdot1\cdot} = 0.0001$	$\delta_{r\cdot\cdot1} = 0.0040$
$\delta_{r2\cdot\cdot} = 0.00050$	$\delta_{r\cdot2\cdot} = 0.0080$	$\delta_{r\cdot\cdot2} = 0.0050$
$\delta_{r3\cdot\cdot} = 0.00040$	$\delta_{r\cdot3\cdot} = 0.035$	$\delta_{r\cdot\cdot3} = 0.0060$
$\delta_{r4\cdot\cdot} = 0^*$	$\delta_{r\cdot4\cdot} = 0^*$	$\delta_{r\cdot\cdot4} = 0.20$
$\delta_{r5\cdot\cdot} = 0.00040$	$\delta_{r\cdot5\cdot} = 0.0040$	
	$\delta_{r\cdot6\cdot} = 0.020$	

*"dummy" methods

Stratum 1 is an input stratum consisting of containers of P_uO_2. Stratum 2 is an output product stratum consisting of containers of sintered pellets. Stratum 3 is an output stratum consisting of dirty powder sent offsite for scrap recovery. Stratum 4 is an output waste stream stratum consisting of containers of solid waste measured by NDA. Stratum 5 is a beginning inventory stratum consisting of containers of mixed oxide powder. Stratum 6 is a beginning inventory stratum containing the same kind of material as output stratum 3. Stratum 7 is a beginning inventory stratum consisting of containers of grinder swarf. Strata 8, 9, and 10 are ending inventory strata containing the same kinds of materials as strata 5, 6, and 7 respectively.

The pertinent parameter values are given in the following table.

	Stratum (k)									
	1	2	3	4	5	6	7	8	9	10
n_k	32	200	1	1	20	1	1	20	1	1
m_k	24	198	10	100	15	4	6	18	5	3
r_k	4	5	1	1	3	1	1	3	1	1
c_k	2	1	1	1	1	1	1	1	1	1
q	1	2	3	4	5	3	3	5	3	3
p	1	2	3	4	5	3	6	5	3	6
t	1	2	3	4	3	3	2	3	3	2
x_{kqpt}(1)	1536	1485	9.0	0.4	112.5	3.6	4.5	135	4.5	2.25

(1) entries are in kg Pu

Equation (3.4.2) is now applied for each stratum.

$$V_r(x_{1111}) = 0.197046 \text{ kg}^2 \text{ Pu}$$

$$V_r(x_{2222}) = 0.198261 \text{ kg}^2 \text{ Pu}$$

$$V_r(x_{3333}) = 0.010215 \text{ kg}^2 \text{ Pu}$$

$$V_r(x_{4444}) = 0.000064 \text{ kg}^2 \text{ Pu}$$

$$V_r(x_{5553}) = 0.014632 \text{ kg}^2 \text{ Pu}$$

$$V_r(x_{6333}) = 0.004086 \text{ kg}^2 \text{ Pu}$$

$$V_r(x_{7362}) = 0.001435 \text{ kg}^2 \text{ Pu}$$

$$V_r(x_{8553}) = 0.017558 \text{ kg}^2 \text{ Pu}$$

$$V_r(x_{9333}) = 0.005108 \text{ kg}^2 \text{ Pu}$$

$$V_r(x_{10,362}) = 0.000717 \text{ kg}^2 \text{ Pu}$$

V_r(MUF) is then computed using (eq. 3.4.3)

$$V_r(\text{MUF}) = 0.197046 + 0.198261 + \cdots + 0.000717$$
$$= 0.449122 \text{ kg}^2 \text{ Pu}$$

EXAMPLE 3.3 (c)

In the example just considered, suppose that after the first 10 batches of P_uO_2 powder receipts were weighed, the scale was replaced by another scale of similar design. Stratum 1 could still be treated as a single stratum, even though the assumption that only one measurement method of each type were used for all items, were violated. A modification in the calculations would be required to accommodate this second scale, which may be identified by q = 6.

The 10 batches of P_uO_2 powder weighed on scale 1 corresponds to 640 kg Pu while the 14 batches weighed on scale 2 corresponds to the remaining 896 kg Pu. In calculating the random error variance of the total element weight in stratum 1, the term:

$$x_{1111}^{2}\ \delta_r{}^2{}_{1..}/n_1m_1 \quad \text{or}$$

$$(1536)^2(0.00025)^2/768 = 0.000192$$

is replaced by

$$(640)^2(0.00025)^2/320 + (896)^2(0.00025)^2/448 = 0.000192$$

The result is the same, as is easy to prove in general. For the random error variance, it doesn't matter how many measurement methods are used in a given stratum as long as they have the same measurement error standard deviations. This statement is not true for systematic error variances, as will be demonstrated later.

3.4.3.4 Systematic Error Variance of MUF

In developing the formulas needed to compute the systematic error variance of MUF, attention will first be focused on the short-term systematic errors. Method 3.4 applies.

Method 3.4

Notation

The notation is given in 3.4.3.2.

Model

See the discussion for the model in Method 3.3.

Results

The calculations indicated need only be performed for those measurements for which the first subscript on δ is g, i.e., for the non-zero short term systematic error variances.

For each combination of values, q(i), calculate

$$M_{q(i)\cdot\cdot} = \sum_{k=1}^{K} A_k\, x_{kq(i)pt} \qquad \text{(eq. 3.4.4)}$$

where A_k = +1 for input and beginning inventory strata and where A_k = -1 for output and ending inventory strata.

For each combination of values, p(i), calculate

$$M_{\cdot p(i)\cdot} = \sum_{k=1}^{K} A_k\, x_{kqp(i)t} \qquad \text{(eq. 3.4.5)}$$

where A_k is defined as above.

For each combination of values, t(i), calculate

$$M_{\cdot\cdot t(i)} = \sum_{k=1}^{K} A_k\, x_{kqpt(i)} \qquad \text{(eq. 3.4.6)}$$

where A_k is defined as above.

The short term systematic error variance of MUF is

$$V_g(\text{MUF}) = \sum_q \delta_{gq\cdot\cdot}^{2} \sum_i M^2_{q(i)\cdot\cdot} + \sum_p \delta_{g\cdot p\cdot}^{2} \sum_i M^2_{\cdot p(i)\cdot}$$

$$+ \sum_t \delta^2_{g\cdot\cdot t} \sum_i M^2_{\cdot\cdot t(i)} \qquad \text{(eq. 3.4.7)}$$

Any stratum in which the element factor is a nominal factor (historical average, stoichiometric) will not have g as a first subscript on δ for the sampling or analytical errors. Hence, the calculations indicated would not be performed in such cases.

Basis

The basis for Method 3.4 is the same as for Method 3.3. The distinction is that for random error variances, one squares quantities and then sums them; for systematic error variances, one sums and then squares.

Examples

EXAMPLE 3.4 (a)

In the facility described in example 3.3 (b), assume that there are short-term systematic errors associated with the analytical methods due, in part, to system recalibrations during the one-year material balance period. Also, with reference to example 3.3 (c), the introduction of the second scale in the bulk measurement of the P_uO_2 receipts can be treated by introducing a short-term systematic error; since this second scale has the same design as the first one, the effect is the same as if the first scale had simply been recalibrated.

The following error parameter values are given.

$$\delta_{g1\cdot\cdot} = 0.00010 \qquad \delta_{g\cdot\cdot1} = 0.0013$$

$$\delta_{g\cdot\cdot2} = 0.0016$$

$$\delta_{g\cdot\cdot3} = 0.0020$$

$$\delta_{g\cdot\cdot4} = 0.06$$

From the information given in example 3.3 (c), for scale (bulk measurement method) 1,

$$X_{11(1)11} = 640 \qquad X_{11(2)11} = 896$$

Then, from (eq. 3.4.4),

$$M_{1(1)\cdot\cdot} = 640 \qquad M_{1(2)\cdot\cdot} = 896$$

For the analytical methods, assume that the following quantities of materials are associated with the various shifts in the systematic errors (all quantities in kg Pu):

<u>stratum 1</u> $\quad x_{1111(1)} = x_{1111(2)} = x_{1111(3)} = 512$

<u>stratum 2</u> $\quad x_{2222(1)} = x_{2222(2)} = x_{2222(3)} = 495$

<u>stratum 3</u> $\quad x_{3333(1)} = 0 \ , \ x_{3333(2)} = 9.0$

<u>stratum 4</u> $\quad x_{4444(1)} = 0.16 \ , \ x_{4444(2)} = 0.24$

<u>stratum 5</u> $\quad x_{5553(1)} = 112.5 \ , \ x_{5553(2)} = 0$

<u>stratum 6</u> $\quad x_{6333(1)} = 3.6 \ , \ x_{6333(2)} = 0$

<u>stratum 7</u> $\quad x_{7362(1)} = 4.5 \ , \ x_{7362(2)} = x_{7362(3)} = 0$

<u>stratum 8</u> $\quad x_{8553(1)} = 0 \ , \ x_{8553(2)} = 135$

<u>stratum 9</u> $\quad x_{9333(1)} = 0 \ , \ x_{9333(2)} = 4.5$

<u>stratum 10</u> $\quad x_{10,362(1)} = x_{10,362(2)} = 0 \ , \ x_{10,362(3)} = 2.25$

Equation 3.4.6 may then be applied for each combination of values t(i):

$M_{\cdot\cdot 1}(1) = M_{\cdot\cdot 1}(2) = M_{\cdot\cdot 1}(3) = 512$

$M_{\cdot\cdot 2}(1) = -495 + 4.5 = -490.5$

$M_{\cdot\cdot 2}(2) = -495$

$M_{\cdot\cdot 2}(3) = -495 -2.25 = -497.25$

$M_{\cdot\cdot 3}(1) = 112.5 + 3.6 = 116.1$

$M_{\cdot\cdot 3}(2) = -9.0 -135 -4.5 = -148.5$

$M_{\cdot\cdot 4}(1) = -0.16$

$M_{\cdot\cdot 4}(2) = -0.24$

Equation 3.4.7 is applied to find the short term systematic error variance of MUF.

$$
\begin{aligned}
& & & \underline{\text{kg}^2\ \text{Pu}} \\
V_g(\text{MUF}) &= (0.00010)^2[(640)^2 + (896)^2] & &= 0.012124 \\
&+ (0.0013)^2[(512)^2 + (512)^2 + (512)^2] & &= 1.329070 \\
&+ (0.0016)^2[(-490.5)^2 + (-495)^2 + (-497.25)^2] & &= 1.876154 \\
&+ (0.0020)^2[(116.1)^2 + (-148.5)^2] & &= 0.142126 \\
&+ (0.06)^2[(-0.16)^2 + (-0.24)^2] & &= \underline{0.000300} \\
& & V_g(\text{MUF}) &= 3.359774\ \text{kg}^2\ \text{Pu}
\end{aligned}
$$

Next, consider the long term systematic error variance of MUF. The calculations described in Method 3.5 follow easily from those in Method 3.4.

Method 3.5

Notation

The notation is given in 3.4.3.2.

Model

See the discussion for the model in Method 3.3.

Results

For each value of q, calculate

$$M_{q\cdot\cdot} = \sum_{k=1}^{K} A_k x_{kqpt} \qquad \text{(eq. 3.4.8)}$$

where A_k = +1 for input and beginning inventory strata and A_k = -1 for output and ending inventory strata. Note that if the calculations indicated by (eq. 3.4.4) are performed for each value of q, then $M_{q\cdot\cdot}$ may be found by summing the $M_{q(i)\cdot\cdot}$ values over i. Similar statements hold for sampling and analytical errors.

For each value of p, calculate

$$M_{\cdot p\cdot} = \sum_{k=1}^{K} A_k\, x_{kqpt} \qquad \text{(eq. 3.4.9)}$$

where A_k is defined as above.

For each value of t, calculate

$$M_{\cdot\cdot t} = \sum_{k=1}^{K} A_k\, x_{kqpt} \qquad \text{(eq. 3.4.10)}$$

where A_k is defined as above.

The long term systematic error variance of MUF is

$$V_s(\text{MUF}) = \sum_q M_{q\cdot\cdot}^2\, \delta_{sq\cdot\cdot}^2 + \sum_p M_{\cdot p\cdot}^2\, \delta_{s\cdot p\cdot}^2 + \sum_t M_{\cdot\cdot t}^2\, \delta_{s\cdot\cdot t}^2 \qquad \text{(eq. 3.4.11)}$$

For any stratum for which the element factor is a nominal factor (historical average, stoichiometric), set $\delta_{s.p.} = 0$. The quantity $\delta_{s..t}$ must be assigned a value that is a measure of the systematic error between the true average element factor for that stratum and the nominal factor. In this event, the subscript t is a dummy index. The quantity $\delta_{s..t}$ could be quite large if a stoichiometric factor is used.

If it is known that the element factor in question is obviously in error by an appreciable amount in a given direction, it would be preferable to attempt to correct the data for the bias in the factor, and reduce the value for $\delta_{s..t}$ accordingly to reflect the systematic error in the residual bias. See example 3.5(c).

Basis

The basis for Method 3.5 is the same as for Method 3.3.

Examples

EXAMPLE 3.5 (a)

Continue with the facility of example 3.3 (a). The following error parameter values are given.

$\delta_{s1\cdot\cdot} = 0.000439$ $\delta_{s\cdot1\cdot} = 0$ $\delta_{s\cdot\cdot1} = 0.000571$

$\delta_{s2\cdot\cdot} = 0.000175$ $\delta_{s\cdot2\cdot} = 0$ $\delta_{s\cdot\cdot2} = 0.000341$

$\delta_{s3\cdot\cdot} = 0$* $\delta_{s\cdot3\cdot} = 0$* $\delta_{s\cdot\cdot3} = 0.0462$

$\delta_{s4\cdot\cdot} = 0.00167$ $\delta_{s\cdot4\cdot} = 0$ $\delta_{s\cdot\cdot4} = 0.00896$

$\delta_{s\cdot5\cdot} = 0.00444$

*"dummy" methods

From the data table given in example 3.3 (a), the $M_{\cdot\cdot\cdot}$ values of (eq. 3.4.8), (eq. 3.4.9), and (eq. 3.4.10) are calculated. It is not necessary to calculate these quantities for q = 3, and p = 1, 2, 3, or 4 since the corresponding error standard deviations are all zero.

$M_{1\cdot\cdot} = 240{,}000$ (all units in kg U)

$M_{2\cdot\cdot} = -238{,}800$

$M_{4\cdot\cdot} = 7200 + 4000 - 7200 - 4000 = 0$

$M_{\cdot5\cdot} = 4000 - 4000 = 0$

$M_{\cdot\cdot1} = 240{,}000$

$M_{\cdot\cdot2} = -238{,}800$

$M_{\cdot\cdot3} = -1200$

$M_{\cdot\cdot4} = 7200 + 4000 - 7200 - 4000 = 0$

Equation (3.4.11) is now applied to find the systematic error variance of MUF:

$$
\begin{aligned}
& & & \underline{\mathrm{kg}^2\mathrm{U}} \\
V_s(\mathrm{MUF}) &= (240{,}000)^2(0.000439)^2 &=& \ 11100.73 \\
&+ (-238{,}800)^2(0.000175)^2 &=& \ 1746.40 \\
&+ (240{,}000)^2(0.000571)^2 &=& \ 18779.96 \\
&+ (-238{,}800)^2(0.000341)^2 &=& \ 6630.98 \\
&+ (-1200)^2(0.0462)^2 &=& \ 3073.59 \\
& & V_s(\mathrm{MUF}) =& \ 41{,}331.66 \ \mathrm{kg}^2\mathrm{U}
\end{aligned}
$$

From example 3.3 (a) $V_r(\text{MUF}) = 3678.35\ \text{kg}^2\text{U}$. Therefore,

$V(\text{MUF}) = 45010.01\ \text{kg}^2\text{U}$

standard deviation of MUF = 212 kg U, or 0.089% of input

EXAMPLE 3.5 (b)

Continue with the mixed oxide fuel fabrication facility of examples 3.3 (b) and 3.4 (a). The following error parameter values are given.

$\delta_{s1\cdot\cdot} = 0.00020$	$\delta_{s\cdot1\cdot} = 0$	$\delta_{s\cdot\cdot1} = 0.0007$
$\delta_{s2\cdot\cdot} = 0.00035$	$\delta_{s\cdot2\cdot} = 0.0010$	$\delta_{s\cdot\cdot2} = 0.0012$
$\delta_{s3\cdot\cdot} = 0.00025$	$\delta_{s\cdot3\cdot} = 0.015$	$\delta_{s\cdot\cdot3} = 0.0015$
$\delta_{s4\cdot\cdot} = 0$*	$\delta_{s\cdot4\cdot} = 0$*	$\delta_{s\cdot\cdot4} = 0.08$
$\delta_{s5\cdot\cdot} = 0.00025$	$\delta_{s\cdot5\cdot} = 0.0024$	
	$\delta_{s\cdot6\cdot} = 0.008$	

*"dummy" method

From the data table given in example 3.3 (b), the $M_{\cdot\cdot\cdot}$ values of (eq. 3.4.8), (eq. 3.4.9), and (eq. 3.4.10) are calculated. For the analytical methods, the $M_{\cdot\cdot\cdot}$'s are easily calculated using the results of example 3.4 (a) for which the short-term systematic error variances were calculated. All units are in kg Pu.

$$M_{1\cdot\cdot} = 1536$$

$$M_{2\cdot\cdot} = -1485$$

$$M_{3\cdot\cdot} = -9.0 + 3.6 + 4.5 - 4.5 - 2.25 = -7.65$$

$$M_{5\cdot\cdot} = 112.5 - 135 = -22.5$$

$$M_{\cdot2\cdot} = -1485$$

$$M_{\cdot3\cdot} = -9.0 + 3.6 - 4.5 = -9.9$$

$$M_{\cdot5\cdot} = 112.5 - 135 = -22.5$$

$M_{\cdot 6 \cdot} = 4.5 - 2.25 = 2.25$

$M_{\cdot\cdot 1} = 1536$

$M_{\cdot\cdot 2} = -490.5 - 495 - 497.25 = -1482.75$

$M_{\cdot\cdot 3} = 116.1 - 148.5 = -32.4$

$M_{\cdot\cdot 4} = -0.4$

Equation (3.4.11) is now applied to find the long-term systematic error variance of MUF.

			kg^2Pu
$V_s(MUF)$	$= (1536)^2(0.00020)^2$	=	0.094372
	$+ (-1485)^2(0.00035)^2$	=	0.270140
	$+ (-7.65)^2(0.00025)^2$	=	0.000004
	$+ (-22.5)^2(0.00025)^2$	=	0.000032
	$+ (-1485)^2(0.0010)^2$	=	2.205225
	$+ (-9.9)^2(0.015)^2$	=	0.022052
	$+ (-22.5)^2(0.0024)^2$	=	0.002916
	$+ (2.25)^2(0.008)^2$	=	0.000324
	$+ (1536)^2(0.0007)^2$	=	1.156055
	$+ (-1482.75)^2(0.0012)^2$	=	3.165908
	$+ (-32.4)^2(0.0015)^2$	=	0.002362
	$+ (-0.4)^2(0.08)^2$	=	0.001024
	$V_s(MUF)$	=	6.920414 kg^2Pu

From examples 3.3 (b) and 3.4 (a), the random and short term systematic error variances are given.

$V_r(MUF) = 0.449122 \ kg^2Pu$

$V_g(MUF) = 3.359774 \ kg^2Pu$

Therefore, summing, the variance of MUF is

$V(MUF) = 10.729310 \text{ kg}^2\text{Pu}$

$\sqrt{V(MUF)} = 3.276$ kg Pu, or 0.213% of input

EXAMPLE 3.5(c)

In examples 3.3(b), 3.4(a) and 3.5(b), say that in stratum 1, a nominal plutonium factor of 0.875 is used for the PuO_2 powder, and further say that X_{kgpt} for stratum 1 is then 1542 kg Pu rather than 1536 kg Pu. In accordance with the notes that follow (eq. 3.4.2), (eq. 3.4.7) and (eq. 3.4.11), the following changes are then made in the calculations.

In example 3.3(b), set $\delta_{r.1.} = \delta_{r..1.} = 0$ and $r_1 = 1$. Then

$Vr\ (X_{1111}) = (1542)^2\ (0.00025)^2/768$

$= 0.000194 \text{ kg}^2$ Pu, (rather than 0.197046 kg^2 Pu).

and $Vr\ (MUF) = 0.252270 \text{ kg}^2$ Pu, (rather than 0.449122 kg^2 Pu)

In example 3.4(a), $\delta_{g..1} = 0$, (rather than 0.0013). Then, in the table of calculations on Page 125, replace the quantity 1.329070 by zero so that:

$Vg\ (MUF) = 2.030704 \text{ kg}^2$ Pu (rather than 3.359774 kg^2 Pu).

In example 3.5(b), set $\delta_{s..1} = 0.0025$, (rather than 0.0007). Then, in the table of calculations on Page 129, replace

$(1536)^2\ (0.0007)^2 = 1.156055$ by

$(1542)^2\ (0.0025)^2 = 14.861025 \text{ kg}^2$ Pu, so that

$Vs\ (MUF) = 20.625384 \text{ kg}^2$ Pu, (rather than 6.920414 kg^2 Pu).

Finally,

$V\ (MUF) = 0.000194 + 2.030704 + 20.625384$

$= 22.656282 \text{ kg}^2$ Pu, (rather than 10.729310 kg^2 Pu).

To continue with this example, the value of $\delta_{s..1} = 0.0025$ is set this large because it is known that the nominal factor of 0.875 is biased high. One could attempt to correct the data for bias by utilizing a "more reasonable" factor of, say, 0.872, based on applicable historical data. For a factor of 0.872, the value for X_{kgpt} for stratum 1 is 1536.71 kg^2 Pu. The new value for Vg (MUF) is essentially unchanged, (0.000192 kg^2 Pu) and the new value for Vg (MUF) is identically the same as before, viz., 2.030704 kg^2 Pu. For the 0.872 factor, assign the value 0.0012 to $\delta_{s..1}$. Then, rather than 14.861025 kg^2 Pu as the contribution to the systematic error variance due to the use of the nominal factor of 0.875, the contribution for the 0.872 factor is:

$(1536.71)^2\ (0.0012)^2 = 3.400528 \text{ kg}^2$ Pu. With this change,

$Vs\ (MUF) = 9.164887 \text{ kg}^2$ Pu, and

$V\ (MUF) = 0.000192 + 2.030704 + 9.164837$

$= 11.195783 \text{ kg}^2$ Pu.

Before leaving this subject, it is noted that in all of the examples, common element concentration factors did not appear in different strata. Should this occur, then the methods suggested in a later section, 3.4.5, may be applied.

3.4.3.5 Case of Constant Absolute Errors

As stated in the assumptions of 3.4.3.1, all errors have thus far been assumed to be constant on a relative basis, and δ with subscripts was used to designate such an error standard deviation. For example, $\delta_{rq\cdot\cdot} = 0.0004$ designates a relative error of 0.04% for the random error standard deviation of bulk measurement method q.

In some applications, and most notably in the case of scales, errors are more likely to be constant on an absolute basis rather than on a relative basis. The error propagation formulas given in the foregoing sections must be modified to account for this. The modifications are very simple if one keeps in mind the relationship between relative errors and absolute errors.

Letting σ with subscripts be the standard deviation in absolute units, then in stratum k, one has the relationship

$$\delta_{rq\cdot\cdot} = \frac{n_k\ m_k\ \sigma_{rq\cdot\cdot}}{x_{kqpt}} \qquad \text{(eq. 3.4.12)}$$

assuming that one is speaking of a bulk measurement. Similar modifications can easily be made for sampling and analytical if needed, but here the common practice is to express all errors relatively.

With (eq. 3.4.12) in mind, then the key equations may be modified as follows:

In (eq. 3.4.2), replace the first term by

$$n_k\ m_k\ \sigma^2_{rq\cdot\cdot}$$

for all strata in which errors are expressed in σ units rather than in δ units. Effectively, this replaces <u>amounts</u> by <u>numbers</u> of weighing.

In (eq. 3.4.4), for the methods for which errors are expressed in absolute units, multiply $x_{kq(i)pt}$ by

$$n_k\ m_k / x_{kq(i)pt}$$

This is the same as replacing $x_{kq(i)pt}$ by the <u>number</u> of weighings (or measurements) performed under condition i.

A similar change is made in (eq. 3.4.8). For the methods in question, multiply x_{kqpt} by

$$n_k \; m_k / x_{kqpt}$$

<u>Example</u>

The examples for the mixed oxide fuel fabrication plant (see examples 3.3 (b), 3.4 (a), and 3.5 (b)) are reconsidered given that for bulk measurement method 1, the error standard deviations are expressed in absolute units rather than relatively. The error standard deviations in question are:

$$\sigma_{r1\cdot\cdot} = 0.0005 \text{ kg} \qquad (0.5 \text{ g})$$

$$\sigma_{g1\cdot\cdot} = 0.0002 \text{ kg} \qquad (0.2 \text{ g})$$

$$\sigma_{s1\cdot\cdot} = 0.0004 \text{ kg} \qquad (0.4 \text{ g})$$

Then, with the other input data unchanged, upon application of the modified (eq. 3.4.2),

$$V_r(x_{1111}) = 768(.0005)^2 + (1536)^2[(.0001)^2/96 + (.004)^2/192] = 0.197046 \quad ,$$

(as before)

Turning to example 3.4 (a), the values for $M_{1(1)\cdot\cdot}$ and $M_{1(2)\cdot\cdot}$ become respectively

$$M_{1(1)\cdot\cdot} = (640)(768)/1536 = 320 \qquad \text{(number of weighings)}$$

$$M_{1(2)\cdot\cdot} = (896)(768)/1536 = 448 \qquad \text{(number of weighings)}$$

Then, the first term of (eq. 3.4.7) becomes, for $q = 1$,

$$(0.0002)^2[(320)^2 + (448)^2] = 0.012124 \text{ kg}^2 \quad , \text{(as before)}$$

From example 3.5 (b),

$$M_{1\cdot\cdot} = (1536)(768)/1536 = 768 \text{ weighings}$$

From the first term of (eq. 3.4.11), for $q = 1$, one gets

$$(.004)^2(768)^2 = 0.094372 \text{ kg}^2$$

It is noted that in this example the results, of course, are unchanged. This is because the scale in question is used in only one stratum. If a scale were used in different strata in which the average weight per item differs, then this would affect the results.

3.4.4 Variance of Cumulative MUF

The cumulative MUF is the sum of individual MUF's over a number of material balance periods. The cumulative MUF has the same model structure as the MUF for a given material balance period, i.e., the MUF components are still the inputs, outputs, beginning and ending inventories. Therefore, the variance of the cumulative MUF is calculated using the same methods as are used for the MUF over a single balance period. In this connection, note that if one were interested in calculating the variance of the cumulative MUF over, say, a three year period, then only the inventories at the beginning and end of this three year period affect the MUF, and hence its variance. It doesn't matter how many physical inventories there are during this period of time; the three year MUF is completely independent of such intermediate inventories.

In a parallel situation, if one were interested in calculating the MUF and its variance over a number of combined material balance areas (or facilities), then the remarks in the foregoing paragraph still apply. The summation is now over space rather than over time. Thus, if the MUF for a state were to be calculated, this quantity, and hence its variance, is affected only by inputs to and outputs from the state, and the beginning and ending inventories in that state. Clearly, transfers among facilities within the state do not affect the MUF, just as transfers among locations within a facility do not affect the facility MUF. The problem is not one of calculating a state MUF or its variance; in concept this is a simple exercise, or at least, no more complicated than that of calculating similar quantities for a facility. The problem is rather one of implementation, since all facilities within the state would have to be inventoried simultaneously. One could back off from this requirement by maintaining records of facility transfers between inventory times, but to be effective, inventories would have to be reasonably close to being simultaneous in time.

The foregoing discussion suggests the related questions as to how frequently one should take inventory to close a material balance and compute a MUF, and how finely one should divide a material balance area into sub-areas. Here is an instance in which one's intuition is perhaps challenged by the facts, for it has been proven that from point of view of maximizing the probability of detecting the removal of material from a given material balance area over a fixed time interval, one should neither subdivide the material balance area into smaller sub-areas, nor should one subdivide the time interval to close the material balance more frequently [3.8]. There are, of course, criteria to consider other than the detection probability. It is clear that the role of more frequent material balance closings (more frequent inventories) is to detect removals of material more quickly. Correspondingly, subdivision of material balance areas into sub-areas will serve to localize the removals. It is unfortunate that these two important kinds of criteria, detection probability and identification of removals in time and space work at cross-purposes; clearly, some balance is needed. See also references [3.9] - [3.11].

3.4.5 Variance of Isotope MUF by General Approach

In the foregoing sections, the variance of element MUF was treated. The isotope MUF for a facility may also be calculated, and so it is necessary to provide methods for calculating its variance.

Before providing such methods, it is worthwhile to note that for facilities other than enrichment plants, primary emphasis should logically be devoted to controlling the element MUF. If adequate control is maintained on the element MUF, then, except for mistakes in booking isotope values, there is no way the isotope MUF could be out of control. Since, on a percentage of throughput basis, the uncertainty in the isotope MUF cannot be smaller than the uncertainty in the element MUF, it follows that the element MUF is the quantity of primary concern. This fact was pointed out, and quantified results were found for a light water reactor fuel fabrication facility in a paper by Nilson, Schneider, and Jaech [3.12]. The authors pointed out that the variance of isotope MUF can easily be twice that of the variance of element MUF on a percentage of throughput basis. In the type of facility they treated, from a materials control viewpoint, the isotope MUF provides no information beyond that provided by the element MUF. (The authors do point out, however, that to guard against substitution diversion strategies, isotope measurements must, of course, be made. The question is not whether or not such measurements need be made, but rather, how the resulting data are to be factored into the decision framework.)

The variance of the isotope MUF is, as implied in the preceding paragraphs, made up of two types of sources of variance. First, any uncertainties in measuring bulks and element concentrations will also result in an uncertainty in the isotope value since the isotope is normally calculated as the product of bulk, element concentration, and isotope concentration. The exception is with NDA measurements in which the isotope is measured directly and the element is calculated from that measurement. Such items normally comprise a small part of the material balance. That part of the variance of the isotope MUF that is due to errors in the measurements of bulk and element concentration is covered in Section 3.4.5.1.

Secondly, errors occur in the measurement of isotope concentration. Here, booking practices play an important role. If a block of material is inputted to a facility at a given measured isotope concentration value, and if nothing in the process changes that concentration, then the output would be at the same concentration as the input. To assure that the concentration remains unchanged, measurements are also made of the output. But in the situation just described, these measurements should be treated as <u>verification</u> measurements and should not be booked unless they provide evidence that the isotope concentration had indeed changed. If such verification type measurements are booked, then they introduce an artifical MUF in the isotope value and also affect the variance of the isotope MUF. This will be illustrated in the example of Section 3.4.5.2 in which section the method is provided for calculating that contribution to the variance of isotope MUF due to uncertainties in the measurements of isotope concentrations.

3.4.5.1 <u>Variance of Isotope MUF Due to Measurement Errors in Bulk and Element Measurements</u>

<u>Method 3.6</u>

<u>Notation</u>

The notation is given in 3.4.3.2 except that x now refers to an isotope weight rather than an element weight.

Model

See the discussion for the model in Method 3.3. For isotope weight, the multiplicative model is still used, since the isotope amount is found by multiplying the element amount by the isotope concentration factor when the amount of isotope is found by the bulk, sampling, analytical measurement route.

Results

Follow methods 3.3, 3.4, and 3.5 after replacing all element weights by isotope weights in the calculations. Before performing these calculations, delete all strata in which the amount of isotope is measured directly by NDA. This step is necessary because for such strata, the uncertainty in the measurement of the element in no way affects the uncertainty in the measurement of the isotope; rather, the situation is reversed.

Basis

The basis for Method 3.6 is the same as for Method 3.3.

Examples

EXAMPLE 3.6 (a)

In Example 3.3 (a), the uranium is at six different enrichments. (This example is continued as Example 3.5 (a)). The pertinent data are tabled.

Stratum	Enrichment of Uranium
1	40 batches at 3.25% U-235 30 batches at 2.67% U-235 10 batches at 1.52% U-235
2	22,900 items at 3.25% U-235 16,720 items at 2.67% U-235 5,900 items at 1.52% U-235 2,240 items at 2.87% U-235
3	The U-235 is measured directly by NDA. Delete this stratum
4	5 batches at 3.12% U-235 1 batch at 2.58% U-235
5	All batches at 2.58% U-235
6	3 batches at 3.25% U-235 2 batches at 2.67% U-235 300 kgs U at 1.52% U-235 900 kgs U at 2.87% U-235
7	1 batch at 3.25% U-235 1 batch at 2.67% U-235 500 kgs U at 1.52% U-235 1500 kgs U at 2.87% U-235

Using the notation of the two cited examples, these data are converted to kgs U-235. From Example 3.3 (a):

$$x_{1111} = (120{,}000)(0.0325) + (90{,}000)(0.0267) + (30{,}000)(0.0152)$$

$$= 6759.00$$

$x_{2222} = 6723.21$ $\quad$ $x_{6444} = 211.47$

$x_{4444} = 218.16$ $\quad$ $x_{7454} = 109.85$

$x_{5454} = 103.20$

From Example 3.5 (a):

$M_{1\cdot\cdot} = 6759.00$ $\quad$ $M_{\cdot\cdot 1} = 6759.00$

$M_{2\cdot\cdot} = -6723.21$ $\quad$ $M_{\cdot\cdot 2} = -6723.21$

$M_{4\cdot\cdot} = 0.04$ $\quad$ $M_{\cdot\cdot 4} = 0.04$

$M_{\cdot 5\cdot} = -6.65$

From Example 3.3 (a), to find $V_r(\text{MUF})$:

$V_r(x_{1111}) = 0.0553$ kg^2 U-235

$V_r(x_{2222}) = 0.0615$ kg^2 U-235

$V_r(x_{4444}) = 0.8556$ kg^2 U-235

$V_r(x_{5454}) = 0.5543$ kg^2 U-235

$V_r(x_{6444}) = 0.8039$ kg^2 U-235

$V_r(x_{7454}) = 0.6280$ kg^2 U-235

Then,

$V_r(\text{MUF}) = 2.9586$ kg^2 U-235

From Example 3.5 (a), to find $V_s(\text{MUF})$:

		kg^{2}U-235
$(6759.00)^2(0.000439)^2$	=	8.8043
$+ (-6723.21)^2(0.000175)^2$	=	1.3843
$+ (0.04)^2(0.00167)^2$	=	0.0000
$+ (-6.65)^2(0.00444)^2$	=	0.0009
$+ (6759.00)^2(0.000571)^2$	=	14.8949

$$+ (-6723.21)^2(0.000341)^2 = 5.2561$$
$$+ (0.04)^2(0.00896)^2 = 0.0000$$

$$V_s(MUF) = 30.3405 \text{ kg}^2\text{U-235}$$

The variance in the U-235 isotope MUF due to errors of measurement for bulk and uranium concentrations is the sum,

$$2.9586 + 30.3405 = 33.2991 \text{ kg}^2\text{U-235}$$

To this must now be added the uncertainties due to the measurement of U-235 concentrations in strata 1, 2, and 4-7, and due to the measurement of the amount of U-235 in stratum 3. The procedure for incorporating these uncertainties is in Method 3.7.

3.4.5.2 Variance of Isotope MUF Due to Measurement Errors in Isotope Measurements

Method 3.7

Notation

S_i = algebraic sum of isotope weights for isotope factor i. In the algebraic sum, amounts in input and beginning inventory strata have a plus sign while those in output and ending inventory strata have a minus sign.

G = total number of isotope factors

r_i^* = number of samples drawn to establish isotope factor i

c_i^* = number of isotopic analyses per sample

δ^*, with subscripts = a relative standard deviation associated with an isotopic measurement. The subscripts are defined as in Section 3.4.3.2. The only errors assumed to be non-zero are:

$\delta^*_{r \cdot p \cdot}$ = random error in sampling for isotope

$\delta^*_{r \cdot \cdot t}$ = random error in isotopic analysis

$\delta^*_{s \cdot \cdot t}$ = long term systematic error in isotopic analysis

T_t = sum of S_i values based on analytical method t

$V_r^*(MUF)$ = random error variance in MUF due to random errors in isotope measurements

$V_s^*(MUF)$ = systematic error variance in MUF due to systematic errors in isotope measurements

Model

See the discussion for the model in Method 3.6.

Results

$$V_r^*(MUF) = \sum_{i=1}^{G} S_i^2(\delta^{*2}_{r \cdot p \cdot}/r_i^* + \delta^{*2}_{r \cdot \cdot t}/r_i^* c_i^*) \qquad \text{(eq. 3.4.13)}$$

$$V_s^*(MUF) = \sum_t T_t^2 \, \delta^{*\,2}_{s \cdot \cdot t} \qquad \text{(eq. 3.4.14)}$$

Basis

The basis for this method is the same as for Method 3.3.

Examples

EXAMPLE 3.7 (a)

Example 3.6 (a) is continued. The following values are given for the parameters. (The nominal factor applies to the NDA measurements in stratum 3.)

Factor (i)	r_i^*	c_i^*	p	t
0.0325	5	2	1	1
0.0267	3	2	1	1
0.0152	2	2	1	1
0.0287	14	1	2	2
0.0312	5	2	1	1
0.0258	4	2	1	1
Nominal	2770	1	-	3

$\delta^*_{r \cdot 1 \cdot} = 0.0005$ $\qquad \delta^*_{r \cdot 2 \cdot} = 0.0005$

$\delta^*_{r \cdot \cdot 1} = 0.0015$ $\qquad \delta^*_{s \cdot \cdot 1} = 0.0008$

$\delta^*_{r \cdot \cdot 2} = 0.0022$ $\qquad \delta^*_{s \cdot \cdot 2} = 0.0010$

$\delta^*_{r \cdot \cdot 3} = 0.07$ $\qquad \delta^*_{s \cdot \cdot 3} = 0.04$

Equations (eq. 3.4.13) and (eq. 3.4.14) may now be applied, but first, the S_i and T_t values must be calculated. The data are from Example 3.6 (a).

$$S_1 = 0.0325\ (120{,}000 - 114{,}500 - 3600 - 1000) = 29.25$$

$S_2 = 80.10$ $\qquad$ $S_5 = 187.20$

$S_3 = -4.56$ $\qquad$ $S_6 = 134.16$

$S_4 = -390.32$ $\qquad$ $S_7 = -36.00$

$$T_1 = 29.25 + 80.10 - 4.56 + 187.20 + 134.16 = 426.15$$

$$T_2 = -390.32$$

$$T_3 = -36.00$$

From (eq. 3.4.13),

$$V_r^*(\text{MUF}) = (29.25)^2[(0.0005)^2/5 + (0.0015)^2/10] + \cdots$$

$$= 0.0766\ \text{kg}^2\text{U-235}$$

From (eq. 3.4.14),

$$V_s^*(\text{MUF}) = (426.15)^2(0.0008)^2 + \cdots$$

$$= 2.3422\ \text{kg}^2\text{U-235}$$

Using the results from Examples 3.6 (a) and the above, the total variance of the isotope MUF is V*(MUF):

$$V^*(\text{MUF}) = 33.2991 + 0.0766 + 2.3422 = 35.7179\ \text{kg}^2\text{U-235}$$

$$\text{standard deviation} = 5.976\ \text{kg U-235}$$

To continue with this example, suppose now that booking practices are changed such that measurements of outputs are not regarded as verification measurements but are actually booked. Redefine the factors as follows:

Stratum	Factors	Description
1	0.03250 0.02670 0.01520	Measurement of input; the shipper's values are booked
2	0.03257 0.02668 0.01521 0.02870	Measurement of output, based on facility measurements

4,5	0.03122 0.02576	Measurements of beginning inventory, based on facility measurements
6,7	0.03257 0.02668 0.01521 0.02870	Measurements of ending inventory, based on facility measurements

The revised table of factors then appears as follows:

Factor (i)	r_i^*	c_i^*	p	t
0.03250	5	2	1	1
0.02670	3	2	1	1
0.01520	2	2	1	1
0.03257	25	1	3	2
0.02668	15	1	3	2
0.01521	10	1	3	2
0.02870	14	1	2	2
0.03122	20	1	3	2
0.02576	12	1	3	2
Nominal	2770	1	-	3

The errors remain the same. One additional error is included, the random error in sampling from material type 3 (sintered pellets). Assume that $\delta^*_{r\cdot 3\cdot} = 0.0005$.

The S_i and T_t values are recalculated. There are now 10 S_i values.

$$S_1 = (0.03250)(120{,}000) = 3900.00$$

$$S_2 = (0.02670)(90{,}000) = 2403.00$$

$$S_3 = (0.01520)(30{,}000) = 456.00$$

$$S_4 = (0.03257)(-114{,}500 -3600 -1000) = -3879.09$$

$$S_5 = (0.02668)(-83{,}600 -2400 -1000) = -2321.16$$

$$S_6 = (0.01521)(-29{,}500 -300 -500) = -460.86$$

$$S_7 = (0.02870)(-11{,}200 -900 -1500) = -390.32$$

$$S_8 = (0.03122)(6000) = 187.32$$

$$S_9 = (0.02576)(1200 + 4000) = 133.95$$

$$S_{10} = -36.00$$

$$T_1 = 3900.00 + 2403.00 + 456.00 = 6759.00$$

$$T_2 = -3879.09 + \cdots + 133.95 = -6730.16$$

$$T_3 = -36.00$$

From (eq. 3.4.13),

$$V_r^*(\mathrm{MUF}) = (3900.00)^2[(0.0005)^2/5 + (0.0015)^2/10] + \cdots$$

$$= 12.0465\ \mathrm{kg}^2\mathrm{U}\text{-}235$$

From (eq. 3.4.14),

$$V_s^*(\mathrm{MUF}) = (6759.00)^2(0.0008)^2 + (-6730.16)^2(0.0010)^2 + (0.04)^2(-36.00)^2$$

$$= 76.6065\ \mathrm{kg}^2\mathrm{U}\text{-}235$$

The total variance of the isotope MUF is then

$$V^*(\mathrm{MUF}) = 33.2991 + 12.0465 + 76.6065$$

$$= 121.9522\ \mathrm{kg}^2\mathrm{U}\text{-}235$$

and the standard deviation is 11.043 kg U-235. Note that this is over twice as large as the corresponding value when the measurements of output are not booked but serve only to verify the inputs. The importance of booking practices is clearly demonstrated by this example.

3.4.6 Effects of Other Factors on MUF and its Variance

In the discussion of Section 3.4.1, it was pointed out that MUF is affected by factors other than errors of measurement. It is emphasized that the variance of MUF, calculated by the procedures in the preceding sections, includes only the effects of measurement errors, and it is implicitly assumed that any measurement system on which such errors are based is functioning properly during the material balance period in question. Thus, there are some limitations as to what conclusions can be drawn about diversion of nuclear materials on the basis of only the MUF and its variance. That is, it does not necessarily follow that if an observed MUF differs significantly from zero based on a hypothesis test and using the variance of MUF as calculated in the preceding sections, this is evidence of diversion. Before such a conclusion is drawn, the possible effects of other factors on MUF and its variance should be taken into consideration. How this may be done objectively is a difficult problem.

Consider the factors that may affect MUF and/or its variance. These are such factors as unmeasured or hidden inventories, improperly modeled measurement biases, misstatements on error variances, and improperly recorded data. Unmeasured inventories will increase the size of MUF but will not affect its calculated variance. Improperly modeled measurement biases may affect both the MUF and its variance, depending upon the nature of the improper modeling. For example, if biases exist but are not corrected for, or if the corrections do not properly reflect the actual biases, then obviously this will affect the value for MUF. Such conditions are, of course, presumed to exist since no measurement system can be free of bias. This is why systematic error variances are a part of the variance of MUF calculations; to reflect these biases. However, it may be difficult to model a system properly. For example, the sampling system for a liquid waste stream may select representative samples the majority of time, but process upsets may perturb the system on occasion so that the sample does not reflect the contents of the waste stream during such events. To continue, misstatements on the sizes of measurement error variances, either under or overstatements, will clearly affect the variance of MUF. Finally, mistakes committed in the recording of data will affect the MUF, but not the calculated variance. Such mistakes are realistically impossible to eliminate completely, and are quite difficult if not impossible to model properly. Their collective effects make it very difficult to distinguish between material diversions and losses that may be explained by innocent causes.

The problems pointed out in the preceding paragraphs are simple to pose but difficult to solve. The function of inspection is to instill confidence in the facility MUF and its variance. In Chapter 4, detailed inspection plans are provided to that end. Further, facility inspection can address itself to other problems just discussed; for example, an assessment can be made of the magnitude of unmeasured inventories, and these can be taken into account in the final MUF evaluation. Even detailed and well conducted facility inspections cannot, however, provide a complete solution.

One approach that has received some attention starts with the premise that there is a tolerable level of lack of material control that will not be included in the standard MUF-LEMUF analysis. This tolerable level is then modeled in some way. Such modeling may be done synthetically, by identifying the various factors that might contribute to MUF and/or its variance, assigning tolerable ranges of values to such factors, and then combining their effects, either analytically or possibly through simulation methods. This does not provide a simple solution; it is difficult to be objective and realistic in this modeling process.

Another approach is to evaluate past MUF data from typical plants considered to have an acceptable level of control. Future MUF performance may then be judged against this past acceptable performance. It is difficult to be objective with this approach also. There is no simple solution to the problem of MUF evaluation; objectivity can only proceed so far in the evaluation process, and semi-objective evaluation must complete any given evaluation, aided by whatever modeling is available. It is relatively straightforward to reach a decision as to whether or not a loss of material is larger than can be explained by errors of measurement; it is quite a different problem to distinguish between losses due to innocent causes, and those due to material diversion.

3.5 CALCULATION OF VARIANCE OF $\hat{D}$, THE DIFFERENCE STATISTIC

It has been pointed out that the primary role of inspection from an accounting viewpoint is to instill confidence in the reported MUF and its variance. In performing this function, the so-called $\hat{D}$ statistic, or the difference statistic, is of prime importance. The quantity $\hat{D}$ is an estimate of the bias in the facility MUF. In actuality, it estimates a relative bias between the facility and the inspection agency, which is interpreted as a bias in the facility MUF when the assumption is made that the agency inspection measurements are unbiased.

The $\hat{D}$ statistic, as defined explicity in the next section, has intuitive appeal in the sense that it seems to be a logical way to compare inspector data with operator data. Beyond this, the importance of this statistic has been demonstrated based on theoretical considerations. In reference [3.13], it is shown that if all items in a given stratum are biased (or falsified) by the same amount, then $\hat{D}$ is an optimal statistic from point of view of maximizing the probability of detecting this bias. The more general case in which items may be biased by variable amounts has also been studied, but an exact solution has not been found for this case. However, in view of the aforementioned important result for the constant bias case, it seems reasonable to base quantitative verification on the $\hat{D}$ statistic for the more general case also; one would not expect to find significant departures from optimality, if any.

In Section 3.5.1, the $\hat{D}$ statistic is defined. The variance of $\hat{D}$ is computed directly by error propagation methods in 3.5.2. In 3.5.3, a general approach to calculating the variance of $\hat{D}$ under specified rather non-restrictive assumptions will be presented.

3.5.1 Definition of $\hat{D}$

A stratum is defined as in 3.4.3.1. Within each stratum, the inspector obtains measured values for a sampled number of items and compares his results on an item by item basis with those of the facility. The inspector may not obtain a completely independently measured value for each sampled item; he may use average concentration factors to apply to a number of items, just as the facility does.

For each item measured by the inspector, let the difference: facility value minus inspector value, be calculated. Then, average these differences in each stratum, letting $\bar{d}_k$ be the average difference in stratum k. This difference is in terms of either element weight or isotope weight. The difference is then extrapolated to apply to the total weight of element or isotope in stratum k.

$$\hat{D}_k = N_k \bar{d}_k \qquad \text{(eq. 3.5.1)}$$

where N_k is the number of items in stratum k.

To determine the net effect of the biases in all strata on MUF, the $\hat{D}$ statistic is defined as

$$\hat{D} = \sum_k A_i \hat{D}_k \qquad \text{(eq. 3.5.2)}$$

where A_i = +1 for input and beginning inventory strata, and where A_i = -1 for output and ending inventory strata. With $\hat{D}$ defined in this way, a positive value of $\hat{D}$ means that MUF is biased on the high side while a negative value means that MUF is biased low (see the discussion in Section 3.6).

In order to make probability statements about $\hat{D}$, i.e., in order to determine whether or not an observed $\hat{D}$ provides evidence that the bias in the facility MUF is different from zero, it is necessary to calculate its variance. This is done by methods discussed and exemplified in the sections to follow.

3.5.2 Variance of $\hat{D}$ by Direct Application of Error Propagation Formulas

If a given calculated $\hat{D}$ is based on a simple model that may be written explicitly, then either Method 3.1 or Method 3.2 already given may be applied directly to calculate its variance. This is illustrated by the following example in which the variance of MUF is calculated by Method 3.2. The example is concerned with a shipper-receiver difference analysis for a single stratum rather than with a facility-inspection comparison. Mathematically, the two problems are equivalent.

EXAMPLE 3.2 (c)

Shipper-receiver data for a receipt of 22 cylinders of low enriched UF_6 are displayed below.

Cylinder	Net Weight UF_6(lbs)		Percent U		Percent U-235		U-235(lbs)	
	S	R	S	R	S	R	S	R
1	4853	4850	67.61	67.590	3.288	3.300	107.88	108.18
2	4855	4851					107.93	108.20
3	4852	4848					107.86	108.13
4	4846	4843					107.73	108.02
5	4817	4818	67.60	67.605	2.394	2.397	77.96	78.08
6	4838	4835					78.30	78.35
7	4506	4504	67.61	67.60	2.832	2.820	86.28	85.86
8	4504	4506					86.24	85.90
9	4503	4496					86.22	85.71
10	4504	4503					86.24	85.84
11	4504	4503					86.24	85.84
12	4502	4502	67.60	67.610	2.821	2.833	85.85	86.23
13	4503	4503					85.87	86.25
14	4498	4499					85.78	86.17
15	4513	4515					86.06	86.48
16	4854	4851	67.58	67.585	3.282	3.294	107.66	108.00
17	4853	4850					107.64	107.97
18	4854	4854					107.66	108.06
19	4849	4848					107.55	107.93
20	4856	4855	67.60	67.595	3.286	3.300	107.87	108.30
21	4853	4850					107.80	108.19
22	4853	4851					107.80	108.21

The quantity $\bar{d}_k$ is the average of the differences in the values for the last two columns. Since N_k is 22, i.e., since every item in this stratum was measured, the quantity $\hat{D}_k$ given by (eq. 3.5.1) is simply the total of the 22 differences. This is -3.48 pounds U-235.

The data layout in the table suggest the following model. The notation is unique to this example.

Shipper Values

$$
\begin{array}{lllllllll}
x_1 & = & \mu_1 & \delta_x & \theta_x & \gamma_x & \varepsilon_{x1} & \omega_{x1} & \nu_{x1} \\
\vdots & & & & \vdots & & & \vdots & \vdots \\
x_4 & = & \mu_4 & \delta_x & \theta_x & \gamma_x & \varepsilon_{x4} & \omega_{x1} & \nu_{x1} \\
 & & & & & & & & \\
x_5 & = & \mu_5 & \delta_x & \theta_x & \gamma_x & \varepsilon_{x5} & \omega_{x2} & \nu_{x2} \\
\vdots & & & & \vdots & & & \vdots & \vdots \\
x_{13} & = & \mu_{13} & \delta_x & \theta_x & \gamma_x & \varepsilon_{x13} & \omega_{x4} & \nu_{x4} \\
\vdots & & & & \vdots & & & \vdots & \vdots \\
x_{22} & = & \mu_{22} & \delta_x & \theta_x & \gamma_x & \varepsilon_{x22} & \omega_{x6} & \nu_{x6}
\end{array}
$$

where x_i = observed amount of U-235 in pounds for shipper

μ_i = true amount of U-235 in pounds

δ_x = shipper's systematic error in weighing

θ_x = shipper's systematic error in analysis for uranium concentrate

γ_x = shipper's systematic error in analysis for U-235 concentration

ε_{xi} = shipper's random error in weighing

ω_{xj} = shipper's random error in analysis for uranium concentrate

ν_{xj} = shipper's random error in analysis for U-235 concentration

For the receiver, the notation is similar except that y replaces x throughout. It is assumed that each error is a random variable with mean equal to one and standard deviations given below. All errors are assumed to be independently distributed.

Standard Deviations

Random Variable	Standard Deviation
δ_x	0.0001 (i.e., 0.01% relative)
θ_x	0.0004
γ_x	0.0006
ε_{xi}	0.0001
ω_{xj}	0.0002
ν_{xj}	0.0010

δ_y	0.0005
θ_y	0.0007
γ_y	0.0008
ε_{yi}	0.0005
ω_{yj}	0.0004
ν_{yj}	0.00025

The quantity $\hat{D}_1$, or $\hat{D}$ in this instance, is of the form:

$$\begin{aligned}\hat{D} = \mu_1 & (\delta_x\theta_x\gamma_x\varepsilon_{x1}\omega_{x1}\nu_{x1} - \delta_y\theta_y\gamma_y\varepsilon_{y1}\omega_{y1}\nu_{y1}) \\ & + \cdots \\ & + \mu_4 (\delta_x\theta_x\gamma_x\varepsilon_{x4}\omega_{x1}\nu_{x1} - \delta_y\theta_y\gamma_y\varepsilon_{y4}\omega_{y1}\nu_{y1}) \\ & + \mu_5 (\delta_x\theta_x\gamma_x\varepsilon_{x5}\omega_{x2}\nu_{x2} - \delta_y\theta_y\gamma_y\varepsilon_{y5}\omega_{y2}\nu_{y2}) \\ & + \cdots \\ & + \mu_{13} (\delta_x\theta_x\gamma_x\varepsilon_{x13}\omega_{x4}\nu_{x4} - \delta_y\theta_y\gamma_y\varepsilon_{y13}\omega_{y4}\nu_{y4}) \\ & + \cdots \\ & + \mu_{22} (\delta_x\theta_x\gamma_x\varepsilon_{x22}\omega_{x6}\nu_{x6} - \delta_y\theta_y\gamma_y\varepsilon_{y22}\omega_{y6}\nu_{y6})\end{aligned}$$

Following Method 3.2, a number of partial derivatives, evaluated at the mean values of the random variables, must be calculated. These are

$$\frac{\partial \hat{D}}{\partial \delta_x} = \frac{\partial \hat{D}}{\partial \theta_x} = \frac{\partial \hat{D}}{\partial \gamma_x} = \sum_{i=1}^{22} \mu_i$$

$$\frac{\partial \hat{D}}{\partial \delta_y} = \frac{\partial \hat{D}}{\partial \theta_y} = \frac{\partial \hat{D}}{\mu \gamma_y} = -\sum_{i=1}^{22} \mu_i$$

$$\frac{\partial \hat{D}}{\partial \varepsilon_{xi}} = \mu_i \quad , i = 1, 2, \cdots, 22$$

$$\frac{\partial \hat{D}}{\partial \varepsilon_{yi}} = -\mu_i \quad , i=1, 2, \cdots, 22$$

$$\frac{\partial \hat{D}}{\partial \omega_{x1}} = \frac{\partial \hat{D}}{\partial \nu_{x1}} = \mu_1 + \mu_2 + \mu_3 + \mu_4$$

$$\frac{\partial \hat{D}}{\partial \omega_{x2}} = \frac{\partial \hat{D}}{\partial \nu_{x2}} = \mu_5 + \mu_6 \quad \text{, etc.}$$

Various sums involving the μ_i are needed. The quantity μ_i is unknown and must be replaced by an estimate. For convenience, estimate μ_i by x_i. (It will make very little difference in propagating the errors whether one estimates μ_i by x_i, or by the mean $(y_i + x_i)/2$. Use the simplest value, x_i (or y_i).) Then

$$x_1 + x_2 + x_3 + x_4 = 431.40$$

$$x_5 + x_6 = 156.26$$

$$x_7 + x_8 + \cdots + x_{11} = 431.22$$

$$x_{12} + x_{13} + x_{14} + x_{15} = 343.56$$

$$x_{16} + x_{17} + x_{18} + x_{19} = 430.51$$

$$x_{20} + x_{21} + x_{22} = 323.47$$

$$\sum_{i=1}^{22} x_i = 2116.42$$

(Eq. 3.3.2) of Method 3.2 may now be applied to find the variance of $\hat{D}$. In the calculations displayed below, the terms having a common coefficient, (b_i^2 in the referenced equation), are combined.

	$\underline{\text{lbs}^2 \text{ U-235}}$
$(2116.42)^2(1+16+36+25+49+64) \times 10^{-8}$	$= 8.5553$
$(107.88^2+107.93^2+\cdots+107.80^2)(1+25) \times 10^{-8}$	$= 0.0537$
$(431.40^2+156.26^2+\cdots+323.47^2)(4+100+16+625) \times 10^{-8}$	$= 5.9934$
$\text{Var } \hat{D}$	$= 14.6024$
$\sigma_{\hat{D}}$	$= 3.82$ lbs U-235

Even for this rather simple model and small set of data, the calculations by the direct approach can quickly become burdensome. The question arises whether some simplifying assumptions can be made without greatly affecting the results. For example, in the second and third lines of the calculations displayed in the example, suppose average values were squared and then multiplied by the number of terms. The second line becomes:

$$22(96.20)^2(1+25) \times 10^{-8} = 0.0529 \quad \text{(compared with 0.0539)}$$

This third line becomes

$$6(352.74)^2(4+100+16+625) \times 10^{-8} = 5.5618 \quad \text{(compared with 5.9934)}$$

With these simplifying assumptions, $\sigma_{\hat{D}}$ would be 3.76 lbs U-235 rather than 3.82 lbs U-235, which is not a difference of major impact.

This example, and others of a similar nature, suggests that, as for the variance of MUF, a general approach for calculating the variance of $\hat{D}$ based on simplifying assumptions can also be developed. This is the subject of the next section.

3.5.3 Variance of $\hat{D}$ by General Approach

General formulas will be developed to permit simple calculation of the variance of $\hat{D}$. These formulas will be based on assumptions set forth in Section 3.5.3.1 As was true for the calculation of the variance of MUF by general formulas, the assumptions will rarely if ever be completely valid in given applications. However, experience has shown that this is not a great difficulty, since in many cases, even moderate departures from the assumptions have very little effect (as was illustrated by the example just presented). Further, if one has concern about the validity of the general formulas in a given instance, they can readily be altered as appropriate to accommodate a different set of circumstances.

3.5.3.1 Assumptions

The assumptions about the facility data were set forth in Section 3.4.3.1. (Again, the variance of $\hat{D}$ for element weight is considered here; see Section 3.5.4 for isotope weight.) The additional assumptions relative to the inspection are as follows:

(1) For samples of items within a stratum, the inspector also makes measurements. He need not necessarily make the same type of measurement as the facility, e.g., he may use nondestructive assay methods to a much greater extent than does the facility operator.

(2) The inspector and the facility use the same material sampling procedures, and hence, systematic errors in sampling will cancel. The effect of changing this assumption on the calculations should be quite obvious.

(3) When there are batches within a stratum, the inspector may first sample batches at random and then measure the same number of items in each batch sampled.

(4) The inspector may utilize a number of laboratories to analyze the samples, but for a given stratum, all use the same analytical method. This idea is carried further in the next section.

3.5.3.2 Notation

The notation is an extension to that given earlier in Section 3.4.3.2. The quantity y_{kqpt} is defined as was x_{kqpt}, except that y refers to an inspector value. The measurement methods q, p, t refer to his methods.

Note: As was the case with the facility operator, it may be that within a stratum, the same systematic error does not affect all items, i.e., there is a short term systematic error. Use parentheses to indicate the total element weight associated with "condition i" for a given measurement.

Since, under assumption (3) of the previous paragraph, the inspector may utilize a number of laboratories, this concept is extended to accommodate this possibility. Specifically,

$y_{kqpt(i(j))}$ = total element weight in stratum k as determined by the inspector using the indicated measurement methods, and for those items measured under condition j within laboratory i.

If need be, this idea can be extended further using additional classifications that may be either crossed or nested. However, the extension just indicated should be adequate to cover the great majority of applications.

To continue with the notation, δ with subscripts still denotes a relative standard deviation. The first subscript of r, s, or g is defined as in 3.4.3.2. If the first subscript is h, this refers to a short term systematic error within another such error, i.e., to a condition or time effect within a laboratory. Subscripts 2, 3, and 4 are defined as in 3.4.3.2. A fifth subscript is either x to refer to an operator standard deviation or y to refer to one for the inspector. Further, let u_k, w_k, v_k, and a_k denote inspector parameters associated with stratum k:

u_k = number of batches sampled by the inspector

w_k = number of items per sampled batch for which the inspector makes bulk measurements

v_k = number of samples drawn by the inspector per sampled batch to determine the element factor

a_k = number of analyses performed by the inspector per sample

With this notation in mind, Method 3.8 will now provide formulas needed to compute the random error variance of $\hat{D}$.

3.5.3.3 Random Error Variance of $\hat{D}$

Method 3.8

Notation

The notation is given in Sections 3.4.3.2 and 3.5.3.2.

Model

The statistic $\hat{D}$ is defined in (eq. 3.5.2). The model for a given term in the indicated sum is exemplified by the model discussed in example 3.2 (c).

Results

For stratum k, the random error variance of $\hat{D}_k$ due to measurement errors committed by the facility is

$$V_{rx}(\hat{D}_k) = x^2_{kqpt}[\delta^2_{rq\cdot\cdot x}/u_k w_k + \delta^2_{r\cdot p\cdot x}/u_k r_k + \delta^2_{r\cdot\cdot tx}/c_k u_k r_k] \quad \text{(eq. 3.5.3)}$$

That due to the inspector is

$$V_{ry}(\hat{D}_k) = y^2_{kqpt}[\delta^2_{rq\cdot\cdot y}/u_k w_k + \delta^2_{r\cdot p\cdot y}/u_k v_k + \delta^2_{r\cdot\cdot ty}/a_k u_k v_k] \quad \text{(eq. 3.5.4)}$$

The variance of $\hat{D}_k$ is

$$V_r(\hat{D}_k) = V_{rx}(\hat{D}_k) + V_{ry}(\hat{D}_k) \quad \text{(eq. 3.5.5)}$$

The variance of $\hat{D}$ is then found by summing $V_r(\hat{D}_k)$ over the strata.

$$V_r(\hat{D}) = \sum_{k=1}^{K} V_r(\hat{D}_k) \quad \text{(eq. 3.5.6)}$$

Basis

The formula for $V_r(\hat{D})$ is found by application of Methods 3.1 and 3.2. The *key equations, (eq. 3.5.3) and (eq. 3.5.4) are simple to remember if it is kept* in mind that the divisor for each variance component is the number of measurement operations affecting that component. For example, in (eq. 3.5.3), the divisor on $\delta^2_{rq\cdot\cdot x}$ is the number of batches sampled by the inspector, u_k, times the number of items weighed per sampled batch by the inspector. Since these items are compared on a one by one basis with the corresponding facility measurements, only those items weighed (i.e., bulk measured) by the inspector affect $\hat{D}_k$. All other items in that stratum that are weighed by the facility do not affect $\hat{D}_k$, and hence, the number of such weighings is not in the divisor. Similar reasoning holds for the second term. Only the numbers of samples drawn by the facility that affect $\hat{D}_k$ are included. This is the number of batches sampled by the inspector times the number of items sampled by the facility per such sampled batch.

Examples

EXAMPLE 3.8 (a)

The facility of examples 3.3 (a) and 3.5 (a) is inspected. The matrix below sets forth the pertinent parameter values for the inspector. The facility data are given in example 3.3 (a). For purposes of error propagation, assume that $x_{kqpt} = y_{kqpt}$.

	Stratum k						
	1	2	3	4	5	6	7
q	1	1	2	1	1	1	1
p	1	2	3	4	5	4	5
t	1	2	3	4	4	4	4
u_k	12	1	10	6	4	6	4
w_k	3	100	1	3	4	3	4
v_k	2	24	1	2	3	2	3
a_k	2	2	1	2	2	2	2

The error standard deviations for the inspector are as follows:

$\delta_{r1\cdot\cdot y} = 0.000658$ $\delta_{r\cdot 1\cdot y} = 0.000531$ $\delta_{r\cdot\cdot 1y} = 0.000433$

$\delta_{r2\cdot\cdot y} = 0$ $\delta_{r\cdot 2\cdot y} = 0$ $\delta_{r\cdot\cdot 2y} = 0.000822$

$\delta_{r\cdot 3\cdot y} = 0$ $\delta_{r\cdot\cdot 3y} = 0.0923$

$\delta_{r\cdot 4\cdot y} = 0.0181$ $\delta_{r\cdot\cdot 4y} = 0.0198$

$\delta_{r\cdot 5\cdot y} = 0.0418$

Equation 3.5.3 is now applied for each stratum

$V_{rx}(\hat{D}_1) = 1143\ kg^2U$

$V_{rx}(\hat{D}_2) = 515\ kg^2U$

$V_{rx}(\hat{D}_3) = 479\ kg^2U$

$V_{rx}(\hat{D}_4) = V_{rx}(\hat{D}_6) = 950\ kg^2U$

$V_{rx}(\hat{D}_5) = V_{rx}(\hat{D}_7) = 839\ kg^2U$

Equation 3.5.4 is now applied for each stratum

$V_{ry}(\hat{D}_1) = 1594\ kg^2U$

$V_{ry}(\hat{D}_2) = 1050\ kg^2U$

$V_{ry}(\hat{D}_3) = 1227\ kg^2U$

$V_{ry}(\hat{D}_4) = V_{ry}(\hat{D}_6) = 2263\ kg^2U$

$V_{ry}(\hat{D}_5) = V_{ry}(\hat{D}_7) = 2591\ kg^2U$

The random error variance of D, given by (eq. 3.5.6), is found by summing all the above values.

$$V_r(\hat{D}) = 19{,}294 \text{ kg}^2\text{U}$$

EXAMPLE 3.8 (b)

The facility of examples 3.3 (b), 3.3 (c), 3.4 (a), and 3.5 (b) is inspected. This is a mixed oxide fuel fabrication facility. The matrix below sets forth the pertinent parameter values for the inspector. The pertinent facility data are in example 3.3 (b). For purposes of error propagation, assume that $x_{kqpt} = y_{kqpt}$.

	Stratum k									
	1	2	3	4	5	6	7	8	9	10
q	1	2	2	3	1	1	1	2	2	2
p	1	2	3	4	5	3	6	5	3	6
t	1	2	3	4	3	3	2	3	3	2
u_k	16	40	4	8	6	2	2	4	2	1
w_k	6	12	1	1	5	1	1	7	1	1
v_k	3	5	1	1	2	1	1	2	1	1
a_k	2	2	2	1	2	2	2	2	2	2

The error standard deviations for the inspector are as follows.

$\delta_{r1\cdot\cdot y} = 0.00050$ $\quad\delta_{r\cdot 1\cdot y} = 0.0001$ $\quad\delta_{r\cdot\cdot 1y} = 0.0050$

$\delta_{r2\cdot\cdot y} = 0.00075$ $\quad\delta_{r\cdot 2\cdot y} = 0.0080$ $\quad\delta_{r\cdot\cdot 2y} = 0.0070$

$\delta_{r3\cdot\cdot y} = 0$ $\quad\delta_{r\cdot 3\cdot y} = 0.035$ $\quad\delta_{r\cdot\cdot 3y} = 0.010$

$\delta_{r\cdot 4\cdot y} = 0$ $\quad\delta_{r\cdot\cdot 4y} = 0.40$

$\delta_{r\cdot 5\cdot y} = 0.0040$

$\delta_{r\cdot 6\cdot y} = 0.020$

Equation 3.5.3 is now applied for each stratum, as is (eq. 3.5.4) for the inspector. All quantities are in kg^2 plutonium.

$$V_{rx}(\hat{D}_1) = 0.296817 \qquad V_{ry}(\hat{D}_1) = 0.621036$$

$$V_{rx}(\hat{D}_2) = 0.982474 \qquad V_{ry}(\hat{D}_2) = 0.978395$$

$$V_{rx}(\hat{D}_3) = 0.025538 \qquad V_{ry}(\hat{D}_3) = 0.025830$$

$$V_{rx}(\hat{D}_4) = 0.000800 \qquad V_{ry}(\hat{D}_4) = 0.003200$$

$$V_{rx}(\hat{D}_5) = 0.036630 \qquad V_{ry}(\hat{D}_5) = 0.069715$$

$$V_{rx}(\hat{D}_6) = 0.008172 \qquad V_{ry}(\hat{D}_6) = 0.008264$$

$$V_{rx}(\hat{D}_7) = 0.004305 \qquad V_{ry}(\hat{D}_7) = 0.004301$$

$$V_{rx}(\hat{D}_8) = 0.079079 \qquad V_{ry}(\hat{D}_8) = 0.150722$$

$$V_{rx}(\hat{D}_9) = 0.012769 \qquad V_{ry}(\hat{D}_9) = 0.012915$$

$$V_{rx}(\hat{D}_{10}) = 0.002152 \qquad V_{ry}(\hat{D}_{10}) = 0.002152$$

The random error variance of $\hat{D}$, given by (eq. 3.5.6), is found by summing all the above values.

$$V_r(\hat{D}) = 3.325266 \text{ kg}^2 \text{ plutonium}$$

3.5.3.4 Systematic Error Variance of $\hat{D}$

In developing the formulas needed to compute the systematic error variance of $\hat{D}$, attention will first be directed at the short term systematic error. Method 3.9 applies.

Method 3.9

Notation

The notation is given in Sections 3.4.3.2 and 3.5.3.2.

Model

See the discussion for the model in Method 3.8.

Results

The calculations indicated need only be performed for those measurements for which the first subscript on δ is g or h, i.e., for the non-zero short term systematic error variances.

For each combination of values q(i), calculate

$$M_{q(i)\cdot\cdot x} = \sum_{k=1}^{K} A_k x_{kq(i)pt} \qquad \text{(eq. 3.5.7)}$$

where A_k = +1 for input and beginning inventory strata and where A_k = -1 for output and ending inventory strata.

For each combination of values t(i), calculate

$$M_{\cdot\cdot t(i)x} = \sum_{k=1}^{K} A_k x_{kqpt(i)} \qquad \text{(eq. 3.5.8)}$$

where the A_k are defined as for (eq. 3.5.7).

The contribution to the short term systematic error variance of $\hat{D}$ due to facility measurements is

$$V_{gx}(\hat{D}) = \sum_{q} \delta^2_{gq\cdot\cdot x} \sum_{i} M^2_{q(i)\cdot\cdot x}$$

$$+ \sum_{t} \delta^2_{g\cdot\cdot tx} \sum_{i} M^2_{\cdot\cdot t(i)x} \qquad \text{(eq. 3.5.9)}$$

For the inspector, for each combination of values q(i), calculate

$$M_{q(i)\cdot\cdot y} = \sum_{k=1}^{K} A_k y_{kq(i)pt} \qquad \text{(eq. 3.5.10)}$$

with the A_k defined as for (eq. 3.5.7).

For each combination of values t(i(j)), calculate

$$M_{\cdot\cdot t(i(j))y} = \sum_{k=1}^{K} A_k y_{kqpt(i(j))} \qquad \text{(eq. 3.5.11)}$$

with A_k defined as above. Finally, for each combination t(i), calculate from (eq. 3.5.11),

$$M_{\cdot\cdot t(i)y} = \sum_{j} M_{\cdot\cdot t(i(j))y} \qquad \text{(eq. 3.5.12)}$$

The contribution to the short term systematic error variance of $\hat{D}$ due to inspector measurements is

$$V_{gy}(\hat{D}) = \sum_q \delta^2_{gq\cdot\cdot y} \sum_i M^2_{q(i)\cdot\cdot y} + \sum_t \delta^2_{h\cdot\cdot ty} \sum_{i,j} M^2_{\cdot\cdot t(i(j))y}$$

$$+ \sum_t \delta^2_{g\cdot\cdot ty} \sum_i M^2_{\cdot\cdot t(i)y} \qquad \text{(eq. 3.5.13)}$$

The total short term systematic error variance of $\hat{D}$ is

$$V_g(\hat{D}) = V_{gx}(\hat{D}) + V_{gy}(\hat{D}) \qquad \text{(eq. 3.5.14)}$$

Basis

The basis for Method 3.9 is the same as for Method 3.8. The distinction is that for random error variances, one first squares quantities and then sums them; for systematic error variances, one sums and then squares.

It is implicitly assumed in this method that both the inspector and the facility apply the same material sampling procedures and hence commit the same systematic errors. For the $\hat{D}$ statistic, these errors would then cancel. This is why, in the method, there are no expressions for $M_{\cdot p(i)\cdot x}$ and $M_{\cdot p(i)\cdot y}$. Should this assumption not be valid, the equations used to compute these two quantities are essentially the same as (eq. 3.5.7) and (eq. 3.5.10), with obvious modifications. The equations (eq. 3.5.9) and (eq. 3.5.13) would each contain the additional set of terms.

Examples

EXAMPLE 3.9 (a)

The low enriched uranium fuel fabrication facility of example 3.8 (a) is continued. Say that there are no short term systematic errors for either the facility or the inspector except for those occurring because the inspector distributes the inspection samples to four laboratories. The values for the error parameters are:

$$\delta_{g\cdot\cdot 1y} = 0.000544$$

$$\delta_{g\cdot\cdot 2y} = 0.000522$$

$$\delta_{g\cdot\cdot 4y} = 0.00711$$

The allocation of samples to the various laboratories is given by the following table. The tabular entries are the amounts of uranium, in kilograms, represented by the samples sent to the laboratories, i.e., they are $y_{kqpt(i)}$ values.

			Laboratory (i)			
Stratum	A_k	t	1	2	3	4
1	1	1	120,000	120,000	0	0
2	-1	2	79,600	79,600	79,600	0
4	1	4	7,200	0	0	0
5	1	4	1,000	1,000	1,000	1,000
6	-1	4	3,600	3,600	0	0
7	-1	4	2,000	2,000	0	0

The quantities $M_{\cdot\cdot t(i)y}$ are calculated from (eq. 3.5.12). First, (eq. 3.5.11) must be applied, where j = 1 in all cases. Since j = 1, $M_{\cdot\cdot t(i(j))y}$ and $M_{\cdot\cdot t(i)y}$ are equivalent, and so the latter quantity is calculated from (eq. 3.5.11).

$$M_{\cdot\cdot 1(1)y} = 120{,}000 \qquad M_{\cdot\cdot 1(2)y} = 120{,}000$$

$$M_{\cdot\cdot 2(1)y} = M_{\cdot\cdot 2(2)y} = M_{\cdot\cdot 2(3)y} = -79{,}600$$

$$M_{\cdot\cdot 4(1)y} = 7200 + 1000 - 3600 - 2000 = 2600$$

$$M_{\cdot\cdot 4(2)y} = 1000 - 3600 - 2000 = -4600$$

$$M_{\cdot\cdot 4(3)y} = M_{\cdot\cdot 4(4)y} = 1000$$

Then, (eq. 3.5.13) is applied.

$$\begin{aligned} V_{gy}(\hat{D}) &= (0.000544)^2[(120{,}000)^2 + (120{,}000)^2] \\ &+ (0.000522)^2[3(-79{,}600)^2] \\ &+ (0.00711)^2[(2600)^2 + (-4600)^2 + 2(1000)^2] \\ &= 15{,}215 \text{ kg}^2\text{U} \end{aligned}$$

Finally, from (eq. 3.5.14), the total short term systematic error variance of $\hat{D}$ is

$$V_g(\hat{D}) = 15{,}215 \text{ kg}^2\text{U}$$

This contribution to the total error is a result of distributing the samples to different laboratories. Had only one laboratory been utilized, the corresponding value would have been

$$\begin{aligned} V_{gy}(\hat{D}) &= (0.000544)^2(240{,}000)^2 + (0.000522)^2(-238{,}800)^2 \\ &= 32{,}584 \text{ kg}^2\text{U} \end{aligned}$$

EXAMPLE 3.9 (b)

Continue with the mixed oxide fuel fabrication facility of example 3.8 (b). For the facility measurements, the information on the short term systematic errors was given in example 3.4 (a). The value calculated for $V_g(\text{MUF})$ in that example is identical with $V_{gx}(\hat{D})$, viz,

$$V_{gx}(\hat{D}) = 3.359774 \text{ kg}^2\text{Pu}$$

This result is true in general as long as the inspector does not commit the same systematic errors. Since it is assumed in Method 3.9 that the facility and the inspector commit the same systematic errors in sampling, but not in analytical, and since the only short term systematic errors assumed to exist in this example are those due to analytical, the result that $V_g(\text{MUF}) = V_{gx}(D)$ holds in this example.

Turning to the inspector measurements, assume that he distributes the samples to only one laboratory, but that the short term systematic error standard deviations for analytical measurements, including NDA measurements, are as follows:

$$\delta_{g\cdot\cdot 1y} = 0.0016$$

$$\delta_{g\cdot\cdot 2y} = 0.0020$$

$$\delta_{g\cdot\cdot 3y} = 0.0025$$

$$\delta_{g\cdot\cdot 4y} = 0.12$$

The table below gives the amounts associated with each measurement error shift by stratum and by measurement method, i.e., the tabled values are $y_{kqpt(i)}$ values, in kg Pu.

			Short Term Measurement Error (i)		
Stratum	A_k	t	1	2	3
1	1	1	768	768	-
2	-1	2	288	765	432
3	-1	3	-	9.0	-
4	-1	4	0.4	-	-
5	1	3	112.5	-	-
6	1	3	3.6	-	-
7	1	2	4.5	-	-
8	-1	3	-	135	-
9	-1	3	-	4.5	-
10	-1	2	-	-	2.25

Since only the one laboratory is involved in this example, the quantities $M_{\cdot\cdot t(i)y}$ are calculated as in the previous example. Physically, these values now represent total amounts of plutonium associated with each measurement error rather than with each laboratory; the error propagation is identical.

$$M_{\cdot\cdot 1(1)y} = M_{\cdot\cdot 1(2)y} = 768$$

$$M_{\cdot\cdot 2(1)y} = -288 + 4.5 = -283.5$$

$$M_{\cdot\cdot 2(2)y} = -765$$

$$M_{\cdot\cdot 2(3)y} = -432 - 2.25 = -434.25$$

$$M_{\cdot\cdot 3(1)y} = 112.5 + 3.6 = 116.1$$

$$M_{\cdot\cdot 3(2)y} = -9.0 - 135 - 4.5 = -148.5$$

$$M_{\cdot\cdot 4(1)y} = -0.4$$

Then, (eq. 3.5.13) is applied.

$$\begin{aligned} V_{gy}(\hat{D}) &= (0.0016)^2[2(768)^2] + (0.0020)^2[(-283.5)^2 + (-765)^2 \\ &\quad + (-434.25)^2] + (0.0025)^2[(116.1)^2 + (-148.5)^2] \\ &\quad + (0.12)^2(-0.4)^2 \\ &= 6.660956 \text{ kg}^2 \text{ Pu} \end{aligned}$$

Finally, from (eq. 3.5.14), the total short term systematic error variance of $\hat{D}$ is

$$V_g(\hat{D}) = 3.359774 + 6.660956 = 10.020730 \text{ kg}^2 \text{ Pu}$$

EXAMPLE 3.9 (c)

In the example just completed, assume that the samples of P_uO_2 drawn from stratum 1 are distributed to three laboratories and that within each laboratory, they are analyzed under two sets of conditions (shift in short-term systematic error). Then, the value for $\delta_{g\cdot\cdot 1y}$ of 0.0016 given in the previous example becomes $\delta_{h\cdot\cdot 1y}$ since $\delta_{g\cdot\cdot 1y}$ must now represent the laboratory effect for analytical method 1. Suppose that

$$\delta_{g\cdot\cdot 1y} = 0.0008$$

and recalculate $V_{gy}(\hat{D})$.

For application in (eq. 3.5.11) and following, the following input values are used:

$$y_{1111(1(1))} = y_{1111(1(2))} = 256$$

$$y_{1111(2(1))} = y_{1111(2(2))} = 256$$

$$y_{1111(3(1))} = y_{1111(3(2))} = 256$$

Applying (eq. 3.5.11), the y values just given are identically the same as the M values of (eq. 3.5.11). From (eq. 3.5.12),

$$M_{\cdot\cdot t(1)y} = M_{\cdot\cdot t(2)y} = M_{\cdot\cdot t(3)y} = 512$$

Then, from (eq. 3.5.13),

$$\begin{aligned} V_{gy}(\hat{D}) &= (0.0016)^2[6(256)^2] + (0.0008)^2[3(512)^2] + C \\ &= 1.509949 \text{ kg}^2 \text{ Pu} + C \end{aligned}$$

The remaining terms are represented by C. From the previous example,

$$C = 6.660956 - (0.0016)^2[2(768)^2] = 3.641057$$

Therefore,

$$V_{gy}(\hat{D}) = 5.151006 \text{ kg}^2 \text{ Pu}$$

and $$V_g(\hat{D}) = 8.510780 \text{ kg}^2 \text{ Pu}$$

Note that the reduction in size from the previous example occurs because now there are six sets of analytical conditions rather than three, causing more of an averaging effect.

Next, consider the long term systematic error variance of $\hat{D}$. The calculations described in Method 3.10 follow easily from those in Method 3.9.

Method 3.10

Notation

The notation is given in Sections 3.4.3.2 and 3.5.3.2.

Model

See the discussion for the model of Method 3.8.

Results

First, consider the systematic errors for the facility measurements. For each value of q, calculate

$$M_{q\cdot\cdot x} = \sum_{k=1}^{K} A_k \, x_{kqpt} \qquad \text{(eq. 3.5.15)}$$

where A_k = +1 for input and beginning inventory strata and A_k = -1 for output and ending inventory strata. Note that if the calculations indicated by (eq. 3.5.7) are performed for each value of q, then $M_{q\cdot\cdot x}$ may be found by summing the $M_{q(i)\cdot\cdot x}$ values over i. Similar statements hold for equations to follow.

For each value of t, calculate

$$M_{\cdot\cdot tx} = \sum_{k=1}^{K} A_k \, x_{kqpt} \qquad \text{(eq. 3.5.16)}$$

with the A_k defined as above.

The long term systematic error variance of D due to facility measurements is

$$V_{sx}(\hat{D}) = \sum_{q} M^2_{q\cdot\cdot x} \, \delta^2_{sq\cdot\cdot x} + \sum_{t} M^2_{\cdot\cdot tx} \, \delta^2_{s\cdot\cdot tx} \qquad \text{(eq. 3.5.17)}$$

For the inspector measurements, for each value of q, calculate

$$M_{q\cdot\cdot y} = \sum_{k=1}^{K} A_k \, y_{kqpt} \qquad \text{(eq. 3.5.18)}$$

and for each value of t, calculate

$$M_{\cdot\cdot ty} = \sum_{k=1}^{K} A_k \, y_{kqpt} \qquad \text{(eq. 3.5.19)}$$

where A_k is again defined as above. The long-term systematic error variance of $\hat{D}$ due to inspector measurements is

$$V_{sy}(\hat{D}) = \sum_{q} M^2_{q\cdot\cdot y} \, \delta^2_{sq\cdot\cdot y} + \sum_{t} M^2_{\cdot\cdot ty} \, \delta^2_{s..ty} \qquad \text{(eq. 3.5.20)}$$

Finally, the total systematic error variance of $\hat{D}$ is

$$V_s(\hat{D}) = V_{sx}(\hat{D}) + V_{sy}(\hat{D}) \qquad \text{(eq. 3.5.21)}$$

Basis

See the discussion for the basis in Method 3.9.

Examples

EXAMPLE 3.10 (a)

Continue with the example of 3.9 (a). For the facility measurements, the information on the long-term systematic errors was given in example 3.5 (a). The value calculated for $V_S(MUF)$ in that example was 41,332 kg^2U. In this particular example, $V_S(MUF)$ and $V_{SX}(\hat{D})$ have identical values because although $V_S(MUF)$ includes the effects of systematic errors in sampling while $V_{SX}(\hat{D})$ does not (under the assumption that the inspector commits the same systematic errors in sampling as does the facility), in this particular example, the contribution to $V_S(MUF)$ from this error source was zero. Therefore,

$$V_{sx}(\hat{D}) = 41{,}332 \text{ kg}^2\text{U}$$

For the inspector measurements, the error parameter values are:

$$\delta_{s1\cdot\cdot y} = 0.000439 \qquad \delta_{s\cdot\cdot 1y} = 0.000172$$

$$\delta_{s\cdot\cdot 2y} = 0.000165$$

$$\delta_{s\cdot\cdot 3y} = 0.0692$$

$$\delta_{s\cdot\cdot 4y} = 0.00225$$

The values for $M_{q\cdot\cdot y}$ and $M_{\cdot\cdot ty}$ are calculated from (eq. 3.5.18) and (eq. 3.5.19).

$$M_{1\cdot\cdot y} = 240{,}000-238{,}800 = 1200$$

$$M_{\cdot\cdot 1y} = 240{,}000$$

$$M_{\cdot\cdot 2y} = -238{,}800$$

$$M_{\cdot\cdot 3y} = -1200$$

$$M_{\cdot\cdot 4y} = 0$$

(Eq. 3.5.20) is applied.

$$V_{sy}(\hat{D}) = (1200)^2(0.000439)^2 + (240{,}000)^2(0.000172)^2$$

$$+ (-238{,}800)^2(0.000165)^2 + (-1200)^2(0.0692)^2$$

$$= 10{,}152 \text{ kg}^2\text{U}$$

The total systematic error variance of $\hat{D}$ is given by (eq. 3.5.21).

$$V_s(\hat{D}) = 41{,}332 + 10{,}152 = 51{,}484 \text{ kg}^2\text{U}$$

The results from examples 3.8 (a), (random error) 3.9 (a), (short term systematic error), and this example are now combined to give the total variance of $\hat{D}$.

$$V(\hat{D}) = V_r(\hat{D}) + V_g(\hat{D}) + V_s(\hat{D})$$

$$= 19{,}294 + 15{,}215 + 51{,}484 = 85{,}993 \text{ kg}^2\text{U}$$

EXAMPLE 3.10 (b)

Continue with the example of 3.9 (b). For the facility measurements, the information on the long term systematic error variance was given in example 3.5 (b). The value calculated for V_s(MUF) in that example was 6.920414 kg^{2}Pu.

Unlike the example just completed, V_s(MUF) and $V_{sx}(\hat{D})$ are not the same value in the current example. This is so because V_s(MUF) includes systematic errors in sampling while $V_{sx}(\hat{D})$ does not, it being assumed that the facility and the inspector commit the same systematic sampling errors. From example 3.5 (b), that part of V_s(MUF) due to sampling is

$$(-1485)^2(0.0010)^2 + (-9.9)^2(0.015)^2 + (-22.5)^2(0.0024)^2 + (2.25)^2(0.008)^2$$

$$= 2.230517 \text{ kg}^2\text{Pu}$$

Therefore,

$$V_{sx}(\hat{D}) = 6.920414 - 2.230517$$

$$= 4.689897 \text{ kg}^2\text{Pu}$$

For the inspector measurements, the error parameter values are:

$\delta_{s1\cdot\cdot y} = 0.00030$ $\qquad$ $\delta_{s\cdot\cdot 1y} = 0.0012$

$\delta_{s2\cdot\cdot y} = 0.00050$ $\qquad$ $\delta_{s\cdot\cdot 2y} = 0.0015$

$\delta_{s\cdot\cdot 3y} = 0.0020$

$\delta_{s\cdot\cdot 4y} = 0.15$

The values for $M_{q\cdot\cdot y}$ and $M_{\cdot\cdot ty}$ are calculated from (eq. 3.5.18) and (eq. 3.5.19). From the data matrices of examples 3.3 (b) and 3.8 (b),

$$M_{1\cdot\cdot y} = 1536 + 112.5 + 3.6 + 4.5 = 1656.6$$

$$M_{2\cdot\cdot y} = -1485 -9.0 -135 -4.5 -2.25 = -1635.75$$

From example 3.9 (b),

$$M_{\cdot\cdot 1y} = 2(768) = 1536$$

$$M_{\cdot\cdot 2y} = -283.5 -765 -434.25 = -1482.75$$

$$M_{\cdot\cdot 3y} = 116.1 -148.5 = -32.4$$

$$M_{\cdot\cdot 4y} = -0.4$$

Then, (eq. 3.5.20) is applied.

$$\begin{aligned} V_{sy}(\hat{D}) &= (1656.6)^2(0.00030)^2 + (-1635.75)^2(0.00050)^2 \\ &+ (1536)^2(0.0012)^2 + (-1482.75)^2(0.0015)^2 \\ &+ (-32.4)^2(0.0020)^2 + (-0.4)^2(0.15)^2 \\ &= 9.267826 \text{ kg}^2\text{Pu} \end{aligned}$$

The total systematic error variance of D is given by (eq. 3.5.21).

$$\begin{aligned} V_s(\hat{D}) &= 4.689897 + 9.267826 \\ &= 13.957723 \text{ kg}^2\text{Pu} \end{aligned}$$

The results from examples 3.8 (b), (random error), 3.9 (b), (short-term systematic error, with all samples sent to the same laboratory), and this example are now combined to give the total variance of $\hat{D}$.

$$\begin{aligned} V(\hat{D}) &= 3.325266 + 10.020730 + 13.957723 \\ &= 27.303719 \text{ kg}^2\text{Pu} \end{aligned}$$

3.5.4 Variance of $\hat{D}$ for Isotope

If the variance of $\hat{D}$ is to be calculated for isotope weight rather than for element weight, then the additional uncertainty associated with the determination of the isotope factors must be included. Before considering the methods for performing these calculations, the discussion in Section 3.4.5 should be reread. This discussion places in perspective the preferred role for measurements of isotope in certain kinds of facilities.

Two methods are given for performing the calculation. Method 3.11 gives the procedures to follow when calculating the variance of isotope $\hat{D}$ caused by uncertainties in the measurement of bulk and in the sampling and analytical measurements for element. Method 3.12 provides the additional calculations needed to factor in the uncertainty due to isotope measurements. In application, of course, both methods must be followed.

Method 3.11

Notation

The notation is given in Section 3.5.3.2 except that the element weights are now isotope weights.

Model

See the discussion for the model of Method 3.8. For isotope weight, the multiplicative model is still used, since the amount of isotope is found by multiplying the amount of element by the isotope concentration factor when the amount of isotope is found by the bulk, sampling, analytical measurement route as opposed to by nondestructive assay.

Results

First, delete from the calculations all strata in which the amount of isotope is determined directly by nondestructive assay, i.e., in which the amount of isotope is not derived from the amount of element by application of a measured isotope concentration factor. If there are strata for which measurements of the amount of isotope are found by bulk-sampling-analytical by one party and by nondestructive assay by the other, delete from the calculations only those that apply to the party using nondestructive assay.

After making the appropriate data deletions, follow Methods 3.8, 3.9, and 3.10, replacing all element weights by isotope weights in the calculations. This may be done by determining the weighted average enrichment per stratum for all the material in the stratum (i.e., not for just those items inspected). Then, in the case of the random error variance, multiply the previously calculated $V_{rx}(\hat{D}_k)$ and $V_{ry}(\hat{D}_k)$ values by the squares of these average enrichments, expressed as fractions rather than as percentages. In the case of the short-term and long-term systematic error variances, the weighted average enrichment fractions must be multiplied by the stratum totals (and by the sub-stratum totals in the case of short term errors) to express these totals in amounts of isotope before finding the sums associated with each systematic error.

Basis

The basis for Method 3.11 is essentially the same as for Method 3.8.

It is implicitly assumed in this method that the weighted average enrichments for the items inspected are the same as for the uninspected items in each stratum (or substratum). More exactly, the coefficient that converts relative error variances to absolute amounts in the error propagation should be the square of the product of the total number of items in the stratum and the average amount of isotope per item inspected. Thus, only if the average amount per item inspected is the same as the average amount per uninspected item will the propagation formulas be correct in the strict sense. This assumption is necessary because the $\hat{D}$ statistic that is in common usage in Agency inspection is implicitly based on a model in which errors are constant on an absolute basis, whereas the error structure generally assumed in measurements of this type is one in which errors are constant on a relative basis. (To be consistent, it would be more appropriate to work with ratios of facility to inspector measurements rather than with differences, but the advantages that would follow from this change in procedure are outweighed by the problems in introducing the ratio statistic to replace the familiar difference statistic.) The problem is quite academic and only becomes important if a given stratum contains material with widely varying enrichments and if the inspection concentrates on certain enrichments in the stratum that are, in total, not representative.

Examples

EXAMPLE 3.11 (a)

Data relative to the inspection of the previously discussed low enriched fuel fabrication plant are given in examples 3.8 (a), 3.9 (a), and 3.10 (a). Material enrichments for the plant are given in example 3.6 (a).

Assume that the inspector obtains quantitive measures of U-235 concentrations using the stabilized assay meter (SAM-2) except in stratum 3, the solid waste output stratum. In stratum 3, nondestructive assay measurements are made directly of the amount of U-235. Since the same is true of the facility measurements, stratum 3 is deleted in the calculations to follow.

Following Method 3.11, the first step is to calculate the average enrichment for all strata but stratum 3. From the data tables of examples 3.3 (a) and 3.6 (a), these weighted enrichments are, by stratum:

Stratum		
1	6759.00/240,000	= 0.02816
2	6723.21/238,800	= 0.02815
4	218.16/7200	= 0.03030
5	103.20/4000	= 0.02580
6	211.47/7200	= 0.02937
7	109.85/4000	= 0.02746

Then, the random error variances are (see example 3.8 (a)):

$$V_r(\hat{D}_1) = (0.02816)^2(1143 + 1594) = \underset{}{\overset{\text{kg}^2\text{U-235}}{2.1704}}$$

$$V_r(\hat{D}_2) = (0.02815)^2(515 + 1050) = 1.2401$$

$$V_r(\hat{D}_4) = (0.03030)^2(950 + 2263) = 2.9498$$

$$V_r(\hat{D}_5) = (0.02580)^2(839 + 2591) = 2.2831$$

$$V_r(\hat{D}_6) = (0.02937)^2(950 + 2263) = 2.7715$$

$V_r(\hat{D}_7) = (0.02746)^2(839 + 2591) = 2.5864$

Summing, $V_r(\hat{D}) = 14.0013$ kg^{2}U-235.

Next, the short term systematic error variance of $\hat{D}$ for isotope is calculated, following example 3.9 (a). In the cited example, the data table indicated the amounts of uranium represented by the samples distributed to the four laboratories. By applying the weighted average enrichment per stratum to that table, these amounts are converted to kilograms of U-235.

			Laboratory (i)			
Stratum	A_k	t	1	2	3	4
1	1	1	3379.20	3379.20	-	-
2	-1	2	2240.74	2240.74	2240.74	-
4	1	4	218.16	-	-	-
5	1	4	25.80	25.80	25.80	25.80
6	-1	4	105.73	105.73	-	-
7	-1	4	54.92	54.92	-	-

The quantities $M_{\cdot\cdot t(i)y}$ are calculated as in example 3.9 (a).

$M_{\cdot\cdot 1(1)y} = M_{\cdot\cdot 1(2)y} = 3379.20$

$M_{\cdot\cdot 2(1)y} = M_{\cdot\cdot 2(2)y} = M_{\cdot\cdot 2(3)y} = 2240.74$

$M_{\cdot\cdot 4(1)y} = 218.16 + 25.80 - 105.73 - 54.92 = 83.31$

$M_{\cdot\cdot 4(2)y} = 25.80 - 105.73 - 54.92 = -134.85$

$M_{\cdot\cdot 4(3)y} = M_{\cdot\cdot 4(4)y} = 25.80$

Then, (eq. 3.5.13) is applied.

$$\begin{aligned} V_{gy}(\hat{D}) &= (0.000544)^2[2(3379.20)^2] + (0.000522)^2[3(2240.74)^2] \\ &+ (0.00711)^2[(83.31)^2 + (-134.85)^2 + 2(25.80)^2] \\ &= 12.2004 \text{ kg}^2 \text{ U-235} \end{aligned}$$

Since, in this example, there is no contribution to the short term systematic error variance from the facility measurements

$V_g(\hat{D}) = V_{gy}(\hat{D}) = 12.2004$ kg^2 U-235

Finally, the long term systematic error variance is calculated following Method 3.10. The calculations of example 3.10 (a) illustrate Method 3.10 for uranium, and these are now repeated for U-235.

First, consider the contribution to this error variance from the facility measurements. Following the same argument as presented in the first paragraph of the example 3.10 (a) discussion, and with reference to the calculations of example 3.6 (a), the result is

$$V_{sx}(\hat{D}) = V_s(\text{MUF}) = 30.3405 \text{ kg}^2\text{U-235}$$

For the inspector measurements, the values for $M_{q\cdot\cdot y}$ and $M_{\cdot\cdot ty}$ are calculated as in example 3.10 (a).

$$M_{1\cdot\cdot y} = 6759.00 - 6723.21 + 218.16 + 103.20$$

$$-211.47 - 109.85$$

$$= 35.83 \quad ,$$

since scale 1 is used in all strata.

$$M_{\cdot\cdot 1y} = 6759.00$$

$$M_{\cdot\cdot 2y} = -6723.21$$

$$M_{\cdot\cdot 4y} = 218.16 + 103.20 - 211.47 - 109.85 = 0.04$$

Note that $M_{\cdot\cdot 3y}$ is not calculated since the stratum 3 measurements have been deleted. Then, from (eq. 3.5.20),

$$V_{sy}(\hat{D}) = (35.83)^2(0.000439)^2 + (6759.00)^2(0.000172)^2$$

$$+ (-6723.21)^2(0.000165)^2 + (0.04)^2(0.00225)^2$$

$$= 2.5824 \text{ kg}^2\text{U-235}$$

The total systematic error variance of $\hat{D}$ is given by (eq. 3.5.21)

$$V_s(\hat{D}) = 30.3405 + 2.5824 = 32.9229 \text{ kg}^2\text{U-235}$$

The random, short term systematic, and long term systematic error variances are now added to give the total variance of $\hat{D}$ due to uncertainties in the measurement of bulk and of element concentrations:

$$14.0013 + 12.2004 + 32.9229$$

$$= 59.1246 \text{ kg}^2\text{U-235}$$

Method 3.11, as exemplified in example 3.11 (a) just completed, gives the variance of $\hat{D}$ in isotope weight that is caused by uncertainties in the measurement of bulk and in the sampling and analytical measurement for element. Thus, the result in example 3.11 (a), viz, $V_t(D)$ = 59.1246 kg U-235, does not include the effects of errors in measuring the isotope that are committed by either the facility or the inspector. Procedures for including these errors are detailed in Method 3.12

Method 3.12

Notation

The notation is an extension to that given earlier in Method 3.7. For the error standard deviations, a slight change is made in the notation to distinguish between facility and inspector errors. Specifically, the following errors are defined.

$\delta^*_{r \cdot p \cdot}$ = random error standard deviation in sampling for isotope, assumed to be the same for the facility and the inspector

$\delta^*_{r \cdot \cdot tx}$ = random error standard deviation in isotopic analysis for the facility's analytical method t

$\delta^*_{s \cdot \cdot tx}$ = systematic error standard deviation in isotopic analysis for the facility's analytical method t

$\delta^*_{r \cdot \cdot ty}$ = random error standard deviation in isotopic analysis for the inspector's analytical method t

$\delta^*_{s \cdot \cdot ty}$ = systematic error standard deviation in isotopic analysis for the inspector's analytical method t

Additional parameters relating to the inspector's measurement are defined:

v_i^* = number of samples drawn by the inspector to verify the facility's isotopic concentration factor i

a_i^* = number of isotopic analyses per sample performed by the inspector

Model

See the discussion for the model in Method 3.11.

Results

For facility measurements, the random and systematic error contributions to the variance of $\hat{D}$ due to measurements of isotopic concentrations are denoted by $V_{rx}(\hat{D})$ and $V_{sx}(\hat{D})$ respectively. They are calculated by:

$$V^*_{rx}(\hat{D}) = \sum_{i=1}^{G} S_i^2 \left(\delta^{*2}_{r \cdot p \cdot}/r^*_i + \delta^{*2}_{r \cdot \cdot tx}/r^*_i c^*_i\right) \qquad \text{(eq. 3.5.22)}$$

$$V^{*2}_{sx}\ (\hat{D}) = \sum_t T^2_{tx}\ \delta^{*2}_{s\cdot\cdot tx} \qquad \text{(eq. 3.5.23)}$$

These equations are identical to those for V^*_r(MUF) and V^*_s(MUF) given by (eq. 3.4.13) and (eq. 3.4.14) respectively. An additional subscript is added to $\delta^*_{r\cdot\cdot t}$, $\delta^*_{s\cdot\cdot t}$, and T_t in the cited equations to designate that these parameters relate to facility measurements, but the numerical calculations are identical.

For the inspector, the random error is

$$V^*_{ry}\ (\hat{D}) = \sum_{i=1}^{G} S^2_i\ (\delta^{*2}_{r\cdot p\cdot}/v^*_i + \delta^{*2}_{r\cdot\cdot ty}/v^*_i\ a^*_i) \qquad \text{(eq. 3.5.24)}$$

To compute the systematic error, first calculate

T_{ty} = sum of S_i values based on the inspector's isotopic analytical method t

$$V^*_{sy}\ (\hat{D}) = \sum_t T^2_{ty}\ \delta^{*2}_{s\cdot\cdot ty} \qquad \text{(eq. 3.5.25)}$$

The total variance of $\hat{D}$ due to isotopic measurements is

$$V_t^*\ (\hat{D}) = V^*_{rx}(\hat{D}) + V^*_{sx}(\hat{D}) + V^*_{ry}(\hat{D}) + V^*_{sy}(\hat{D}) \qquad \text{(eq. 3.5.26)}$$

To compute the total variance of the isotope D due to all sources of error, the results of Methods 3.11 and 3.12 are combined. That is, find the sum

$$V^*(\hat{D}) = V\ (\hat{D}) + V_t^*\ (\hat{D}) \qquad \text{(eq. 3.5.27)}$$

<u>Basis</u>

The basis for this method is essentially the same as for Method 3.8.

Some important assumptions on which the computational equations are based should be emphasized. First, since $\delta^*_{r\cdot p\cdot}$ has no x or y subscript, it is implicitly assumed that both parties use the same sampling equipment. Further, it is assumed that systematic errors in sampling for isotope are zero. Finally, no provision is made for calculating a short term systematic error variance for isotope measurements; such errors are assumed to be zero.

If any of these assumptions should be invalid in a given application, the additional calculations required should be apparent from similar applications detailed by the methods of this chapter.

<u>Examples</u>

EXAMPLE 3.12 (a)

Continue with example 3.11 (a). Additional information relative to the inspector's measurements is required.

In stratum 1, of the 12 batches inspected, 6 have 3.25% enrichment, 4 have 2.67%, and 2 have 1.52%. To measure the enrichment, the inspector uses the stabilized assay meter (SAM-2) on 5 items per sampled batch.

In stratum 2, SAM-2 measurements are made on 50 items. Twenty of these are at 3.25% U-235, 15 at 2.67%, 5 at 1.52%, and 10 at 2.87%.

In stratum 3, the inspector measures the amount of U-235 directly on the 10 items. In stratum 4, all 6 batches are measured with SAM-2 measurements made on 3 items per batch. In stratum 5, all batches are again measured with SAM-2 measurements made on 4 items per batch. No measurements are made of enrichment in strata 6 and 7.

The inspector's error standard deviations are

$\delta^*_{r \cdot \cdot 1y} = 0.01$ $\qquad$ $\delta^*_{r \cdot \cdot 2y} = 0.10$

$\delta^*_{s \cdot \cdot 1y} = 0.004$ $\qquad$ $\delta^*_{s \cdot \cdot 2y} = 0.06$

where the subscript 1 refers to the SAM-2 and the subscript 2 to the nondestructive assay instrument used in measuring the solid waste in stratum 3.

The pertinent data for the facility are given in example 3.7 (a), and use is made of some of the results calculated in that example.

In applying (eq. 3.5.22) to calculate $V^*_{rx}(\hat{D})$ and (eq. 3.5.23) to calculate $V^*_{sx}(\hat{D})$, it was noted that these equal $V^*_r(\text{MUF})$ and $V^*_s(\text{MUF})$ respectively, quantities which were already calculated for this facility in example 3.7 (a).

$$V_{rx}^*(\hat{D}) = 0.0766 \text{ kg}^2 \text{ U-235}$$

$$V_{sx}^*(\hat{D}) = 2.3422 \text{ kg}^2 \text{ U-235}$$

For the inspector measurements, a_i^* of (eq. 3.5.23) equals one for all i, while values for v_i^* are derived from the information given about the inspector's measurements per stratum

i	Factor	v_i^*
1	0.0325	30 + 20 = 50
2	0.0267	20 + 15 = 35
3	0.0152	10 + 5 = 15
4	0.0287	10
5	0.0312	15
6	0.0258	3 + 16 = 19
7	Nominal	10

From (eq. 3.5.24),

$$V_{ry}^{*}(\hat{D}) = (29.25)^2[(0.0005)^2 + (0.01)^2]/50$$

$$+ (80.10)^2[(0.0005)^2 + (0.01)^2]/35$$

$$+ (-4.56)^2[(0.0005)^2 + (0.01)^2]/15 + (-390.32)^2[(0.0005)^2 + (0.01)^2]/10$$

$$+ (187.20)^2[(0.0005)^2 + (0.01)^2]/15 + (134.16)^2[(0.0005)^2 + (0.01)^2]/19$$

$$+ (-36.00)^2(0.10)^2/10$$

$$= 3.1727 \text{ kg}^2\text{U-235}$$

To calculate $V_{sy}^{*}(\hat{D})$ from (eq. 3.5.25), it is first necessary to compute T_{ty} for t = 1 (SAM-2) and t = 2 (NDA for solid waste).

$$T_{1y} = 29.25 + 80.10 - 4.56 + 187.20 + 134.16 - 390.32 = 35.83$$

$$T_{2y} = -36.00$$

$$V_{sy}^{*}(\hat{D}) = (35.83)^2(0.004)^2 + (-36.00)^2(0.06)^2$$

$$= 4.6861 \text{ kg}^2\text{U-235}$$

From (eq. 3.4.15),

$$V_{t}^{*}(\hat{D}) = 0.0766 + 2.3422 + 3.1727 + 4.6861$$

$$= 10.2776 \text{ kg}^2\text{U-235}$$

The total variance of the isotope $\hat{D}$ is found using (eq. 3.5.27) and the results from this current example and from example 3.12 (a).

$$V^{*}(\hat{D}) = 59.1246 + 10.2776 = 69.4022 \text{ kg}^2\text{U-235}$$

3.6 THE (MUF-$\hat{D}$) STATISTIC

3.6.1 Application of (MUF-$\hat{D}$)

There are two statistics, or quantities, that are used to evaluate the material balance data for a given facility. The first is the facility MUF, the amount of material that is unaccounted for based on the facility's measurements of inputs, outputs, and inventories. Methods for calculating the variance of MUF are given in Section 3.4. The second statistic is the so-called difference statistic, or D statistic, which measures the relative bias between the facility and inspection measurements, or better stated, between the values recorded by the facility and the corresponding values based on measurements made by the inspector. Methods for calculating the variance of $\hat{D}$ are given in Section 3.5.

In making inferences based on the observed or reported values for MUF and $\hat{D}$, one is faced with a choice as to how to proceed. On the one hand, a significance test can be made of the bias between the facility and inspector values by using the $\hat{D}$ statistic. If the test outcome is that there is no evidence of bias, then one may accept the facility's stated MUF and test it for significance, making no further use of $\hat{D}$. The other approach is to correct the facility MUF for bias using the $\hat{D}$ statistic. The quantity, $\hat{D}$, is defined in such a way that if $\hat{D}$ is positive, the MUF is overstated while if $\hat{D}$ is negative, MUF is understated. This suggests that (MUF-$\hat{D}$) is the statistic which may be regarded as the inspector's estimate of the facility MUF.

In the next chapter, in Section 4.6, comparisons are made of the two evaluation approaches: (1) first test for the significance of $\hat{D}$ and, if not significant, then test for the significance of MUF; (2) test for the significance of (MUF-$\hat{D}$). These comparisons are made for specific examples, and on the basis of these examples it is inferred that (MUF-$\hat{D}$) is the preferred statistic if, in fact, the criterion for selection of a statistic is based on the probability of detection.

The fact that the probability of detection is larger for (MUF-$\hat{D}$) than for the $\hat{D}$ and MUF tests applied separately, to be illustrated in the examples of Section 4.6, has, in fact, been shown to be true in general [3.14]. This fact is consistent by analogy with the earlier reported finding discussed in Section 3.4.4 in which the statistical advantages of making a single test of diversion without making subdivisions by time or by space was pointed out. Thus, the (MUF-$\hat{D}$) statistic, being a global statistic in the same sense that the overall MUF is (i.e., the MUF over the total finite time period and over the total material balance area) would be expected to have the same advantage as the MUF global statistic.

For other studies in which comparisons are made of (MUF-$\hat{D}$) with D and MUF applied separately, see [3.15] and [3.16].

Consider now the calculation of the variance of (MUF-$\hat{D}$), a quantity needed in the chapter to follow.

3.6.2 Variance of (MUF-$\hat{D}$)

Having calculated the variance of MUF and of $\hat{D}$ separately by the methods of Sections 3.4 and 3.5, it is a simple matter to calculate the variance of (MUF-$\hat{D}$). This calculation is given by Method 3.13.

Method 3.13

Notation

The notation is consistent with that in previous sections. Specifically,

V(MUF) = variance of element MUF

V*(MUF) = variance of isotope MUF

$V(\hat{D})$ = variance of element $\hat{D}$

$V^*(\hat{D})$ = variance of isotope $\hat{D}$

Additional notation includes:

$V(MUF-\hat{D})$ = variance of element $(MUF-\hat{D})$

$V^*(MUF-\hat{D})$ = variance of isotope $(MUF-\hat{D})$

V_0 = that part of the variance of element MUF due to systematic errors that are common to both the facility and the inspector

V_0^* = defined as V_0, but for isotope MUF

One additional quantity is needed. This is the covariance between MUF and $\hat{D}$, denoted by

cov (MUF, $\hat{D}$) for element

cov* (MUF, $\hat{D}$) for isotope

Model

A simple additive model will be used to provide the bases for the results. In this model, all errors of a given type, (i.e., random or systematic) are combined and represented by a single error. In this exposition, short-term systematic errors are not included, for simplicity. For the same reason, a very simple material balance involving only two strata (an input and an output stratum) is considered. Further, the items within a stratum are presumed to be so ordered such that the first n_i items out of the total of N_i items are the ones inspected. None of these simplifying assumptions affect the validity of the results.

The model is written as follows. For inputs, the facility measurement of item i is

$$x_{1i} = X_{1i} + \theta_1 + \Delta_1 + \varepsilon_{1i} \qquad \text{(eq. 3.6.1)}$$

and for outputs it is

$$x_{2i} = X_{2i} + \theta_2 + \Delta_2 + \varepsilon_{2i} \qquad \text{(eq. 3.6.2)}$$

In stratum 1, i runs from 1 to N_1 and in stratum 2, i runs from 1 to N_2. The small x's represent observed values and the large X's represent true values. The quantities θ_1, Δ_1, Θ_2, and Δ_2 are systematic errors, while ε_{1i} and ε_{2i} are random errors. All errors are presumed to be distributed with zero means and variances denoted by $\sigma^2_{\theta 1}$, $\sigma^2_{\Delta 1}$, $\cdots$, $\sigma^2_{\varepsilon 2}$ respectively.

The corresponding inspector values are

$$y_{1i} = X_{1i} + \theta_1 + \psi_1 + \eta_{1i} \qquad \text{(eq. 3.6.3)}$$

and

$$y_{2i} = X_{2i} + \theta_2 + \psi_2 + \eta_{2i} \qquad \text{(eq. 3.6.4)}$$

Note that θ_1 and θ_2 are the same as for the facility measurements, i.e., they represent those systematic errors that are committed by both parties. In (eq. 3.6.3), i runs from 1 to n_1 while in (eq. 3.6.4), i runs from 1 to n_2. The small y's represent inspector measurements.

Results

For element,

$$V(MUF-\hat{D}) = V(\hat{D}) - V(MUF) + 2\ V_0 \qquad \text{(eq. 3.6.5)}$$

$$cov(MUF,\hat{D}) = V(MUF) - V_0 \qquad \text{(eq. 3.6.6)}$$

For isotope,

$$V^*(MUF-\hat{D}) = V^*(\hat{D}) - V^*(MUF) + 2\ V_0^* \qquad \text{(eq. 3.6.7)}$$

$$cov^*(MUF,\hat{D}) = V^*(MUF) - V_0^* \qquad \text{(eq. 3.6.8)}$$

Basis

Equations (eq. 3.6.1) - (eq. 3.6.4) form the basis for the results. They are written in general terms to apply to either element or isotope so that by deriving (eq. 3.6.5) and (eq. 3.6.6) for element, the corresponding equations for isotope will follow.

The approach is to write the model for MUF, for $\hat{D}$, and for (MUF-$\hat{D}$) using the model equations, find the variance of these quantities by error propagation, and demonstrate that the variances are related as indicated by (eq. 3.6.5). If (eq. 3.6.5) is shown to be true, (eq. 3.6.6) will follow easily, as will be shown.

First, consider MUF.

$$\begin{aligned} MUF &= \sum_{i=1}^{N_1} x_{1i} - \sum_{i=1}^{N_2} x_{2i} \\ &= \sum_{i=1}^{N_1} X_{1i} + N_1\theta_1 + N_1\Delta_1 + \sum_{i=1}^{N_1} \varepsilon_{1i} \\ &- \sum_{i=1}^{N_2} X_{2i} - N_2\theta_2 - N_2\Delta_2 - \sum_{i=1}^{N_2} \varepsilon_{2i} \end{aligned} \qquad \text{(eq. 3.6.9)}$$

The variance of MUF is

$$V(MUF) = N_1{}^2\sigma^2_{\theta 1} + N_1{}^2\sigma^2_{\Delta 1} + N_1\sigma^2_{\varepsilon 1} + N_2^2\sigma^2_{\theta 2} + N_2^2\sigma^2_{\Delta 2} + N_2\sigma^2_{\varepsilon 2} \qquad \text{(eq. 3.6.10)}$$

Consider $\hat{D} = N_1\bar{d}_1 - N_2\bar{d}_2$ where

$$\bar{d}_1 = \sum_{i=1}^{n_1} X_{1i}/n_1 + \theta_1 + \Delta_1 + \sum_{i=1}^{n_1} \varepsilon_{1i}/n_1$$

$$- \sum_{i=1}^{n_1} X_{1i}/n_1 - \theta_1 - \psi_1 - \sum_{i=1}^{n_1} \eta_{1i}/n_1$$

$$= \Delta_1 - \psi_1 + \sum_{i=1}^{n_1} \varepsilon_{1i}/n_1 - \sum_{i=1}^{n_1} \eta_{1i}/n_1 \qquad \text{(eq. 3.6.11)}$$

Similarly,

$$\bar{d}_2 = \Delta_2 - \psi_2 + \sum_{i=1}^{n_2} \varepsilon_{2i}/n_2 - \sum_{i=1}^{n_2} \eta_{2i}/n_2 \qquad \text{(eq. 3.6.12)}$$

The variance of $\hat{D}$ follows immediately:

$$V(\hat{D}) = N_1{}^2\sigma^2_{\Delta_1} + N_1{}^2\sigma^2_{\psi 1} + N_1{}^2\sigma^2_{\varepsilon 1}/n_1 + N_1{}^2\sigma^2_{\eta 1}/n_1$$

$$+ N_2{}^2\sigma^2_{\Delta 2} + N_2{}^2\sigma^2_{\psi 2} + N_2{}^2\sigma^2_{\varepsilon 2}/n_2 + N_2{}^2\sigma^2_{\eta 2}/n_2 \qquad \text{(eq. 3.6.13)}$$

From (eq. 3.6.9), (eq. 3.6.11), and (eq. 3.6.12), and including only the error terms,

$$(MUF-\hat{D}) = N_1\theta_1 + N_1\psi_1 - N_2\theta_2 - N_2\psi_2$$

$$+ \sum_{i=n_1+1}^{N_1} \varepsilon_{1i} + (1-\frac{N_1}{n_1})\sum_{i=1}^{n_1} \varepsilon_{1i} - \sum_{i=n_2+1}^{N_2} \varepsilon_{2i} - (1-\frac{N_2}{n_2}) \sum_{i=1}^{n_2} \varepsilon_{2i}$$

$$+ \frac{N_1}{n_1} \sum_{i=1}^{n_1} \eta_{1i} - \frac{N_2}{n_2} \sum_{i=1}^{n_2} \eta_{2i} \qquad \text{(eq. 3.6.14)}$$

$$V(MUF-\hat{D}) = N_1{}^2\sigma^2_{\theta 1} + N_1{}^2\sigma^2_{\psi 1} + N_2{}^2\sigma^2_{\theta 2} + N_2{}^2\sigma^2_{\psi 2}$$

$$+ N_1(\frac{N_1}{n_1} - 1)\sigma^2_{\varepsilon 1} + N_2(\frac{N_2}{n_2} - 1)\sigma^2_{\varepsilon 2} + \frac{N_1{}^2}{n_1} \sigma^2_{\eta 1} + \frac{N_2{}^2}{n_2} \sigma^2_{\eta 2} \qquad \text{(eq. 3.6.15)}$$

Finally, from (eq. 3.6.10), (eq. 3.6.13), and (eq. 3.6.15), and noting that $V_0 = N_1{}^2\sigma^2_{\theta 1} + N_2{}^2\sigma^2_{\theta 2}$, that the truth of (eq. 3.6.5) is demonstrated. That is

$$V(\hat{D}) - V(MUF) + 2V_0 = N_1{}^2\sigma^2_{\psi 1} + N_2{}^2\sigma^2_{\psi 2} + N_1{}^2\sigma^2_{\theta 1} + N_2{}^2\sigma^2{}_{\theta 2}$$

$$+ N_1(\frac{N_1}{n_1} - 1)\ \sigma^2_{\varepsilon 1} + N_2(\frac{N_2}{n_2} - 1)\ \sigma^2_{\varepsilon 2} + \frac{N_1{}^2}{n_1} \sigma^2_{\eta 1} + \frac{N_2{}^2}{n_2} \sigma^2_{\eta 2} \quad ,$$

which is the expression for $V(MUF-\hat{D})$, completing the proof.

The result of (eq. 3.6.6) then follows immediately. Write

$$V(MUF-\hat{D}) = V(MUF) + V(\hat{D}) - 2\ cov(MUF,\ \hat{D})$$

and equate this to the right hand side of (eq. 3.6.5). Solving for cov(MUF, $\hat{D}$) yields the solution given by (eq. 3.6.6).

Before leaving this section, it is noted that the key result, (eq. 3.6.5), is also contained in reference [3.17] for a general model that also included short term systematic errors. In the reference, make the equivalence

$$\hat{T}_p = (MUF-\hat{D})$$

$$\hat{T}_t = MUF$$

$$\hat{T}_t-\hat{T}_p = \hat{D}$$

With this equivalence in notation, (eq. 3.6.5) follows immediately from equations (10), (20), (23), and (24) of [3.17], keeping in mind that A_1 of [3.17] corresponds to V_0 of this section. The result (eq. 3.6.5) is also contained in reference [3.18], except that it is assumed in that reference that the facility and the inspector do not commit common systematic errors, so that $V_0 = 0$.

Examples

EXAMPLE 3.13 (a)

Consider the low enriched uranium fuel fabrication facility discussed in previous examples in this chapter. The following results were found:

from example 3.5 (a)	$V(MUF) = 45{,}010\ kg^2U$
from example 3.10 (a)	$V(\hat{D}) = 85{,}993\ kg^2U$
and	$V_0 = 0$

(The result that $V_0 = 0$ follows from the fact that although both the facility and the inspector use the same sampling technique there was zero contribution to V(MUF) due to systematic errors in sampling.)

From (eq. 3.6.5),

$$V(MUF-\hat{D}) = 85{,}993 - 45{,}010 = 40{,}983\ kg^2U$$

From (eq. 3.6.6),

$$cov(MUF,\ \hat{D}) = 45{,}010\ kg^2U$$

Note that this size covariance indicates that the correlation coefficient between MUF and D is

$$\rho = 45{,}010/\sqrt{(45{,}010)(85{,}993)}$$

$$= 0.72$$

For isotope,

from example 3.7 (a), $V^*(MUF) = 35.7179\ kg^2U\text{-}235$

from example 3.12 (a), $V^*(\hat{D}) = 69.4022\ kg^2U\text{-}235$

and $V_0^* = 0$

From (eq. 3.6.7),

$$V^*(MUF-\hat{D}) = 69.4022 - 35.7179 = 33.6843\ kg^2U\text{-}235$$

From (eq. 3.6.8),

$$cov^*(MUF,\hat{D}) = 35.7179\ kg^2U\text{-}235$$

The correlation coefficient is

$$\rho = 35.7179/\sqrt{(35.7179)(69.4022)}$$

$$= 0.72 \quad \text{, as for uranium.}$$

EXAMPLE 3.13 (b)

Consider the mixed oxide fuel fabrication facility discussed in previous examples in this chapter. The following results were found:

from example 3.5 (b), $V(MUF) = 10.729310\ kg^2Pu$

from example 3.10 (b), $V(\hat{D}) = 27.303719\ kg^2Pu$

and $V_0 = 2.230517\ kg^2Pu$

From (eq. 3.6.5),

$$V(MUF-\hat{D}) = 27.303719 - 10.729310 + 2(2.230517)$$

$$= 21.035443\ kg^2Pu$$

From (eq. 3.6.6)

$$cov(MUF,\hat{D}) = 10.729310 - 2.230517$$

$$= 8.498793\ kg^2Pu$$

The correlation coefficient is

$$\rho = 8.498793/\sqrt{(10.729310)(27.303719)}$$

$$= 0.50$$

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ANNEX 3.1

PROGRAM ORDER FORM

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Chapter 4

DESIGN OF INSPECTION PLANS

4.1 PURPOSE OF INSPECTION

Stated very simply and in somewhat general terms, the purpose of Agency inspection is to provide assurance that the material balance data for a facility properly reflect the state of material control that exists in that facility, and further, that the state of control is satisfactory, i.e., it provides no indication of unaccounted for losses of material. In order to provide objectivity, quantitative criteria are established, and statements about the inspection activities are framed in terms of these criteria.

Inspection plans are to be derived to apply to a given facility and for a given material balance period. The inspection plan is not restricted to include only those activities that take place to verify amounts of material in inventory at a given time; the plan must also include provisions for monitoring flow streams, i.e., inputs to and outputs from the facility.

This discussion on inspection activities is limited to those activities that bear on material accountancy. There are other activities that take place during an inspection, e.g., checking on containment and surveillance devices, but such activities, while certainly of importance in helping to provide assurance that the material balance data are acceptable when evaluated against the criteria, are not considered to be a part of those activities aimed directly at providing this assurance in quantitative terms.

In planning for inspection, it is assumed that the facility accounting data may misrepresent the actual amounts of material in discrete items. Although such data misrepresentations may clearly occur because of innocent reasons, e.g., because of mistakes in recording the measured data, it is assumed for planning purposes that data misrepresentations occur intentionally in order to mask diversion. This assumption is made in order to provide assurance that the inspection is effective and credible against all possible combinations of understatements and overstatements of material. To be effective and credible, the inspection must guard against the worst possible set of circumstances; this worst possible set corresponds to actions that would be taken by a diverter attempting to conceal diversion through data falsification. More is said to this point in later sections.

It is noted here that the assumed existence of an adversary, the diverter, is the basis for game-theoretic developments of inspector strategies. Throughout this chapter, reference will be made to results derived from game-theoretic considerations, and they will be compared with results found using other approaches, which also assume the existence of an adversary. Generally speaking, there is close agreement in results among the different approaches to the inspection problem. This is a comfortable result, and permits some freedom for inspection planners to choose among the various approaches to inspection planning that are proposed. For a rather complete treatment of the game theoretic approach to safeguards problems, see Avenhaus [4.1].

4.1.1 Response of Accountancy Statistics to Diversion Scenarios

Before proceeding further, it is worthwhile to discuss different diversion scenarios and how the accountancy measures available to the inspector: MUF, $\hat{D}$, and (MUF-$\hat{D}$) react to these scenarios. It is perhaps most enlightening to consider this problem in the context of a specific example. In this example, errors of measurement are ignored, total inspection and a perfect material balance are assumed, i.e., in the absence of diversion, the MUF would be zero. (In the presence of measurement errors and random sampling corresponding to less than total inspection, the results of the following discussion are interpreted for the expectations of the statistics.)

For the example facility material balance, assume the following:

BI = 1000 units (beginning inventory)

R = 500 units (receipts)

S = 800 units (shipments plus waste streams)

EI = 700 units (ending inventory)

MUF = BI + R - S - EI = 0 units

Some representative diversion scenarios are now constructed, and the responses of the test statistics to these scenarios are noted.

Scenario 1

100 units are moved from BI after having been verified by the inspector. The facility is therefore short 100 units at the end of the balance period, i.e., there are actually only 600 units in EI.

Strategy 1

Do not falsify EI, i.e., book the true 600 unit value for ending inventory. Then, the MUF being the booked amount gives MUF = 1000 + 500 - 800 - 600 = 100. The $\hat{D}_k$ statistics for k = 1, 2, 3, and 4 are all zero since the actual amounts agree with the book amounts. Thus

$$\hat{D} = 0$$

$\text{MUF}-\hat{D}$ = 100 units, reflecting 100 units diverted into MUF and 0 units diverted through data falsification.

Strategy 2

Falsify the EI, i.e., book the EI as 700 units even though there are only 600 units present. Then,

$$\text{MUF} = 1000 + 500 - 800 - 700 = 0$$

$$\hat{D}_1 = \hat{D}_2 = \hat{D}_3 = 0$$

$$\hat{D}_4 = 700 - 600 = 100$$

$\hat{D} = -\hat{D}_4 = -100$ (The minus sign occurs because this represents a data falsification in a negative component of the MUF equation.)

$\text{MUF}-\hat{D} = 0 -(-100)$ = 100 units, reflecting 0 units diverted into MUF and 100 units diverted through data falsification, i.e., through overstating the ending inventory.

Strategy 3

Partially falsify the records to hide the diversion. Overstate the shipments by 20 units and the ending inventory by 30 units. Then,

$$MUF = 1000 + 500 - 820 - 630 = 50$$

$$\hat{D}_1 = \hat{D}_2 = 0$$

$$\hat{D}_3 = 820 - 800 = 20 \quad , \qquad \hat{D}_4 = 630 - 600 = 30$$

$$\hat{D} = -20 - 30 = -50$$

$MUF\text{-}\hat{D} = 50 - (-50) = 100$ units, reflecting 50 units diverted into MUF and 50 units diverted through data falsification.

Scenario 2

Although the true BI is 1000 units, this is booked as 900 units. The 100 units are then removed from the process so that at the end, the EI is actually 600 units.

Strategy 1

Do not falsify the EI, i.e., book the 600 value.

$$MUF = 900 + 500 - 800 - 600 = 0$$

$$\hat{D}_1 = 900 - 1000 = -100$$

$$\hat{D}_2 = \hat{D}_3 = \hat{D}_4 = 0$$

$$\hat{D} = -100$$

$MUF\text{-}\hat{D} = 0 -(-100) = 100$ units, reflecting 0 units diverted into MUF and 100 units diverted through data falsification.

Strategy 2

Being concerned that the inspector will detect the understatement in the beginning inventory, also understate the ending inventory somewhat so that a detected discrepancy could be explained as a measurement bias. Specifically, book 560 units in EI.

$$MUF = 900 + 500 - 800 - 560 = 40$$

$$\hat{D}_1 = 900 - 1000 = -100$$

$$\hat{D}_2 = \hat{D}_3 = 0$$

$$\hat{D}_4 = 560 - 600 = -40$$

$$\hat{D} = -100 - (-40) = -60$$

$MUF\text{-}\hat{D} = 40 - (-60) = 100$ units, reflecting 40 units diverted into MUF and 60 units diverted through data falsification.

These examples should make it clear that the (MUF-$\hat{D}$) statistic reacts to the total amount diverted, no matter what diversion strategy is employed. This is always true, no matter which diversion scenarios may be constructed. In this connection, data falsification can be said to occur only if the material in question is inspectable, i.e., available to the inspector for verification. Thus, with scenario 1, for example, it is not an admissible strategy to remove the 100 units from beginning inventory before it is offered for inspection. Clearly, such a diversion would not be detected within the closed loop of the facility material balance in question. Material not made available for inspection is clearly not safeguarded. Also, if material is diverted into MUF, i.e., simply removed from the process and then shipped to a clandestine facility, such a shipment is not a declared shipment and hence, the fact that there is a discrepancy between the amount shipped and the amount booked as shipment does not constitute a data falsification. This is because the amount booked is in perfect agreement with the amount presented for inspection.

Since the diverter may not restrict his strategies to the "admissible" ones as defined here, for complete protection other statistics would also need to be included, such as shipper-receiver differences. For a complete discussion, see Reference [4.2].

4.2 INSPECTION ACTIVITIES

Inspection activities, while perhaps quite varied in a number of respects, e.g., measurement complexity, cost, accuracy, etc. may be broadly classified as falling into one of two categories--attributes or variables inspection. In attributes inspection, the item inspected is classified as being either acceptable or not acceptable (i.e., a defect) on the basis of the measurement. Attributes inspection has nothing to do with the quality of the measurement, but rather, with the end use to which the measurement is put. Variables inspection, on the other hand, assigns a measured value to each item inspected, and the measured values for a group of items are combined in some way to provide a statistic, or a function of the observations, used in the evaluation in some predetermined way. As was the case with attributes inspection, the quality of the measurement in question is not the feature that determines that an inspection measurement is variable in nature, but rather, it is the end use to which the measurement is put.

The definitions of attributes and variables inspection in the widely referenced U.S. Military Standards are quoted here to close out this discussion.

> From [4.3]: "Inspection by attributes is inspection whereby either the unit of product is classified simply as defective or nondefective, or the number of defects in the unit of product is counted, with respect to a given requirement or set of requirements."

> From [4.4]: "Inspection by variables is inspection wherein a specified quality characteristic on a unit of product is measured on a continuous scale, such as pounds, inches, feet per second, etc., and a measurement is recorded."

A more detailed discussion of inspection activities as they relate to this field of application follows.

4.2.1 Attributes Inspection Activities

The specific attributes inspection activities that may be performed in a given inspection depend on the circumstances of that inspection. Some activities, although attributes in nature, are performed on 100% of the items and hence are not a part of the statistical planning or evaluation function. They are included in the discussion to follow for completeness.

In broad terms, the inspection activities discussed in this chapter fall in one of two categories. One category consists of records examination and the other relates to inspection of measurements data. Within each category there is a further subdivision of possible inspections. Those that might be a part of attributes inspection are discussed below.

In connection with the records examination, consider the following activities:

1) Inspection to detect recording and/or calculational mistakes in the stratum subtotals and in combining stratum subtotals to calculate the MUF. This would normally be a 100% inspection effort.

2) Reconciliation of data for receipts, shipments, and inventories as stated in the records with those submitted in reports to the Agency. This is a post-inspection activity and should be a 100% inspection with discrepancies noted and corrected.

3) Inspection to detect recording and/or calculational mistakes for individual items, including checking for proper application of the source data. This will normally be done on a sampling basis, and is part of the statistical attributes inspection.

Activities of types 1-3 are a part of the records examination. In addition, two activities related to the verification of amounts of nuclear material are a part of attributes inspection.

4) Counting of items to verify that the numbers of items located agree with the number listed on the facility records. This could be a 100% inspection, or it could be a part of the statistical attributes inspection, combined with activities 3 and 5.

5) Inspection with an attributes tester to detect discrepancies, or defects, that are larger than can be explained by the combined errors of measurement for the facility and for the inspector's attributes tester.

Some further discussion of this activity is helpful. Consider the attributes tester. This is a measurement device of some sort that will classify an inspected item as being either a defect or not a defect. The definition of an item defect is different from a planning viewpoint than it is from an implementation viewpoint. For planning purposes, a defect is defined in Section 4.3.1. From an implementation viewpoint, a defect is defined in Chapter 5. The fact that different criteria for defining a defect are used should not really be a bothersome point; in planning, it is important to establish criteria such that the amount of inspection performed is sufficient to provide protection against the best diversion strategies that might be used by a diverter. (This point was made previously in Section 4.1).

Measurement devices used as the attributes tester may not even be testers in the true sense of the word. For example, the attributes test may consist of tipping a container to verify that the actual weight does not differ greatly from that listed by the facility. The stabilized assay meter (SAM) may also be regarded as an attributes tester since it verifies that the item contains uranium of roughly the proper enrichment.

In summary, statistical attributes inspection might consist of randomly selecting items from the facility listing, locating those items, checking the calculations from the source data for those items, and performing attributes measurements such as tipping the containers and/or measuring with an NDA instrument such as the SAM. In any particular inspection, and for a given stratum in that inspection, the definition of an attributes test will, of course, have to be explicitly set forth in implementation. This is not necessary in planning, however. It is only required that some type of attributes inspection be anticipated.

4.2.2 Variables Inspection Activities

Variables inspection presupposes the existence of a variables measuring instrument, or a variables tester. Unlike attributes inspection, it is necessary to have in mind the specific tester to be used in each stratum at the planning stage, because the measurement error variances affect the planning. A variables tester may actually consist of a number of distinct measurement operations such as weighing, material sampling, and chemical analysis, or possibly, NDA analysis of the sample.

For planning purposes, variables inspection consists of two types of activities, both of which are related to the inspection of measurement data (as opposed to records examination). To continue with the list of inspection activities begun in the previous section, activities 6 and 7 defined below comprise the variable inspection.

6) Inspection to detect defects that are sufficiently small so as to escape detection with the attributes tester. This inspection activity is referred to as variables inspection in the attributes mode. As was the case with attributes inspection, the definition of a defect used in planning, given in Section 4.3.2.1, differs from that used in implementation, given in Chapter 5, and for the same reason. Interestingly enough, for planning purposes, it will be seen that the definition of a defect is related to the measurement error of the attributes tester. This means that, as stated above, although it is not necessary to specify the attributes tester when planning for attributes inspection, it is at least necessary to have some idea of its measurement error in order to plan for the variables inspection in the attributes mode.

7) Inspection to detect small defects or biases that may exist in all or some items in the given strata. This inspection activity is aimed at developing data for the difference statistic, $\hat{D}$, discussed in detail in Chapter 3.

For a discussion of how activities 6 and 7, both relating to variables inspection, are united in the planning stage, see Section 4.3.2.

4.3 CRITERIA FOR INSPECTION PLAN DESIGN

Having outlined the types of inspection activities for both the attributes and the variables inspection, the criteria used in planning the inspection are now considered. First these criteria are discussed for attributes inspection.

4.3.1 Attributes Inspection Criteria

In determining the number of items to be inspected, attention is focussed first on a single stratum. Essential to the inspection design is some quantity M.

M = goal amount of element (or isotope) established by external considerations and supplied as an input to the planning.

The attributes inspection criterion is stated very simply. If an amount, M, is missing from the stratum in question, this fact should be detected with (high) probability, $(1-\beta)$. The parameter β is also an input parameter.

Several comments are relevant. First, the definition of "detected" is considered. In order to minimize the amount of inspection that must be performed, a zero-acceptance number plan is used. This means that if even a single defect is found among the items inspected, this constitutes "detection". The action to follow in the event of this detection is yet another matter that must be defined. One action might be to 100% inspect the stratum in question on an attributes basis, correcting all records for the defective items. This is generally not possible, and some other action might be required, such as giving the facility the opportunity to improve the data base prior to another inspection to follow soon. A third possibility, and, in fact, the action most likely to be taken is that the effects on MUF of the detected discrepancy will be evaluated in some way. Section 5.1.3 describes a procedure to calculate the confidence interval for the number of defects in a stratum population. Section 5.1.2.1 describes a procedure to calculate the quantitative effect of defects on MUF. Whatever the follow-up action, detection is clearly and objectively defined here, and such a definition is all that is needed in planning.

Next, consider the definition of a defect. It is clearly to a diverter's benefit to falsify the smallest number of items necessary in order to accumulate the goal amount of M units. Thus, he would choose to falsify each item falsified by the maximum amount possible. Assume that this maximum amount corresponds to the nominal or average amount of element (or isotope) per item, denoted by $\bar{x}_k$, and expressed in the same units as M. Thus, the number of items that would have to be falsified in order to accumulate M units (i.e., the number of "defects") is $M/\bar{x}_k$ in stratum k. With this in mind, the criterion for attributes inspection stated above may now be restated in its equivalent form, as follows:

If the number of gross or large defects in stratum k is $M/\bar{x}_k$, choose the sample size, n_{ak}, large enough such that at least one of these defects will appear in the items inspected with probability $(1-\beta)$.

At this point in the discussion, a possibly worrisome point should be brought up. The inspection is designed to detect M units if all M units are missing in stratum k. In actual fact, the adversary would likely not attempt to remove all M units by this mechanism, i.e., by large data falsifications restricted to a single stratum. Clearly, if some amount less than M is diverted from stratum k in this fashion, then the probability of detection in that stratum is some amount smaller than the desired value, $(1-\beta)$. This is a valid point of concern and touches on the general problem of how the diverter would choose to divert M units by various means so as to escape detection. This general problem is treated in detail in later sections (see Sections 4.4.1.2 and 4.6). At this point in the discussion, one need only accept the fact that the inspection criterion as stated is appropriate; its appropriateness will be demonstrated later.

4.3.2 Variables Inspection Criteria

With variables inspection, there are two kinds of criteria. One set is relevant to variables inspection in the attributes mode and the other to variables inspection in the variables mode. The inspection sample size for stratum k will be derived based on each set of criteria. In implementing the plan, the larger of the two sample sizes will be used in each stratum. This means that the actual protection afforded by the plan, i.e., the plan's ability to detect the goal amount, M, will be least as large as that calculated based on the design criteria as applied separately in the attributes and variables mode. This point is discussed further in Section 4.6.

Before presenting inspection criteria for variables inspection in each of its two modes, it is helpful to discuss briefly why the two separate sample sizes must be calculated. The reason is that an inspection plan must counter the best strategy of a diverter. Specifically, it may well happen that if variables inspection in a variables mode is all that's planned for, then the sample size in a given stratum may be too small to counter the strategy in which a diverter falsifies not all items by a small amount, but rather, a selected smaller number of items by an amount just small enough to escape detection by attributes inspection with the attributes tester. The argument used by the diverter is that if any such defected item is included among the items inspected by variables inspection, it is sure to be labelled a defect; however, chances are that the item would not be included among the items to be inspected were variables inspection only planned to develop data for the $\hat{D}$ statistic. To combat that argument, variables inspection sample sizes must also be large enough to detect these medium sized defects, those sufficiently small so as not to be detected by the attributes tester, but much larger than would be needed to accumulate M units by small defects were all items in the stratum to be falsified.

It is remarked that rather than defining these two categories of defects (medium and small), one could treat both categories of defects simultaneously on a continuous scale. This is done by defining two quantities for a given stratum, one being the number of items to be falsified by the diverter, and the other being the size of the falsification per item. In a game theoretical sense, then, the diverter chooses these two quantities in some optimum way while the inspector also chooses his best strategy to combat the diverter. Limited calculations indicate that the two different approaches to the problem lead to basically the same results from an inspection planning viewpoint [4.1].

Attention is now directed at presenting criteria for variables inspection, first in the attributes mode.

4.3.2.1 Criteria for Variables Inspection in Attributes Mode

The discussion in Section 4.3.1 on attributes inspection criteria is relevant to variables inspection in the attributes mode also. The only difference is in the definition of a defect. As with attributes inspection, it is in the best interest of a diverter to falsify any item selected for falsification by the largest amount feasible. In the case of attributes inspection, this amount was $\bar{x}_k$; in the case of variables inspection, it is $\gamma_k \bar{x}_k$, where γ_k is some number less than 1.

The parameter γ_k is an input design parameter. It describes the ability of the attributes tester to detect a given degree of falsification. Specifically, γ_k may be defined by the following statement: if a discrepancy exceed $\gamma_k \bar{x}_k$ in size, it will be labelled a gross defect in the attributes tester inspection with probability one; if it is smaller than $\gamma_k \bar{x}_k$, it will be detected with probability zero.

In defining γ_k in this fashion, there is no need to define a measurement error as such associated with the attributes tester. For example, if the attributes test involves tipping a container to see if it contains about the recorded amount of gross weight, it is difficult to assign a measurement error to this operation. However, the argument can be made that if the item contains say less than half its recorded amount, then this would be detected by the tipping operation. In this event, γ_k would be assigned the value 0.5.

In the event the attributes tester produces a quantitative response or reading, then γ_k may be defined in terms of the measurement error standard deviation of the difference between the facility and the inspection measurements. Often, this reduces to the measurement error standard deviation for the inspector measurement, which will likely be the dominant error in attributes inspection. If the error standard deviation of this difference is denoted by δ_k on a relative basis, then γ_k might be defined to be, say, $4\delta_k$. For example, a 5% relative error standard deviation results in a value for γ_k of 0.20.

When there is some doubt as to the value of γ_k in a given application, it is always preferable to err on the high side. By erring in this direction, the resulting inspection sample size will be larger than needed, i.e., the error will be on the conservative side. This is one reason that the definition $\gamma_k = 4\delta_k$ is used; even though it is quite likely that a discrepancy smaller than $\gamma_k \bar{x}_k$ would be detected by the attributes tester, the probability of this event is implicitly assumed to be zero in order to be sure that the sample size for variables inspection in the attributes mode is conservatively large.

With these thoughts in mind, the criterion for variables inspection in the attributes mode may now be stated as follows:

If the number of medium defects in stratum k is $M/\gamma_k \bar{x}_k$, choose the sample size, n_{v1k}, large enough such that at least one of these defects will appear in the items inspected with probability $(1-\beta)$.

4.3.2.2 Criteria for Variables Inspection in Variables Mode, Using $\hat{D}$

It was mentioned earlier that the data collected from the variables inspection are used to calculate $\hat{D}$, the difference statistic defined explicitly in Section 3.5.1. This fact implies that the criteria for the variables inspection sample sizes, when variables inspection is in the variables mode, is related in some way to this $\hat{D}$ statistic.

The design criteria in this application are again related to the goal amount, M. In general terms, a sufficient number of items must be inspected such that if the mean value of $\hat{D}$ is -M, this fact is detected using the $\hat{D}$ statistic with probability $(1-\beta)$. (Note: The mean of $\hat{D}$ carries the minus sign, -M, because a negative value for $\hat{D}$ benefits the diverter, as $\hat{D}$ has been defined.)

A number of comments must be made on this general criterion. First, the precise definition of "detected" must be given. This is related to the probability of a "false alarm", i.e., of claiming detection when in fact the true mean of $\hat{D}$ is zero. This probability should be small. It is an input value in planning, and is labelled α.

Secondly, it is noted that, unlike attributes inspection, the sample size for variables inspection in the variables mode is calculated over all strata, and not for a particular stratum. In finding this total sample size, it is implicitly assumed that the items to be inspected are allocated among the strata in some optimum fashion. Specifically, in this discussion, they are allocated to result in a minimum variance for $\hat{D}$ for a fixed total sample size. Other bases for optimum allocation could be factored in, such as the cost of analysis and the attractiveness of the material from a diversion standpoint, but this is not done formally here [4.5].

As a third comment, it is noted that it may not be possible to meet the criterion as stated because the variance of $\hat{D}$ is limited by the systematic error variance. The random part of the variance of $\hat{D}$ will continue to decrease with increased inspection sample sizes; the systematic part will not. Further, even in the case that the design criterion can be met, the effect on $(1-\beta)$ of measuring additional items beyond a certain point may not be worth the effort. Therefore, the emphasis in this chapter will not be so much on determining a specific variables inspection sample size, but rather, in examining the relationship between sample size and $(1-\beta)$ as an aid in choosing the sample size.

Fourthly, it is recognized that the variance of $\hat{D}$ under the hypothesis that there is no diversion through small data falsifications may be smaller than that under the alternative hypothesis that some material may be thus diverted. This is because under the alternative hypothesis, the diverter will likely choose to not falsify all items by the same amount. Thus a statistical sampling error will be introduced because the variance of $\hat{D}$ will depend on which items were selected to be inspected. (See eq. (3.38) of [4.1] for an exact expression for this variance under a specified model.) The parameter C_0^2 is introduced to relate the variance of $\hat{D}$ under the null hypothesis to its variance under the alternative hypothesis during the planning stage. Specifically, for planning purposes, it is assumed that the random error variance of $\hat{D}$ under the alternative hypothesis is C_0^2 times the corresponding variance under the null hypothesis.

As a final comment, it has been noted previously that measurements made with a variables tester may well consist of a number of measurement operations, e.g., a weighing, a sampling, and an analytical measurement. For simplicity, it is assumed in this chapter that the sample size in question is taken to be the number of items sampled to determine the element concentration factor. In calculating the random error variance of $\hat{D}$ in planning, the number of items to be weighed is set equal to the number of items to be sampled. The number of analytical determinations to be made per sampled item may be arbitrarily inputted.

4.3.2.3 Criteria for Variables Inspection in Variables Mode, Using (MUF-$\hat{D}$)

In Section 3.6, the (MUF-$\hat{D}$) statistic was introduced. It was pointed out that (MUF-$\hat{D}$) may be regarded as the inspector's estimate of the facility MUF. Further it was suggested that the separate tests on $\hat{D}$ and MUF could be combined into one test using (MUF-$\hat{D}$).

Clearly, the variance of MUF is independent of inspection sample sizes. In the Section 4.3.2.2 just preceding this one, criteria for determining the variable inspection sample sizes using the $\hat{D}$ statistic were given. The aim here was to detect M units through use of $\hat{D}$. If the (preferred) test statistic, (MUF-$\hat{D}$), is used in the evaluation phase, then it might be logical that (MUF-$\hat{D}$) also be the statistic used in inspection planning, rather than $\hat{D}$. This latter statistic is aimed only at detection of diverted amounts through small data falsifications; the (MUF-$\hat{D}$) statistic responds to the combination of two strategies of diversion, one through small data falsifications, and the other through diversion into MUF. Diversion into MUF means that material is simply removed from the process with no attempts made to alter any records.

When planning is with respect to (MUF-$\hat{D}$) rather than to $\hat{D}$, then the criteria discussed in the preceding Section 4.3.2.2 still apply, the only difference being that (MUF-$\hat{D}$) replaces $\hat{D}$ as the test statistic.

4.4 SELECTION OF INSPECTION SAMPLE SIZES

The methods for deriving the inspection sample sizes may now be given. First, the attributes inspection sample sizes are considered.

4.4.1 Attributes Inspection Sample Sizes

In Section 4.4.1.1, the method for determining the attributes inspection sample size in stratum k is given. In Section 4.4.1.2, it is demonstrated that this procedure provides protection against all diverter strategies in which M units are diverted through large data falsifications but allocated among the various strata.

4.4.1.1 Attributes Inspection Sample Size in Given Stratum

Method 4.1

Notation

The notation is given in Section 4.3.1 in part. It is repeated here for convenience, and some additional notation is included.

M = goal amount of element (or isotope)

N_k = number of items in stratum k

n_{ak} = number of items to be inspected in stratum k

β = probability of failing to detect the amount M, if this amount is missing from the stratum

$\bar{x}_k$ = average amount of element (or isotope) per item in stratum k expressed in the same units as M

Model

The random variable is the number of defects found in a sample of n_{ak} items selected at random from a population of N_k items containing a given number of defects. It is well known that this random variable follows a hypergeometric density function [4.6].

Results

The required sample size, n_{ak}, is given by

$$n_{ak} = N_k(1-\beta^{\bar{x}_k/M}) \qquad \text{(eq. 4.4.1)}$$

The result should always be rounded up to the next integer.

Basis

A zero acceptance plan is used to minimize the amount of inspection required. The sample size, n_{ak}, is the solution of the equation:

$$\text{Prob (0 defects} \mid M/\bar{x}_k \text{ defects in the population)} = \beta$$

Using the hypergeometric model, this equation is written

$$\frac{\binom{M/\bar{x}_k}{0}\binom{N_k - M/\bar{x}_k}{n_{ak}}}{\binom{N_k}{n_{ak}}} = \beta \qquad \text{(eq. 4.4.2)}$$

It is shown in [4.7] that an approximate solution to (eq. 4.4.2) is

$$n_{ak} = 0.5\ (1-\beta^{\bar{x}_k/M})\ (2N_k - M/\bar{x}_k + 1) \qquad \text{(eq. 4.4.3)}$$

When $(M/\bar{x}_k - 7)$ is small relative to N_k, it may be deleted from (eq. 4.4.3), and (eq. 4.4.1) follows immediately. When $(M/\bar{x}_k - 1)$ is not insignificant relative to N_k, the use of (eq. 4.4.1) will result in a conservatively large sample size.

However, it follows from (eq. 4.4.2) that detection is certain $(\beta = 0)$ if $n_{ak} > N_k - M/\bar{x}_k$ - i.e., if the sample size is greater than the number of nondefective items in the population so that the sample size never need exceed $N_k - M/\bar{x}_k + 1$. Also if cost of sampling is an important factor more precise approximation in (eq. 4.4.3) may be warranted.

Note: When computing β for given N_k, M, $\bar{x}_k$, and n_{ak} the approximation of (eq. 4.4.1) may seriously overestimate β unless the condition that $(M/\bar{x}_k - 1)$ is small compared to N_k is satisfied, thus indicating a lower probability of detection than that actually achieved. If the condition is not clearly met the use of (eq. 4.4.3) is to be preferred for calculating β when $n_{ak} \le N_k - M/\bar{x}_k$. If $n_{ak} > N_k - M/\bar{x}_k$, β is identically 0.

Examples

EXAMPLE 4.1 (a)

Let the stratum to be inspected be the feed stratum in the low enriched uranium fabrication plant of Example 3.3 (a). From the data of that example,

$$N_1 = 12000$$

$$\bar{x}_1 = 20 \text{ kg U}$$

Say that the input design parameter values are

$$M = 1500 \text{ kg U}$$

$$\beta = 0.05$$

Then, using (eq. 4.4.1),

$$n_{a_1} = 12000(1- 0.05^{1/75})$$

$$= 469.87 = 470 \text{ items}$$

EXAMPLE 4.1 (b)

Let the stratum to be inspected be stratum 8 of the mixed oxide fuel fabrication facility of Example 3.3 (b). This is the ending inventory stratum of mixed oxide powder items. From the cited example,

$$N_8 = 360$$

$$\bar{x}_8 = 0.375 \text{ kg Pu}$$

Say that the input design parameter values are

$$M = 8 \text{ kg Pu}$$

$$\beta = 0.05$$

Then, from (eq. 4.4.1),

$$n_{a8} = 360\ (1-0.05^{0.375/8})$$

$$= 47.16 = 48 \text{ items}$$

4.4.1.2 Attributes Inspection Sample Sizes Over All Strata

As was mentioned in Section 4.3.1, a diverter would likely not divert all M units from a given stratum and, as a result, the probability of detecting an amount less than M diverted from a given stratum will be less than $(1-\beta)$. Assume, initially, that the diverter diverts the goal amount M through gross data falsifications only, but that he diverts an amount M_k from stratum k so that the sum of M_k over the k strata is M. (This assumption is relaxed later to permit the accumulation of the goal amount through a combination of strategies in which not all of it need be diverted through gross data falsifications.) What is then the probability of detecting this diversion?

To answer this question, "detection" must be defined. In the case of a single stratum, the diversion was detected if one or more defects appeared in the sampled items. To extend this definition over all the strata, detection will be said to occur if at least one such defect is found in one or more strata.

Let β_k = probability of finding 0 defects in the sample of n_{ak} items in stratum k

The overall probability of nondetection is simply the product of the β_k values over all the strata. The n_{ak} in this definition of β_k is given by (eq. 4.4.1).

The expression for β_k is found. The solution to (eq. 4.4.1) with β_k replacing β and M_k replacing M provides this expression.

$$\beta_k = (1 - n_{ak}/N_k)^{M_k/\bar{x}_k} \qquad \text{(eq. 4.4.4)}$$

But with n_{ak} given by (eq. 4.4.1), the expression $(1 - n_{ak}/N_k)$ may be replaced by

$$\beta^{\bar{x}_k/M} ,$$

so (eq. 4.4.4) becomes

$$\beta_k = \beta^{M_k/M} \qquad \text{(eq. 4.4.5)}$$

The product of the β_k values is then

$$\prod_k \beta_k = \beta^{\sum_k M_k/M} = \beta \qquad \text{(eq. 4.4.6)}$$

This gives the probability of nondetection, which is exactly the design value for a given stratum. Thus, even though the probability of detection is not $(1-\beta)$ in any given stratum (unless M_k = M for that stratum), the overall probability of detection will be $(1-\beta)$, given that detection consists of finding at least one defect in at least one stratum.

This result may be more clearly understood if a numerical example is considered. Again consider the facility of Example 3.3 (a) and, following Method 4.1, find the sample size for each of the seven strata. Use M = 1500 kg U. The sampling plan information is given in the table below.

Stratum (k)	N_k	$\bar{x}_k$(kg U)	n_{ak}
1	12000	20	470
2	47760	5	475
3	2770	0.4332	3
4	1800	4	15
5	800	5	8
6	1800	4	15
7	800	5	8

The sample sizes, n_{ak}, are calculated from (eq. 4.4.1) for β = 0.05 with all results rounded up to the nearest integer.

Now, consider three (of the infinitely large number) diversion strategies by which 1500 kg U can be diverted through gross data falsification. These are listed below, with entries in kg U.

	M_k (kg U)		
Stratum	Strategy 1	Strategy 2	Strategy 3
1	600	400	760
2	600	1000	220
3	120	0	0
4	80	0	280
5	0	0	40
6	0	100	0
7	100	0	200
Total	1500	1500	1500

From (eq. 4.4.4), β_k is calculated for each stratum, and for each strategy. The β_k values are displayed in the following table.

	β_k = Probability of Nondetection in Stratum k		
Stratum	Strategy 1	Strategy 2	Strategy 3
1	0.3016	0.4497	0.2191
2	0.3014	0.1355	0.6442
3	0.7407	1.0000	1.0000
4	0.8459	1.0000	0.5567
5	1.0000	1.0000	0.9044
6	1.0000	0.8112	1.0000
7	0.7778	1.0000	0.6050
Product =	0.0443	0.0494	0.0430

The key result is that even though the probability of nondetection in any one stratum is quite large, yet the overall probability of nondetection is at or below the design value of 0.05.

4.4.1.3 Game Theoretic Results

One may approach the problem using game-theoretic considerations as a starting point. This is a logical and appealing framework on which to develop an inspection strategy because of the presumed conflict between the adversary (the diverter) and the inspector (see Section 4.1).

The minimax problem is treated in reference [4.1]. Briefly stated, the diverter chooses a strategy of deciding which items to falsify that will maximize the probability of non-detection (i.e., minimize the probability of detection). The inspector then selects his sample sizes, subject to cost constraints, that will minimize this maximum probability of non-detection.

The minimax results are compared with those described in the preceding sections, i.e., with results based on the criterion that the sample size in a given stratum is that required to detect M units in that stratum. First, consider the case in which the cost of inspection is the same in all strata. Then, using the notation of the preceding sections, the minimax solution (value for n_i^0 on page 58 of [4.1]) is

$$n_{ak} = \frac{N_k \bar{x}_k \sum_k n_{ak}}{\sum_k N_k \bar{x}_k} \qquad \text{(eq. 4.4.7)}$$

where $\sum_k n_{ak}$ is the total sample size, so chosen to provide a probability of detection of $1-\beta$. Specifically, it is the solution of the equation

$$\beta = (1 - M/\sum_k N_k \bar{x}_k)^{\sum_k n_{ak}} \qquad \text{(eq. 4.4.8)}$$

The solution is

$$\sum_k n_{ak} = \frac{\ln\beta}{\ln(1 - M/\sum_k N_k \bar{x}_k)} \qquad \text{(eq. 4.4.9)}$$

Thus, n_{ak} of (eq. 4.4.7) may be rewritten

$$n_{ak} = \frac{N_k \bar{x}_k \ln\beta}{(\sum_k N_k \bar{x}_k)\ln(1-M/\sum_k N_k \bar{x}_k)} \qquad \text{(eq. 4.4.10)}$$

Let us compare this game-theoretic result with the earlier result of (eq. 4.4.1), which appears on the surface to be quite different.

First, note that since M would normally be much smaller than $\sum_k N_k \bar{x}_k$, the total amount available for possible diversion, the denominator of (eq. 4.4.10) simply reduce to -M, using the result that for small a, $\ln(1-a)$ is approximately equal to -a. Thus, (eq. 4.4.10) may be rewritten

$$n_{ak} = -N_k \bar{x}_k \ln\beta/M \qquad \text{(eq. 4.4.11)}$$

Note further that since N_k is common to both (eq. 4.4.1) and (eq. 4.4.11), it remains to determine if

$$(1-\beta^{\bar{x}_k/M}) \quad \text{and}$$

$$- \bar{x}_k \ell n \beta / M$$

are approximately equal to establish that the two formulas for n_{ak} give the same results. To demonstrate this approximate equality, expand

$$\beta^{\bar{x}_k/M}$$

in a Maclaurin's series, retaining only the first order term, and with $\bar{x}_k/M$ as the variable. Thus, approximately,

$$\beta^{\bar{x}_k/M} = \beta^0 + \beta^0 \ell n \beta \left(\frac{\bar{x}_k}{M}\right) = 1 + \ell n \ \beta(\bar{x}_k/M) \qquad \text{(eq. 4.4.12)}$$

Thus,

$$(1-\beta^{\bar{x}_k/M}) = -\bar{x}_k \ \ell n \ \beta/M$$

thereby establishing the identity. The conclusion is that for all practical purposes, (eq. 4.4.1) and (eq. 4.4.10), giving expressions for n_{ak} that appear to be quite different, and that are based on totally different solution paths are, in fact, equivalent.

For those readers who prefer numerical demonstrations of equivalence rather than mathematical ones, the example of Section 4.4.1.2 is reworked using (eq. 4.4.10) rather than (eq. 4.4.1). For this example,

$$\sum_k N_k \bar{x}_k = 502{,}400$$

$$n_{ak} = \frac{N_k \bar{x}_k \ \ell n \ (0.05)}{(502{,}400) \ \ell n \ (1-.002986)} = 0.001994 \ N_k \bar{x}_k$$

$n_{a1} = 479$ (470) $\qquad n_{a4} = n_{a6} = 15$ (15)

$n_{a2} = 477$ (475) $\qquad n_{a5} = n_{a7} = 15$ (15)

$n_{a3} = 3$ (3)

Note the very close agreement with the results found by application of (eq. 4.4.1), which are given in the parentheses.

Thus far in the development, it has been assumed that the cost of inspection per sample is the same in all strata. The approximate minimax solution has also been found for the case in which the cost of inspection may differ from one stratum to the next. Letting ε_k be the effort per measurement in stratum k, the sample size in stratum k is given by

$$n_{ak} = \frac{N_k \bar{x}_k \sum_k \varepsilon_k n_{ak}}{\sum_k \varepsilon_k N_k \bar{x}_k} \qquad \text{(eq. 4.4.13)}$$

where

$$\sum_k \varepsilon_k n_{ak} = -(\sum_k \varepsilon_k N_k \bar{x}_k) \ln \beta/M \qquad \text{(eq. 4.4.14)}$$

These equations follow from (3.42 a) of [4.1]. It is seen that the formulas for n_{ak} may be rewritten:

$$n_{ak} = -N_k \bar{x}_k \ln\beta/M \qquad \text{(eq. 4.4.15)}$$

This is identical to (eq. 4.4.11) for the case in which inspection cost is constant. Hence, from point of view of planning, once M and β are fixed, and for given N_k and $\bar{x}_k$, the cost of inspection plays no role.

In summarizing these results, the conclusion is that the game theoretic minimax approach to the problem produces sample sizes in very good agreement with those found using the criterion that M units diverted in each stratum are to be detected. The minimax approach does provide additional information, namely, it does specify optimum strategies for the diverter, but this is not of interest to the inspector. Either the (eq. 4.4.1) formula or the (eq. 4.4.11) formula may be applied when finding n_{ak}; there is little to choose between them from point of view of simplicity of application.

4.4.2 Variables Inspection Sample Size

The discussion in Section 4.3.2 should be reviewed as background. This provides the motivation for determining the two sample sizes for variables inspection, one sample size relating to variables inspection in the attributes mode, and the other to variables inspection in the variables mode.

4.4.2.1 Variables Inspection Sample Size--Attributes Mode

The procedure for finding the variables inspection sample size in stratum k, when the variables inspection is in the attributes mode, is given by Method 4.2.

Method 4.2

Notation

The quantities M, N_k, β, and $\bar{x}_k$ are defined as in Method 1. The quantity γ_k was defined in Section 4.3.2.1 by the following statement: if a discrepancy exceeds $\gamma_k \bar{x}_k$ in size, it is sure to be labelled a gross defect in the attributes tester inspection; if it is smaller than $\gamma_k \bar{x}_k$, it will not be detected. The last quantity to be defined is

n_{V1k} = number of items to be inspected in stratum k with variables inspection in the attributes mode

Model

Same as for Method 4.1.

Results

The required sample size is given by

$$n_{V1k} = N_k\left(1-\beta^{\gamma_k \bar{x}_k/M}\right) \qquad \text{(eq. 4.4.16)}$$

The result should always be rounded up to the next integer.

Basis

The basis is the same as for Method 4.1, with $\gamma_k \bar{x}_k$ replacing $\bar{x}_k$.

Examples

EXAMPLE 4.2 (a)

Let the stratum to be inspected be the feed stratum in the low enriched uranium fabrication plant of Example 3.3 (a). For attributes inspection, it was found in Example 4.1 (a) that the sample size was 470 items with M = 1500 kg U and β = 0.05. For variables inspection in the attributes mode, find the sample size if γ_1 = 0.20.

Recalling that N_1 = 12000 and $\bar{x}_1$ = 20 kg U, the solution is given by (eq. 4.4.16).

$$n_{v_{11}} = 12000\ (1-0.05^{1/375})$$

$$= 95.48 = 96 \text{ items}$$

EXAMPLE 4.2 (b)

Let the stratum to be inspected be stratum 8 of the mixed oxide fuel fabrication facility of Example 3.3 (b). For attributes inspection, it was found in Example 4.1 (b) that the sample size was 48 items with M = 8 kg Pu and β = 0.05. For variables inspection in the attributes mode, find the sample size if γ_8 = 0.04.

Recalling that N_8 = 360 and $\bar{x}_8$ = 0.375 kg Pu, the solution is given by (eq. 4.4.7).

$$N_{v18} = 360\ (1-0.05^{0.015/8})$$

$$= 2.02 = 3 \text{ items}$$

This example provides the opportunity to indicate a course of action to follow when n_{v1k} is very small, as here. In some applications, n_{v1k} may well be less than 1 which, when rounded up, gives an effective sample of size 1. Since variables data are to be used for a number of purposes, it seems advisable to set a lower limit on its size, recognizing that a single paired comparison certainly provides limited information. In striking a balance between inspection resources and data requirements, a reasonable minimum sample size for the number of variables measurements per strata would normally be about three; this minimum is recommended as a working minimum (subject, of course, to specific circumstances).

As was the case with attributes inspection, the sample size for variables inspection in the attributes mode in stratum k will detect an amount M with probability $(1-\beta)$. If an amount smaller than M units were diverted through medium falsification in stratum k, then the detection probability will clearly be less than $(1-\beta)$ in that particular stratum. However, by an argument very similar to that given in Section 4.4.1.2, if detection involves finding at least one defect in at least one stratum, then no matter how the M units are allocated among the strata, the overall probability of detection will be $(1-\beta)$.

An extension to this result will be given in Section 4.5.2 where the probability of finding at least one gross or medium defect in at least one stratum for attributes and/or variables inspection in the attributes mode is calculated. The goal quantity M, or more generally, the amount diverted through gross and medium data falsifications, which may be less than M, may be allocated arbitrarily among the strata, and into either gross or medium falsifications.

4.4.2.2 Variables Inspection Sample Size--Variables Mode

Sections 4.3.2.2 and 4.3.2.3 should be reviewed. The sample size for variables inspection in the variables mode will utilize either the $\hat{D}$ statistic or the $(MUF-\hat{D})$ statistic for planning purposes. Method 4.3 provides the equations needed to compute the sample size.

Method 4.3

Notation

Much of the notation needed for this method has been defined in prior sections. It is repeated here to facilitate application of the method.

$V_g(\hat{D})$ = short term systematic error variance of $\hat{D}$ defined by (eq. 3.5.14)

$V_s(\hat{D})$ = long term systematic error variance of $\hat{D}$ defined by (eq. 3.5.21)

$V_{s1}(\hat{D}_k)$ = random error variance of $\hat{D}_k$ due to the facility sampling and analytical errors, computed for those strata in which the number of items measured by the inspector exceeds the number of batches. In this event, the error, although random in origin, behaves like a systematic error.

$V_{r1}(\hat{D}_k)$ = random error variance of $\hat{D}_k$ per item due to facility sampling and analytical errors, computed for those strata for which $V_{s1}(\hat{D}_k)$ is not computed.

$V_{r2}(\hat{D}_k)$ = random error variance of $\hat{D}_k$ per item due to the facility bulk measurement

$V_{r3}(\hat{D}_k)$ = random error variance of $\hat{D}_k$ per item due to the inspector's sampling and analytical errors.

σ_s^2 = The sum of the components of the variance of $\hat{D}$ whose values do not decrease with additional inspector measurements (i.e., systematic errors)

s_k^2 = random error variance of $\hat{D}_k$ per item measured by the inspector

$V_r(\hat{D})|H_0$ = random error variance of $\hat{D}$ under the hypothesis that nothing has been diverted and obscured by small data falsifications

$V_r(\hat{D})|H_1$ = random error variance of $\hat{D}$ under the alternative that small data falsifications have been introduced to mask diversion

$C_0{}^2$ = variance inflation factor relating $V_r(\hat{D})|H_1$ to $V_r(\hat{D})|H_0$

θ = ratio of $V_r(\hat{D})|H_0$ to σ_s^2
(Note: θ is inversely proportional to the total sample size)

V(MUF) = variance of element MUF

V_0 = that part of the variance of element MUF due to systematic errors that are common to both the facility and the inspector

STAT = test statistic, either $\hat{D}$ or (MUF-$\hat{D}$)

α = significance level of test

β = probability of failing to detect an amount M diverted and masked by small data falsifications

M = goal quantity in amount of element

t_α = defined such that the area under the standardized normal curve t_α to ∞ is α

t_β = defined such that the area under the standardized normal curve from t_β to ∞ is β

u_k = number of batches sampled by the inspector

w_k = number of items per sampled batch for which the inspector makes bulk measurements

v_k = number of samples drawn by the inspector per sampled batch to determine the element factor

a_k = number of analyses performed by the inspector per sample in stratum k

m_k = number of batches in stratum k

r_k = number of sample drawn by the facility per batch to determine the element factor

c_k = number of analyses performed by the facility per sample in stratum k

Model

It is assumed that the test statistic, either D or (MUF-D), is normally distributed. For each statistic, the mean is zero under the hypothesis of no diversion, and M under the alternative hypothesis. Also, for each statistic, the random error variance under the alternative hypothesis is assumed to be larger than under the null hypothesis.

Results

The quantities σ_S^2 and s_k must be calculated. To calculate σ_S^2, the first step is to calculate $V_g(\hat{D})$ and $V_S(\hat{D})$ from (eq. 3.5.14) and (eq. 3.5.21) respectively. Next, $V_{s_1}(\hat{D}_k)$ is computed for each stratum for which the number of items to be measured by the inspector exceeds the number of batches. It is assumed that the inspector allocates his samples evenly among batches, or as near evenly as possible (i.e., he may sample one additional item from certain batches in order to achieve his required total number of samples). In some instances, it may not be known a priori whether or not $V_{s_1}(\hat{D}_k)$ should be calculated for a certain stratum. One could, of course, iterate to a solution, but the impact of including a given stratum in this calculation when it should not be included, or vice-versa, will usually not be very large, and iteration should normally not be required. The quantity $V_{s_1}(\hat{D}_k)$ is calculated from:

$$V_{s_1}(\hat{D}_k) = x_{kqpt}^2 \left(\delta_{r\cdot p\cdot x}^2/m_k r_k + \delta_{r\cdot\cdot tx}^2/c_k m_k r_k\right) \qquad \text{(eq. 4.4.17)}$$

Then, σ_s^2 is given by

$$\sigma_s^2 = V_g(\hat{D}) + V_s(\hat{D}) + \sum_{k=1}^{K} V_{s_1}(\hat{D}_k) \qquad \text{(eq. 4.4.18)}$$

To calculate s_k, the first step is to calculate $V_{r_1}(\hat{D}_k)$ for all strata for which $V_{s_1}(\hat{D}_k)$ was not computed.

$$V_{r_1}(\hat{D}_k) = x_{kqpt}^2 \left(\delta_{r\cdot p\cdot x}^2/r_k + \delta_{r\cdot\cdot tx}^2/c_k r_k\right) \qquad \text{(eq. 4.4.19)}$$

Next, for each stratum, $V_{r_2}(\hat{D}_k)$ is computed.

$$V_{r_2}(\hat{D}_k) = x_{kqpt}^2 \, \delta_{rq\cdot\cdot x}^2 \qquad \text{(eq. 4.4.20)}$$

Similarly,

$$V_{r_3}(\hat{D}_k) = y_{kqpt}^2 \left(\delta_{rq\cdot\cdot y}^2 + \delta_{r\cdot p\cdot y}^2 + \delta_{r\cdot\cdot ty}^2/a_k\right) \qquad \text{(eq. 4.4.21)}$$

Then, s_k^2 is computed for each stratum

$$s_k^2 = V_{r_1}(\hat{D}_k) + V_{r_2}(\hat{D}_k) + V_{r_3}(\hat{D}_k) \qquad \text{(eq. 4.4.22)}$$

The quantities V(MUF) and V_0 are computed using Methods 3.3, 3.4, and 3.5, keeping in mind the systematic errors that are common to both the facility and the inspector when computing V_0. Then calculate k_2

$$k_2 = \frac{V(MUF) - 2V_0}{\sigma_s^2} \qquad \text{(eq. 4.4.23)}$$

With M known, compute

$$m = M/\sigma_s \qquad \text{(eq. 4.4.24)}$$

The quantities α and β (and hence, t_α and t_β) are chosen, as is C_0^2, the variance inflation factor. Finally, a decision is made as to the choice of the test statistic to be used in planning, i.e., either $\hat{D}$ or (MUF-$\hat{D}$). Define

$$\begin{aligned} a &= 1 \text{ if STAT is } \hat{D} \\ &= 1-k \text{ if STAT is } (MUF-\hat{D}) \end{aligned} \qquad \text{(eq. 4.4.25)}$$

The quantity θ, which is inversely proportional to the sample size, is then calculated.

$$\theta = (A-B)/C \qquad \text{(eq. 4.4.26)}$$

where

$$\begin{aligned} A = [&(C_0^2+1)at_\alpha^2 t_\beta^2 + m^2t_\alpha^2 + C_0^2m^2t_\beta^2 \\ &- at_\alpha^4 - C_0^2at_\beta^4] \end{aligned} \qquad \text{(eq. 4.4.27)}$$

$$B = 2mt_\alpha t_\beta \sqrt{a(C_0^2-1)(C_0^2t_\beta^2-t_\alpha^2)+C_0^2m^2} \qquad \text{(eq. 4.4.28)}$$

$$C = (t_\alpha^2-C_0^2t_\beta^2)^2 \qquad \text{(eq. 4.4.29)}$$

Two special cases are considered. First, if $C_0^2 = 1$, then

$$\theta = m^2/(t_\alpha+t_\beta)^2 - a \qquad \text{(eq. 4.4.30)}$$

If $C = 0$, i.e., if $C_0^2 = (t_\alpha/t_\beta)^2$, then

$$\theta = \frac{a^2(t_\alpha^2 - t_\beta^2)^2 - 2am^2(t_\alpha^2 + t_\beta^2) + m^4}{4m^2t_\alpha^2} \qquad \text{(eq. 4.4.31)}$$

Having computed θ, the total sample size, n_{V_2}, is found by

$$n_{V_2} = (\sum_{k=1}^{K} s_k)^2/\theta\sigma_s^2 \qquad \text{(eq. 4.4.32)}$$

In stratum k, the sample size is

$$n_{v2k} = n_{v2}\, s_k / \sum_{k=1}^{K} s_k \qquad \text{(eq. 4.4.33)}$$

Note 1: The quantity θ as computed by (eq. 4.4.26), (eq. 4.4.30), or (eq. 4.4.31) may be negative. This means that the input value for β is unattainable because of the limitations due to the systematic error. A larger value of β must then be selected and θ recomputed.

Note 2: The number of items inspected in stratum k, n_{v2k}, is interpreted to be the total number of weighings or, equivalently, the total number of samples drawn to estimate element concentration factors. That is,

$$n_{v2k} = u_k v_k = u_k w_k \qquad \text{(eq. 4.4.34)}$$

It is quite unlikely that v_k and w_k will be integers, unless m_k, the number of batches, is either equal to one or to the total number of items in the stratum. Thus, for some strata, it is more correct to write

$$n_{v2k} = \sum_{i=1}^{m_k} v_{ki} = \sum_{i=1}^{m_k} w_{ki} \qquad \text{(eq. 4.4.35)}$$

where v_{ki} is the number of samples drawn by the inspector in batch i of stratum k, and w_{ki} is similarly defined for bulk measurements. For any two such batches, i and j, the quantities v_{ki} and v_{kj} would not normally differ by more than one, nor would w_{ki} and w_{kj}.

Note 3: The results given by this method are quite general in nature. If certain parameter values are fixed, then simpler graphical solutions for θ are obtained. Specifically, commonly used values are

$$\alpha = 0.05 \qquad C_0^2 = 4 \qquad a = 1 \text{ (i.e., use } \hat{D} \text{ statistic)}$$

For these values of the parameters, the figures in Annexes 4.1 (a) and 4.1 (b) show the relationship between θ (or more exactly, $1/\theta$), and β (or, more exactly, $1-\beta$). Having selected a value for θ corresponding to the desired, or feasibly attainable value for β, the corresponding sample sizes are again given by (eq. 4.4.32) and (eq. 4.4.33).

A comment is in order on the value assigned C_0^2. Commonly, the value $C_0^2 = 4$ has been used in planning. This means that the random error variance of $\hat{D}$ under the alternative hypothesis of diversion is four times that under the null hypothesis of no diversion. The variance is inflated when not all items are falsified, or when the items that are falsified (which may be all the items) are falsified by varying amounts. For the first case, an expression for the variance of $\hat{D}$ under H is given by equation (3.38) in [4.1], and a similar expression, modified to reflect the different amounts by which items are falsified, would describe the second diversion strategy. Examination of the cited equation (3.38) would indicate that $C_0^2 = 4$ is a reasonable upper bound; this corresponds to falsifying items by an amount equal to about 3.5 times the random error standard deviation for each item element weight. Since the value of 4 represents an upper bound, it is conservative for planning purposes in that the sample sizes determined for this value of C_0^2 will tend to err on the high side. (Also see Section 6.3 of this Manual.)

Basis

The sample size is selected using amount of element as the variable rather than amount of isotope. First, let the test statistic be $\hat{D}$ rather than (MUF-$\hat{D}$). The variance of $\hat{D}$ under the hypothesis of no diversion is

$$V(\hat{D})|H_0 = \sigma_s^2 + V_r(\hat{D})|H_0 \qquad \text{(eq. 4.4.36)}$$

where

$$V_r(\hat{D})|H_0 = \theta\sigma_s^2 \qquad \text{(eq. 4.4.37)}$$

defines the quantity θ .

Under the alternative hypothesis, H_1, the variance of $\hat{D}$ is

$$V(\hat{D})|H_1 = \sigma_s^2 + C_0^2\theta\sigma_s^2 \qquad \text{(eq. 4.4.38)}$$

The quantity $V_r(\hat{D})|H_0$ is a function of the total sample size and also of the allocation of this total sample size among the K strata. From (eq. 3.5.6), and recognizing that $V_r(\hat{D})|H_0$ is the same as $V_r(\hat{D})$,

$$V_r(\hat{D})|H_0 = \sum_{k=1}^{K} V_r(\hat{D}_k) \qquad \text{(eq. 4.4.39)}$$

where $V_r(\hat{D}_k)$ may be written in the form:

$$V_r(\hat{D}_k) = x_{kqpt}\,(\delta^2_{rq\cdot\cdot x}/n_{v2k} + \delta^2_{r\cdot p\cdot x}/\text{DIV} + \delta^2_{r\cdot\cdot tx}/c_k\text{DIV})$$

$$+\; y^2_{kqpt}\,(\delta^2_{rq\cdot\cdot y}/n_{v2k} + \delta^2_{r\cdot p\cdot y}/n_{v2k} + \delta^2_{r\cdot\cdot ty}/a_k n_{v2k}) \qquad \text{(eq. 4.4.40)}$$

where

$$\text{DIV} = \min\,(n_{v2k},\, m_k r_k) \qquad \text{(eq. 4.4.41)}$$

and where a_k is a known input value (i.e., the number of analyses per sample performed by the inspector is known)

From (eq. 4.4.19) - (eq. 4.4.23), it is seen that $V_r(\hat{D})|H_0$ may be written in the form

$$V_r(\hat{D})|H_0 = \sum_{k=1}^{K} s_k^2/n_{v2k} \qquad \text{(eq. 4.4.42)}$$

keeping in mind that $V_{r_1}(\hat{D}_k)$ is included in the definition of s_k only for those strata where DIV of (eq. 4.4.41) is n_{v_2k}.

In allocating the samples among the strata, this allocation should be performed so as to minimize $V_r(\hat{D})|H_0$. This is seen from (eq. 4.4.48) to follow in which the aim is to maximize t_β. In this equation, θ is directly proportional to $V_r(\hat{D})|H_0$. When all items are falsified by the same amount, the variance inflation factor, C_0^2, is one, and t_β is clearly maximized as θ is minimized. At values of C_0^2 greater than one, t_β is maximized as θ is minimized as long as $\beta<0.50$, which is the region of interest. For values of $\beta>0.50$, it would benefit the inspector to have a large variance of $\hat{D}$, but this region of β is of little practical importance.

Minimization of $V_r(\hat{D})|H_0$ is a standard minimization problem, and it is easily shown that the n_{v_2k} should be chosen by the equation:

$$n_{v2k} = n_{v2}\, s_k / \sum_{k=1}^{K} s_k \qquad \text{(eq. 4.4.43)}$$

(This result is derived in [4.8] where it is also shown how other criteria for optimization may be included.)

From (eq. 4.4.42) and (eq. 4.4.43), along with (eq. 4.4.37), the key relationship between n_{v2} and θ, given by (eq. 4.4.32), follows easily. If some other optimization criterion were imposed, then the relationship between θ and n_{v2} would have to be altered accordingly, but the equations to follow, which lead to the solution for θ, would remain unchanged.

The value for θ is selected such that the goal amount, M, is detected with probability $(1-\beta)$ under the alternative. Further, the significance level of the test is set at α. Before writing down the equations that lead to the solution for θ, it should be noted that as $\hat{D}$ is defined, large <u>negative</u> values of $\hat{D}$ are evidences of diversion, and not large positive values. It is convenient to replace $\hat{D}$ by its negative counterpart, $\hat{D}'$, such that large positive values of $\hat{D}'$ lead to rejection of the null hypothesis of no diversion. Thus, define

$$\hat{D}' = -\hat{D} \qquad \text{(eq. 4.4.44)}$$

It is obvious that the variance of $\hat{D}'$ will, of course, be the same as the variance of $\hat{D}$. The two equations to solve for θ and the critical value, D_0, are

$$\text{Prob } (\hat{D}'>D_0 | E(\hat{D}') = 0) = \alpha \qquad \text{(eq. 4.4.45)}$$

and

$$\text{Prob } (\hat{D}'>D_0 | E(\hat{D}') = M) = 1-\beta \qquad \text{(eq. 4.4.46)}$$

From (eq. 4.4.45), (eq. 4.4.36), and (eq. 4.4.37),

$$D_0 = t_\alpha \sigma_s \sqrt{1 + \theta} \qquad \text{(eq. 4.4.47)}$$

where t_α is defined in the Notation section. From (eq. 4.4.46), (eq. 4.4.38), and (eq. 4.4.47),

$$\frac{t_\alpha \sigma_S \sqrt{1+\theta} - M}{\sigma_S \sqrt{1+C_0^2\theta}} = -t_\beta \qquad \text{(eq. 4.4.48)}$$

or, replacing M/σ_S by m, solve the following equation for θ.

$$t_\alpha \sqrt{1+\theta} - m = -t_\beta \sqrt{1+C_0^2\theta} \qquad \text{(eq. 4.4.49)}$$

Before indicating the solution to this equation, note the effect of replacing the test statistic $\hat{D}$ by (MUF-$\hat{D}$) while keeping all other parameters fixed. In this event, from (eq. 3.6.5), the variance of the test statistic, (MUF-$\hat{D}$), under the null hypothesis is

$$V(MUF-\hat{D})|H_0 = \sigma_S^2(1+\theta-k_2) \qquad \text{(eq. 4.4.50)}$$

while its variance under the alternative hypothesis is

$$V(MUF-\hat{D})|H_1 = \sigma_S^2(1+C_0^2\theta-k_2) \qquad \text{(eq. 4.4.51)}$$

with k_2 defined by (eq. 4.4.53). It follows immediately that for (MUF-$\hat{D}$) as the test statistic, (eq. 4.4.49) becomes

$$t_\alpha \sqrt{a+\theta} - m = -t_\beta \sqrt{a+C_0^2\theta} \qquad \text{(eq. 4.4.52)}$$

where $$a = 1-k_2 \qquad \text{(eq. 4.4.53)}$$

Thus, in general terms, (eq. 4.4.52) may be solved for θ, where a = 1 if the sample size is based on detecting M units with the $\hat{D}$ statistic, and $a = 1-k_2$ if the (MUF-$\hat{D}$) statistic is used. It is shown in [4.9] that the solution to (eq. 4.4.52) is given by the key results: (eq. 4.4.26), (eq. 4.4.30), and (eq. 4.4.31).

Examples

EXAMPLE 4.3 (a)

Consider the low enriched uranium facility of Examples 3.3 (a) and 3.5 (a). Follow Method 4.3 to determine the variables inspection sample size when inspection is in the variables mode. Set M = 1500 kg U, and assume that duplicate analyses are performed in all strata but stratum 3. The inspection samples are distributed to a single laboratory as indicated in Example 3.9 (a). From that example, and from Example 3.10 (a),

$$V_g(\hat{D}) = 32{,}584 \text{ kg}^2\text{U}$$

$$V_s(\hat{D}) = 51{,}484 \text{ kg}^2\text{U}$$

The quantities $V_{s1}(\hat{D}_k)$ are then computed using (eq. 4.4.17) for those strata for which the numbers of items to be measured by the inspector exceed the numbers of batches. These strata include strata 2, 4, 5, 6, and 7.

$$V_{s1}(\hat{D}_2) = (238{,}800)^2[(0.000568)^2/240] = \quad 77$$

$$V_{s1}(\hat{D}_4) = V_{s1}(\hat{D}_6)$$
$$= (7200)^2[(0.0181)^2/60 + (0.0274)^2/60] = \quad 932$$

$$V_{s1}(\hat{D}_5) = V_{s1}(\hat{D}_7)$$
$$= (4000)^2[(0.0418)^2/48 + (0.0274)^2/48] = \quad 833$$

From (eq. 4.4.18),

$$\sigma_s^2 = 32{,}584+51{,}484+77+2(932)+2(833) = \quad 87{,}675 \text{ kg}^2\text{U}$$

Next, $V_{r1}(\hat{D}_1)$ and $V_{r_1}(\hat{D}_3)$ are calculated from (eq. 4.4.19).

$$V_{r1}(\hat{D}_1) = (240{,}000)^2[(0.000531)^2/5 + (0.000433)^2/5] = \quad 5408$$

$$V_{r1}(\hat{D}_3) = (1200)^2(0.0577)^2 = \quad 4794$$

For each stratum, $V_{r2}(\hat{D}_k)$ and $V_{r3}(\hat{D}_k)$ are found using (eq. 4.4.20) and (eq. 4.4.21), and assuming that $x_{kqpt} = y_{kqpt}$ for all k.

$$V_{r2}(\hat{D}_1) = (240{,}000)^2(0.000658)^2 = \quad 24{,}939$$

$$V_{r2}(\hat{D}_2) = (238.800)^2(0.000877)^2 = \quad 43{,}860$$

$$V_{r2}(\hat{D}_3) = 0$$

$$V_{r2}(\hat{D}_4) = V_{r2}(\hat{D}_6) = (7200)^2(0.00250)^2 = \quad 324$$

$$V_{r2}(\hat{D}_5) = V_{r2}(\hat{D}_7) = (4000)^2(0.00250)^2 = \quad 100$$

$$V_{r3}(\hat{D}_1) = (240{,}000)^2[(0.000658)^2+(.000531)^2+(0.000433)^2/2] = 46{,}579$$

$$V_{r3}(\hat{D}_2) = (238{,}800)^2[(0.000658)^2+(0.000822)^2/2] = \quad 43{,}956$$

$$V_{r3}(\hat{D}_3) = (1200)^2(0.0923)^2 = \quad 12{,}268$$

$$V_{r3}(\hat{D}_4) = V_{r3}(\hat{D}_6) = (7200)^2[(0.000658)^2+(0.0181)^2$$
$$+ (0.0198)^2/2] = \quad 27{,}167$$

$$V_{r3}(\hat{D}_5) = V_{r3}(\hat{D}_7) = (4000)^2[(0.000658)^2+ (0.0418)^2$$
$$+ (0.0198)^2/2] = \quad 31{,}099$$

The quantity s_k is now computed for each stratum using (eq. 4.4.22).

$$s_1^2 = 5{,}408 + 24{,}939 + 46{,}579 = 76{,}926 \quad ; s_1 = 277$$

$$s_2^2 = 43{,}860 + 43{,}956 = 87{,}816 \quad ; s_2 = 296$$

$$s_3^2 = 4{,}794 + 12{,}268 = 17{,}062 \quad ; s_3 = 131$$

$$s_4^2 = s_6^2 = 324 + 27{,}167 = 27{,}491 \quad ; s_4 = s_6 = 166$$

$$s_5^2 = s_7^2 = 100 + 31{,}099 = 31{,}199 \quad ; s_5 = s_7 = 177$$

$$\sum_{k=1}^{7} s_k = 1390$$

Suppose that the following criteria are set:

$\alpha = 0.05$ (significance level)

$c_0^2 = 4$ (variance inflation factor)

$a = 1$ ($\hat{D}$ statistic used in planning)

Then, following Note 3, the graphical solution for θ is found. Here,

$$M/\sigma_S = 1500/296 = 5.07$$

Set $\beta = 0.05$, or $1-\beta = 0.95$. From Annex 4.1 (b),

$$^1/\theta \simeq 1.75$$

The total sample size is then given by (eq. 4.4.32).

$$n_{V_2} = (1390)^2(1.75)/(87{,}675)$$
$$= 39$$

In each stratum, the sample size is given by (eq. 4.4.33).

$$n_{V21} = (39)(277)/1390 = 8$$

$$n_{V22} = (39)(296)/1390 = 8$$

$$n_{V23} = 4$$

$$n_{V24} = n_{V26} = 5$$

$$n_{V25} = n_{V27} = \underline{5}$$

$$\text{Total} = 40$$

To illustrate application of the more general methodology suppose now that the (MUF-$\hat{D}$) statistic is to be used in planning, and that β = 0.025 with all other parameters the same. It is necessary to calculate k_2 from (eq. 4.4.23) to obtain the value for a from (eq. 4.4.25). From Example 3.5 (a),

$$V(MUF) = 45{,}010 \text{ kg}^2\text{U}$$

From Example 3.10 (a), V_0 = 0 since, although both the facility and the inspector commit the same systematic errors in sampling, the contribution to the variance of MUF due to these sources of error was zero in this example. Thus,

$$k_2 = 45{,}010/87{,}675 = 0.5134$$

$$a = 1-k_2 = 0.4866$$

The quantity θ is now calculated from (eq. 4.4.26), but first, A, B, and C are found using (eq. 4.4.27), (eq. 4.4.28), and (eq. 4.4.29).

$$A = [(5)(0.4866)(1.645)^2(1.960)^2+(5.07)^2(1.645)^2+(4)(5.07)^2(1.960)^2 -(0.4866)(1.645)^4-(4)(0.4866)(1.960)^4] = 457.5541$$

$$B = (2)(5.07)(1.645)(1.960)\sqrt{(0.4866)(3)[(4)(1.960)^2-(1.645)^2]+(4)(5.07)^2} = 360.0746$$

$$C = [(1.645)^2 - (4)(1.960)^2]^2 = 160.2851$$

$$\theta = \frac{457.5541 - 360.0746}{160.2851} = 0.6082$$

The total sample size in this instance is

$$n_{V2} = (1390)^2/(0.6082)(87{,}675) = 37$$

Finally, suppose that the variance inflation factor, C_0^2, were set equal to 1 rather than 4. Then, θ is given directly by (eq. 4.4.30).

$$\theta = (5.07)^2/(1.645+1.960)^2 - 0.4866$$

$$= 1.4913 \quad ,$$

and the sample size is 15 rather than 37.

EXAMPLE 4.3 (b)

Consider the mixed oxide fuel fabrication facility first introduced in Example 3.3 (b) and considered in a number of examples following that one. Set M = 8 kg Pu, and assume that duplicate analyses are performed in all strata but stratum 4. Further assume that the inspection samples are distributed to only one laboratory, and that the short term systematic error structure is as indicated in Example 3.9 (b). Find the inspection sample size for two cases:

Case 1 $\alpha = 0.05$ $\beta = 0.05$ $C_0^2 = 4$

Use $\hat{D}$ statistic

Case 2 $\alpha = 0.025$ $\beta = 0.20$ $C_0^2 = 4$

Use (MUF-$\hat{D}$) statistic

The quantities $V_g(\hat{D})$ and $V_s(\hat{D})$ were calculated in Examples 3.9 (b) and 3.10 (b) respectively.

$$V_g(\hat{D}) = 10.020730 \text{ kg}^2\text{Pu}$$

$$V_s(\hat{D}) = 13.957723 \text{ kg}^2\text{Pu}$$

The quantities $V_{s1}(\hat{D}_k)$ are then computed using (eq. 4.4.17) for those strata for which the numbers of items to be measured by the inspector exceed the numbers of batches. These strata likely include strata 1, 3, 6, 7, 9, and 10

$$V_{s1}(\hat{D}_1) = (1536)^2[(0.0001)^2/96+(0.0040)^2/192] = \quad 0.196854$$

$$V_{s1}(\hat{D}_3) = (9.0)^2[(0.035)^2/10+(0.0060)^2/10] = \quad 0.010214$$

$$V_{s1}(\hat{D}_6) = (3.6)^2[(0.035)^2/4+(0.0060)^2/4] = \quad 0.004086$$

$$V_{s1}(\hat{D}_7) = (4.5)^2[(0.020)^2/6+(0.0050)^2/6] = \quad 0.001434$$

$$V_{s1}(\hat{D}_9) = (4.5)^2[(0.035)^2/5+(0.0060)^2/5] = \quad 0.005107$$

$$V_{s1}(\hat{D}_{10}) = (2.25)^2[(0.020)^2/3+(0.0050)^2/3] = \quad 0.000717$$

From (eq. 4.4.18),

$$\sigma_s^2 = 10.020730+13.957723+\cdots+0.000717 = \quad 24.196865$$

Next, $V_{r1}(\hat{D}_2)$, $V_{r1}(\hat{D}_4)$, $V_{r1}(\hat{D}_5)$, and $V_{r1}(\hat{D}_8)$ are calculated using (eq. 4.4.19).

$$V_{r1}(\hat{D}_2) = (1485)^2[(0.0080)^2/5+(0.0050)^2/5] = \quad 39.253005$$

$$V_{r1}(\hat{D}_4) = (0.4)^2(0.20)^2 = \quad 0.006400$$

$$V_{r1}(\hat{D}_5) = (112.5)^2[(0.0040)^2/3+(0.0060)^2/3] = \quad 0.219375$$

$$V_{r1}(\hat{D}_8) = (135)^2[(0.0040)^2/3+(0.0060)^2/3] = \quad 0.315900$$

For each stratum, $V_{r2}(\hat{D}_k)$ and $V_{r3}(\hat{D}_k)$ are found using (eq. 4.4.20) and (eq. 4.4.21), and assuming that $x_{kqpt} = y_{kqpt}$ for all k.

$$V_{r2}(\hat{D}_1) = (1536)^2(0.00025)^2 = \quad 0.147456$$

$$V_{r2}(\hat{D}_2) = (1485)^2(0.00050)^2 = \quad 0.551306$$

$V_{r_2}(\hat{D}_3) = (9.0)^2(0.00040)^2 =$ 0.000013

$V_{r_2}(\hat{D}_4) = (0.4)^2\ (0) =$ 0

$V_{r_2}(\hat{D}_5) = (112.5)^2(0.00040)^2 =$ 0.002025

$V_{r_2}(\hat{D}_6) = (3.6)^2(0.00040)^2 =$ 0.000002

$V_{r_2}(\hat{D}_7) = (4.5)^2(0.00040)^2 =$ 0.000003

$V_{r_2}(\hat{D}_8) = (135)^2(0.00040)^2 =$ 0.002916

$V_{r_2}(\hat{D}_9) = (4.5)^2(0.00040)^2 =$ 0.000003

$V_{r_2}(\hat{D}_{10}) = (2.25)^2(0.00040)^2 =$ 0.000001

$V_{r_3}(\hat{D}_1) = (1536)^2[(0.00050)^2+(0.0001)^2+(0.0050)^2/2] =$ 30.104617

$V_{r_3}(\hat{D}_2) = (1485)^2[(0.00075)^2+(0.0080)^2+(0.0070)^2/2] =$ 196.402852

$V_{r_3}(\hat{D}_3) = (9.0)^2[(0.00075)^2+(0.035)^2+(0.010)^2/2] =$ 0.103321

$V_{r_3}(\hat{D}_4) = (0.4)^2(.40)^2 =$ 0.025600

$V_{r_3}(\hat{D}_5) = (112.5)^2[(0.00050)^2+(0.0040)^2+(0.010)^2/2] =$ 0.838477

$V_{r_3}(\hat{D}_6) = (3.6)^2[(0.00050)^2+(0.035)^2+(0.010)^2/2] =$ 0.016527

$V_{r_3}(\hat{D}_7) = (4.5)^2[(0.00050)^2+(0.020)^2+(0.0070)^2/2] =$ 0.008601

$V_{r_3}(\hat{D}_8) = (135)^2[(0.00075)^2+(.0040)^2+(0.010)^2/2] =$ 1.213102

$V_{r_3}(\hat{D}_9) = (4.5)^2[(0.00075)^2+(0.035)^2+(0.010)^2/2] =$ 0.025830

$V_{r_3}(\hat{D}_{10}) = (2.25)^2[(0.00075)^2+(0.020)^2+(0.0070)^2/2] =$ 0.002152

The quantity s_k^2 is now computed for each stratum using (eq. 4.4.22).

$s_1^2 = 0.147456 + 30.104617 = 30.252073;\ s_1 = 5.500$

$s_2^2 = 39.253005+0.551306+196.402852 = 236.207163;\ s_2 = 15.369$

$s_3^2 = 0.000013 + 0.103321 = 0.103334;\ s_3 = 0.321$

$s_4^2 = 0.006400 + 0.025600 = 0.032000;\ s_4 = 0.179$

$s_5^2 = 0.219375 + 0.002025 + 0.838477 = 1.059877;\ s_5 = 1.030$

$s_6^2 = 0.000002 + 0.016527 = 0.016529;\ s_6 = 0.129$

$s_7^2 = 0.000003 + 0.008601 = 0.008604;\ s_7 = 0.093$

$s_8^2 = 0.315900 + 0.002916 + 1.213102 = 1.531918$; $s_8 = 1.238$

$s_9^2 = 0.000003 + 0.025830 = 0.025833$; $s_9 = 0.161$

$s_{10}^2 = 0.000001 + 0.002152 = 0.002153$; $s_{10} = 0.046$

$$\sum_{k=1}^{10} s_k = 24.066$$

For the Case 1 criteria, Note 3 of the method may be followed to provide the graphical solution for θ, noting that

$$M/\sigma_s = 8/4.919 = 1.626$$

From Annex 4.1 (a), it is evident that it is impossible in this example to meet the $\beta = 0.05$ criterion. Even with very large sample sizes, the β value for $M/\sigma_S = 1.626$ is about 0.50. Somewhat arbitrarily set $^1/\theta = 10$ which yields a β value slightly smaller than 0.50. For $^1/\theta = 10$, the total sample size is given by (eq. 4.4.32).

$$n_{v2} = (24.066)^2(10)/(24.196865) = 240$$

In each stratum, the sample size is given by (eq. 4.4.33).

n_{v21}	$= (240)(5.500)/24.066$	= 55
n_{v22}	$= (240)(15.369)/24.066$	= 153
n_{v23}		= 3
n_{v24}		= 2
n_{v25}		= 10
n_{v26}		= 1
n_{v27}		= 1
n_{v28}		= 12
n_{v29}		= 2
$n_{v2,10}$		= 1
	Total	= 240

(Note that there are 5 strata with less than 3 samples. Refer back to the comments following Example 4.2 (b) relative to minimum numbers of samples per stratum.)

This completes the calculations for Case 1. For Case 2, it is first necessary to assign values to V(MUF) and V_0. From Examples 3.5 (b) and 3.10 (b) respectively,

$$V(MUF) = 10.729310 \text{ kg}^2\text{Pu}$$

$$V_0 = 2.230517 \text{ kg}^2\text{Pu}$$

Then, from (eq. 4.4.23),

$$k_2 = [10.729310 - 2(2.230517)]/24.196865$$

$$= 0.2591$$

and, from (eq. 4.4.25),

$$a = 0.7409$$

The quantity θ is then calculated from (eq. 4.4.26) and following, where

$$m = 1.626 \qquad t\alpha = 1.960$$

$$C_0^2 = 4 \qquad t\beta = 0.842$$

$$A = [(5)(0.7409)(1.960)^2(0.842)^2+(1.626)^2(1.960)^2+(4)(1.626)^2(0.842)^2 -(0.7409)(1.960)^4-(4)(0.7409)(0.842)^4] = 15.320064$$

$$B = (2)(1.626)(1.960)(0.842)\ \sqrt{(0.7409)(3)[(4)(0.842)^2-(1.960)^2]+4(1.626)^2} = 15.498965$$

Since B<A for this value of β, a larger value of β must be inputted to provide a positive value for θ. Trial and error calculations show that β must exceed 0.50. At β = 0.55 (or 1-β = 0.45), t_β = -0.126. The revised values for A and B are

$$A = -0.384323 \qquad B = -1.185220$$

Also, $$C = [(1.960)^2-4(-0.126)^2]^2 = 14.274009$$

and

$$\theta = (-0.384323+1.185220)/14.274009$$

$$= 0.0561$$

At this value for θ, the required sample size is

$$n_{V2} = (24.066)^2/(0.0561)(24.196865) = 427$$

4.5 EVALUATION OF INSPECTION PLAN--INDIVIDUAL TESTS

In this section and in Section 4.6 to follow, methods are given for evaluating the inspection plan along with the accountability measures taken by the facility. Evaluation is measured by the probability that the statistical test in question will return a significant result as a function of the amount of material unaccounted for by the mechanism to which the test is designed to be responsive.

In applying the statistical tests in question, a distinction is made between principal and supplemental tests, as discussed in the next section.

4.5.1 Distinction Between Principal and Supplemental Tests

As will be discussed in Section 4.6, of chief concern or interest to the inspection planner is the overall probability of detection of a specified goal amount. Detection occurs if at least one of the statistical tests in question returns a positive response. The statistical tests that are applied collectively in calculating this overall probability of detection are called principal tests. One important characteristic of the principal tests is that, taken as a group, there is control over the value for α, the significance level (false alarm probability) associated with the collection of statistical tests to be applied.

There are two combinations of tests that are identified as principal tests. These are as follows.

Combination 1:

1) The tests for gross defects performed with the attributes tester. There are K such tests.

2) The tests for medium defects performed with the variables tester used in the attributes made. There are K such tests.

3) The test for the significance of (MUF-$\hat{D}$), the facility MUF adjusted for the bias (small defects) that is estimated by the $\hat{D}$ statistic.

Combination 2:

1) Same as above.

2) Same as above.

3) The test for the significance of $\hat{D}$, the measure of facility bias as it affects the MUF.

4) The test for the significance of the facility MUF.

These principal tests are discussed separately in Sections 4.5.2, 4.5.4, 4.5.5, and 4.5.6. The tests that comprise Combination 1 are then considered collectively in Section 4.6.1 while those that comprise Combination 2 are treated in Section 4.6.2.

In addition to these principal tests, the data analyst (inspector) would want to perform other statistical tests when analyzing the data for a given facility over a given material balance period (see Section 4.1.1). These other tests are referred to as supplemental tests to distinguish them from the principal tests. Examples of supplemental tests that may be applied include:

1) Tests for the significance of $\hat{D}_k$, the measure of facility bias in stratum k.

2) Tests for the significance of shipper-receiver differences.

3) Miscellaneous tests on distributional properties, such as outlier tests and tests for normality.

4) Tests for randomness of small calculational mistakes.

5) Tests for randomness of data over time and other data groupings, including CUSUM plots and analyses of variance.

This listing is not intended to be all inclusive, but rather to portray the kinds of supplemental tests that the data analyst may apply. The statistical tests are described in Chapter 5. In this chapter, Chapter 4, the emphasis is not on the test itself, but rather on its ability to detect a specified missing amount of material. Not all of the supplemental tests identified above may be evaluated in this fashion because for some tests, it is difficult to specify an alternative hypothesis that relates directly and in a meaningful way to a missing amount of material. Tests 1 and 2, the tests on $\hat{D}_k$ and on the shipper-receiver difference can be so evaluated, and are discussed in Section 4.5.3.

As the above listing suggests, and as will become more evident in Chapter 5, a large number of statistical tests might be performed in the course of evaluating inspection and material balance data for a given facility. With so many tests, it would be expected that a few would give positive signals due to chance alone, and one should not become unduly concerned when this occurs. The important emphasis should be placed on the principal tests, using either Combination of tests 1 or 2 identified above. For these principal tests, the false alarm rate, α, can be controlled as mentioned earlier. The functions of the supplemental tests are essentially twofold: to provide some degree of assurance that the assumptions underlying the application of the primary tests are valid; and to isolate causes of significant results returned by the primary tests.

4.5.2 Attributes Inspection Tests

Two attributes inspection tests are performed in each stratum, one using the attributes tester to check for gross defects, and one using the variables tester in the attributes mode to check for medium defects. For the K strata, there are a total of 2K attributes tests. There is no problem in performing each test since the existence of one or more defects in any stratum corresponds to detection, i.e., to rejection of the null hypothesis that no defects (gross on medium in size as the case may be) exist. Further, there is no concern with controlling the false alarm rate either since α is zero for each attributes test (clearly, if there are no defects in the population there is zero probability that any will be found in the sample.)

Consider the power of the attributes test taken as a whole, i.e., consider the probability that one or more of the tests will lead to rejection of the hypothesis through observing one or more defects. This is called the probability of detection. In calculating the probability of detection, it is simpler to find its complement, the probability of nondetection. Nondetection implies that no defects are found in any of the strata. To calculate the nondetection probability, define

h_k = amount of element falsified as gross defects in stratum k

g_k = amount of element falsified as medium defects in stratum k

Summed over all strata, the total amount falsified (i.e., diverted and obscured through data falsification) is

$$\sum_{k=1}^{K} (h_k + g_k) = a_2 M \qquad \text{(eq. 4.5.1)}$$

where $0 \leq a_2 \leq 1$

The sample sizes for attributes inspection using the attributes tester and the variables tester in the attributes mode, n_{ak} and n_{v1k} respectively, are given by (eq. 4.4.1) and (eq. 4.4.16). Let p_{ak} be the probability of not finding a defect in stratum k with the attributes tester, and let p_{vk} be similarly defined for the variables tester. Then, letting $\beta_k = \beta$ for all strata and for both testers, from (eq. 4.4.5),

$$\left.\begin{aligned} p_{ak} &= \beta^{h_k/M} \\ p_{vk} &= \beta^{g_k/M} \end{aligned}\right\} \qquad \text{(eq. 4.5.2)}$$

Since the tests are independent, over all strata, the probability of nondetection, denoted by Q_2 is

$$\begin{aligned} Q_2 &= \prod_k p_{ak} p_{vk} \\ &= \beta^{a_2} \end{aligned} \qquad \text{(eq. 4.5.3)}$$

where the last step is a consequence of (eq. 4.5.1).

This result indicates that no matter how the a_2M units of element are distributed among the strata and between gross and medium defects, the overall probability of nondetection is β^{a_2}, where the sample size in each stratum and for each tester is based on the same value for β.

It is important to note that if β is not the same for all strata-tester combinations, then the largest value for β must be used in (eq. 4.5.3). This is so because the optimum strategy of the diverter is to falsify the data in the strata with the largest β values. To illustrate the importance of this point, suppose that $a_2 = 0.6$ (60% of the goal amount M is diverted and obscured through gross and/or medium defects) and that $\beta = 0.05$ for all strata-tester combinations. Then, by (eq. 4.5.3), the non-detection probability is

$$Q_2 = (0.05)^{0.6} = 0.1657 \quad , \text{(detection probability = 0.8343)}.$$

Suppose, however, that constraints on the inspection require that $\beta = 0.50$, say, for the attributes tester in a given stratum. Then, assuming that the stratum in question is sufficiently large to permit diversion of the entire amount, a_2M, from this stratum, the nondetection probability is

$$Q_2 = (0.50)^{0.6} = 0.6598, \text{(detection probability = 0.3402)}.$$

This small detection probability occurs in spite of the fact that $\beta = 0.05$ for all other strata-tester combinations. The example illustrates the need to cover all strata with comparable intensity of inspection (i.e., to achieve the same value for β).

4.5.3 Tests on $\hat{D}_k$ and Shipper/Receiver Difference Tests

From a mathematical viewpoint, there is no distinction between a test on $\hat{D}_k$ (defined by (eq. 3.5.1)), and a shipper/receiver difference test. In both cases, measurements for one party are compared with measurements for the second party. Thus, it is permissible to discuss the $\hat{D}_k$ test and have this discussion apply to the shipper/receiver test as well.

Before giving the method for finding the probability that the $\hat{D}_k$ test will detect a given amount of material diverted from stratum k through small data falsifications, two points must be made. First, in Section 4.4.2.2, the concept of the variance inflation factor, C_0^2, was introduced. That is, the random error variance of $\hat{D}$ under the alternative hypothesis was assumed to be C_0^2 times the corresponding variance under the null hypothesis. A similar inflation factor would apply to $\hat{D}_k$, the $\hat{D}$ statistic for an individual stratum. For planning purposes, it is reasonable to assign a conservatively large value to C_0^2 in order to assure that the corresponding sample size is conservative on the high side. From an evaluation viewpoint, however, it is more reasonable that the inflation factor be a function of the amount diverted. An assumed empirical relationship between the variance inflation factor and the amount diverted under the alternative model is used in the method to follow.

As a second point, the facility value minus the inspector value is the basis for the $\hat{D}_k$ statistic. If the stratum in question is either an input or a beginning inventory stratum, then it is to the diverter's advantage to introduce falsifications that will result in a negative value for $\hat{D}_k$, i.e., the mean of $\hat{D}_k$ under the alternative would be a negative value. On the other hand, if the stratum in question is either an output or an ending inventory stratum, then a positive value of $\hat{D}_k$ would benefit the diverter. To avoid repetition, and to provide consistency with the development in Section 4.5.4 to follow, assume that the stratum in question is an output stratum so that the mean of $\hat{D}_k$ under the alternative is positive. (It would be a simple matter to apply Method 4.4 to an input or a beginning inventory stratum; simply change the sign of $\hat{D}_k$, or, equivalently, redefine $\hat{D}_k$ to be based on inspector minus facility values for such a stratum).

Method 4.4 is now given. This provides the equations needed to calculate the probability of nondetection and its complement, the probability of detection, for a specified alternative.

Method 4.4

Notation

The notation is consistent with that given in Methods 4.3, and 3.8-3.10. In addition,

C_1^2 = variance inflation factor under the alternative hypothesis of diversion through small data falsifications

a_1 = fractional amount diverted through small data falsifications in stratum k

a_1M = total amount of element thus diverted

Q_1 = probability of nondetection of the amount a_1M diverted through small data falsifications in stratum k

Model

The random variable, $\hat{D}_k$, is assumed to be normally distributed with variance $V(\hat{D}_k)|H_0$ under the null hypothesis of no diversion (i.e., when $E(\hat{D}_k) = 0$) and with variance $V(\hat{D}_k)|H_1$ under the alternative hypothesis, i.e., when $E(\hat{D}_k) = a_1M$.

Results

The quantities $V_r(\hat{D}_k)$, $V_g(\hat{D}_k)$, and $V_s(\hat{D}_k)$ are calculated using Methods 3.8, 3.9, and 3.10 respectively. In applying Methods 3.9 and 3.10, which relate to $\hat{D}$ rather than $\hat{D}_k$, include only the single stratum in question. Then

$$V(\hat{D}_k)|H_0 = V_r(\hat{D}_k)+V_g(\hat{D}_k)+V_s(\hat{D}_k) \qquad \text{(eq. 4.5.4)}$$

and

$$V(\hat{D}_k)|H_1 = C_1^2V_r(\hat{D}_k)+V_g(\hat{D}_k)+V_s(\hat{D}_k) \qquad \text{(eq. 4.5.5)}$$

where

$$C_1^2 = \min\left[4, 1+a_1M/\sqrt{V_g(\hat{D}_k)+V_s(\hat{D}_k)}\right] \qquad \text{(eq. 4.5.6)}$$

Next, compute

$$t_1 = \frac{t_\alpha\sqrt{V(\hat{D}_k)|H_0} - a_1M}{\sqrt{V(\hat{D}_k)|H_1}} \qquad \text{(eq. 4.5.7)}$$

Q_1 is then the area under the standardized nominal curve from $-\infty$ to t_1, i.e., Q_1 is defined by

$$Q_1 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t_1} e^{-z^2/2}dz \qquad \text{(eq. 4.5.8)}$$

Basis

The formula for the variance inflation factor, C_1^2, in (eq. 4.5.6) is an assumed relationship. It is felt that a more exact relationship could be derived under certain reasonable assumptions with regards to diversion strategies, but the (eq. 4.5.6) model is regarded as being an adequate one and a preferred one because of its simplicity (see Section 6.3 in this regard).

The null hypothesis, $H_0{:}E(\hat{D}_k) = 0$, is rejected if $\hat{D}_k$ is too large, i.e., if it exceeds some critical value D_{k0}, where D_{k0} is chosen such that it is exceeded with probability α when H_0 is true. Thus, D_{k0} is defined by

$$\text{Prob } (\hat{D}_k > D_{k0})|H_0 = \alpha \quad \text{, which reduces to}$$

$$D_{k0} = t_\alpha \sqrt{V(\hat{D}_k)|H_0}$$

Under the alternative, find the probability of nondetection:

$$Q_1 = \text{Prob}\left[\hat{D}_k < t_\alpha \sqrt{V(\hat{D}_k)|H_0}\right]|H_1 =$$

$$\text{Prob}\left(Z < \frac{t_\alpha \sqrt{V(\hat{D}_k)|H_0} - a_1 M}{\sqrt{V(\hat{D}_k)|H_1}}\right) \qquad \text{(eq. 4.5.9)}$$

where z is normally distributed with zero mean and unit standard deviation. This is the basis for (eq. 4.5.7).

Examples

EXAMPLE 4.4 (a)

Consider the input UO_2 powder stratum for the facility of example 3.8 (a) and find the probability of detecting 400 kg uranium if diverted in this stratum through small falsifications. Set $\alpha = 0.01$.

From example 3.8 (a),

$$V_r(\hat{D}_1) = 1143 + 1594 = 2737 \text{ kg}^2\text{U}$$

From example 3.9 (a),

$$V_g(\hat{D}_1) = (0.000544)^2(2)(120{,}000)^2$$

$$= 8{,}523 \text{ kg}^2\text{U}$$

From example 3.10 (a), and with reference to the earlier example 3.5 (a),

$$V_s(\hat{D}_1) = 11100.73 + 18779.96 + (240{,}000)^2(0.000439)^2$$
$$+ (240{,}000)^2(0.000172)^2$$
$$= 42{,}685 \text{ kg}^2\text{U}$$

By (eq. 4.5.4),

$$V(\hat{D}_1)|H_0 = 53{,}945 \text{ kg}^2\text{U}$$

By (eq. 4.5.6),

$$C_1^2 = \min\left(4,\ 1 + 400/\sqrt{51208}\right)$$

$= 3.7676$, so that, from (eq. 4.5.5),

$$V(\hat{D}_1)|H_1 = 61{,}520 \text{ kg}^2\text{U}$$

By (eq. 4.5.7),

$$t_1 = \frac{2.326\sqrt{53{,}945} - 400}{\sqrt{61{,}520}} = 0.5654$$

The nondetection probability, Q_1, is the area under the standardized normal curve from $-\infty$ to 0.5654, or

$$Q_1 = 0.7141$$

so the probability of detection is $1-Q_1 = 0.2859$.

4.5.4 Test on $\hat{D}$

The test on $\hat{D}$ is very similar to that on $\hat{D}_k$. As with $\hat{D}_k$, it is assumed that the variance inflation factor, C_1^2, is a function of the amount diverted through small data falsifications, i.e., of $E(\hat{D})$ under the alternative. As explained in Section 4.4.2.2, in order to work with positive values of the test statistic under the alternative, it is convenient to replace $\hat{D}$ as defined in (eq. 3.5.2) by its negative counterpart, $\hat{D}'$. In what follows, then, $\hat{D}'$ of (eq. 4.4.44) is the test statistic.

Method 4.5 is now given. This provides the equations needed to calculate the probability of nondetection and its complement, the probability of detection, for a specified alternative.

Method 4.5

Notation

The notation is consistent with that given in Method 4.3. The variance inflation factor, C_1^2, has the same interpretation as in Method 4.4. In addition,

a_3 = fractional amount diverted through small data falsification

a_3M = total amount of element thus diverted

Q_3 = probability of nondetection of the amount a_3M diverted through small data falsifications

Model

The discussion for Method 4.4 applies, with $\hat{D}_K$ replaced by $\hat{D}'$ and a_1 replaced by a_3. For the $\hat{D}'$ statistic, the emphasis is on the variance of $\hat{D}'$ as determined in the planning stage, i.e., by following Method 4.3 to determine θ and hence, the sample size. If the evaluation is rather to be performed with actual sample sizes used in implementation, and assuming that these may differ appreciably from those developed in the planning stages, then Method 4.4 may be applied with slight modification: replace $\hat{D}_K$ by $\hat{D}$, a_1 by a_3, t_1 by t_3, and Q_1 by Q_3.

Results

Following Method 4.3, calculate σ_S^2 from (eq. 4.4.18) and θ from (eq. 4.4.26), (eq. 4.4.30), or (eq. 4.4.31). Then, compute

$$t_3 = \frac{t_\alpha \sqrt{1+\theta} - a_3 m}{\sqrt{1 + C_1{}^2\theta}} \qquad \text{(eq. 4.5.10)}$$

where $m = M/\sigma_S$, and where

$$C_1{}^2 = \min\,(4,\ 1 + a_3 m) \qquad \text{(eq. 4.5.11)}$$

Q_3 is then the area under the standardized normal curve from $-\infty$ to t_3, i.e., Q_3 is defined by

$$Q_3 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t_3} e^{-z^2/2} dz \qquad \text{(eq. 4.5.12)}$$

Basis

The basis for Method 4.5 is developed in the Method 4.3 basis. The key equation, (eq. 4.5.10), follows directly from (eq. 4.4.49) with t_β replaced by t_3 and $C_0{}^2$ replaced by $C_1{}^2$.

Examples

EXAMPLE 4.5 (a)

Consider the low enriched uranium fuel fabrication facility of example 4.3 (a). Find the probability that the difference statistic, $\hat{D}$, will detect 400 kg U diverted, with the diversion obscured by small data falsifications. Use α = 0.05.

From the cited example,

$$\sigma_s = \sqrt{87{,}675} = 296 \text{ kg U}$$

$$\theta = (1.75)^{-1} = 0.571$$

$$a_3m = 400/296 = 1.351$$

Then, from (eq. 4.5.10), since $C_1^2 = 2.351$ from (eq. 4.5.11),

$$t_3 = \frac{1.645\sqrt{1.571} - 1.351}{\sqrt{2.342}} = 0.464$$

and

$$Q_3 = 0.679 \quad \text{, nondetection probability}$$

$$1\text{-}Q_3 = 0.321 \quad \text{, probability of detection}$$

EXAMPLE 4.5 (b)

Consider the mixed oxide fuel fabrication facility of example 4.3 (b). Using the Case 2 criteria in that example, find the probability that the $\hat{D}$ statistic will detect 5 kg Pu diverted, with the diversion obscured by small data falsifications. From the cited example,

$$\sigma_s = 4.919 \text{ kg Pu} \quad ; \theta = 0.0561 \quad ; a_3m = 5/4.919 = 1.016$$

From (eq. 4.5.11), $C_1^2 = 2.106$, so, applying (eq. 4.5.10),

$$t_3 = \frac{1.960\sqrt{1.0561} - 1.016}{\sqrt{1.1131}} = 0.946$$

$$Q_3 = 0.828 \quad \text{, nondetection probability}$$

$$1\text{-}Q_3 = 0.172 \quad \text{, probability of detection}$$

EXAMPLE 4.5 (c)

For the same mixed oxide fuel fabrication facility in the previous example, suppose the inspection plan is as indicated in example 3.8 (b), and rework the example 4.5 (b).

In applying Method 4.5 to solve this problem, the development of Method 4.4 is followed, with $\hat{D}_k$ replaced by $\hat{D}$, a_1 by a_3, t_1 by t_3, and Q_1 by Q_3. From example 3.8 (b),

$$V_r(\hat{D}) = 3.325266 \text{ kg}^2\text{Pu}$$

From example 3.9 (b),

$$V_g(\hat{D}) = 10.020730 \text{ kg}^2\text{Pu}$$

From example 3.10 (b),

$$V_s(\hat{D}) = 13.957723 \text{ kg}^2\text{Pu}$$

By (eq. 4.5.4), appropriately modified,

$$V(\hat{D}')|H_0 = 27.303719 \text{ kg}^2\text{Pu}$$

By (eq. 4.5.6), appropriately modified,

$$C_I^2 = \min(4,\ 1 + 5/\sqrt{23.978453})$$
$$= 2.021$$

so that, from the modified (eq. 4.5.5),

$$V(\hat{D}')|H_1 = 30.698816$$

Then,

$$t_3 = \frac{1.960\ \sqrt{27.303719} - 5}{\sqrt{30.698816}} = 0.946$$

$$Q_3 = 0.828$$

$$1\text{-}Q_3 = 0.172$$

Note that the answers are precisely the same (to three decimals) as in the previous example. In the previous example, the total sample size was 427; in example 3.8 (b), the number of samples was 287 and the number of weighings was 653. The fact that the detection probabilities agree to within 3 decimals is a bit fortuitous, but close agreement would be expected because of the dominance of systematic errors in this example.

4.5.5 Test on MUF

Although the inspector has no direct control over the size of the variance of MUF, this quantity has an impact on the ability to detect diversion, especially as it affects the variance of (MUF-$\hat{D}$) as considered in the next section. The test on MUF is considered in Method 4.6.

Method 4.6

Notation

The notation is given in section 3.4.3.2. In addition,

a_4 = fractional amount of element diverted into MUF

a_4M = total amount of element thus diverted

Q_4 = probability of nondetection of the amount a_4M diverted into MUF

Model

The quantity MUF is assumed to be a random variable which is normally distributed with variance V(MUF). Under the null hypothesis, the mean of MUF is zero, while it is a_4M under the alternative.

Results

Compute $t_4 = t_\alpha - a_4M/\sqrt{V(MUF)}$ (eq. 4.5.13)

Q_4 is the area under the standardized normal curve from $-\infty$ to t_4, i.e., Q_4 is defined by

$$Q_4 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t_4} e^{-z^2/2} dz \qquad \text{(eq. 4.5.14)}$$

Basis

The null hypothesis that E(MUF) = 0 is rejected if MUF exceeds some critical value M_0, where M_0 is chosen such that it is exceeded with probability α when H_0 is true. Thus, M_0 is defined by

$\text{Prob}\ (MUF > M_0)|(E(MUF) = 0) = \alpha$, which reduces to

$$M_0 = t_\alpha \sqrt{V(MUF)}$$

Under the alternative, $E(MUF) = a_4M$. Find the probability of nondetection:

$$Q_4 = \text{Prob}\ (MUF < t_\alpha \sqrt{V(MUF)}|(E(MUF) = a_4M)$$

$$= \text{Prob}\ (z < t_\alpha - a_4M/\sqrt{V(MUF)}) \qquad \text{(eq. 4.5.15)}$$

where z is normally distributed with zero mean and unit standard deviation. This is the basis for (eq. 4.5.15).

Examples

EXAMPLE 4.6 (a)

Consider the mixed oxide fuel fabrication facility of examples 3.3 (b), 3.4 (a), and 3.5 (b), find the probability that the MUF test would detect a diverted amount of 3 kg Pu if α = 0.05.

From example 3.5 (b),

$$V(MUF) = 10.729310 \text{ kg}^2\text{Pu}$$

Therefore, by (eq. 4.5.13),

$$t_4 = 1.645 - 3/\sqrt{10.729310}$$

$$= 0.729$$

$$Q_4 = 0.767 \quad \text{, nondetection probability}$$

$$1\text{-}Q_4 = 0.233 \quad \text{, probability of detection}$$

4.5.6 Test on (MUF-$\hat{D}$)

Consider the (MUF-$\hat{D}$) statistic introduced in Section 3.6 and discussed further in Sections 4.3.2.3 and 4.4.2.2. The detection probability employing (MUF-$\hat{D}$) as the test statistic is covered by Method 4.7. This statistic is responsive to the combination of diversions employing two strategies: diversion into MUF and diversion obscured by small data falsifications.

Method 4.7

Notation

The notation is given in Method 3.13. In addition,

Q_5 = probability of nondetection of the amount $(a_3+a_4)M$ diverted into MUF and small data falsifications (a_3 and a_4 are defined in Methods 4.5 and 4.6 respectively)

Model

The random variable, (MUF-$\hat{D}$), is assumed to be normally distributed with mean zero and variance $V(MUF-\hat{D})|H_0$ under the null hypothesis, and with mean $(a_3+a_4)M$ and variance $V(MUF-\hat{D})|H_1$ under the alternative.

Two approaches to finding the detection probability for the (MUF-$\hat{D}$) statistic are used. First, it is assumed that the inspection sample sizes are those developed in the planning stage, i.e., following Method 4.3. In the second approach, actual sample sizes used in the inspection are utilized, the assumption being that they may differ appreciably from the planned sample sizes.

Results

First approach--

Following Method 4.3, a number of quantities must be calculated. These include:

σ_s^2 using (eq. 4.4.18)

V(MUF) and V_0 using Methods 3.3, 3.4, and 3.5

k_2 using (eq. 4.4.23)

$a = 1-k_2$

θ using (eq. 4.4.26), (eq. 4.4.30), or (eq. 4.4.31)

Then, compute

$$t_5 = \frac{t_\alpha \sqrt{a+\theta} - (a_3 + a_4)m}{\sqrt{a+C_1{}^2\theta}} \qquad \text{(eq. 4.5.16)}$$

where $m = M/\sigma_s$

and where $C_1{}^2 = \min(4,\ 1+a_3m)$, as given by (eq. 4.5.11).

Q_5 is the area under the standardized normal curve from $-\infty$ to t_5, i.e., Q_5 is defined by

$$Q_5 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t_5} e^{-z^2/2} dz \qquad \text{(eq. 4.5.17)}$$

Second approach--

Compute the quantities $V_r(\hat{D})$, $V_g(\hat{D})$, and $V_s(\hat{D})$ using Methods 3.8, 3.9, and 3.10 respectively. Compute $V(\hat{D})|H_0$ and $V(\hat{D})|H_1$ using (eq. 4.5.4) and (eq. 4.5.5) with $\hat{D}_k$ replaced by $\hat{D}$, and with $C_1{}^2$ defined by (eq. 4.5.6), again with $\hat{D}_k$ replaced by $\hat{D}$. The quantities $V(\text{MUF}-\hat{D})|H_0$ and $V(\text{MUF}-\hat{D})|H_1$ are then calculated using (eq. 3.6.5), first with $V(\hat{D})|H_0$ in place of $V(\hat{D})$, and then using $V(\hat{D})|H_1$. The quantity t_5 is then computed from the formula:

$$t_5 = \frac{t_\alpha \sqrt{V(\text{MUF}-\hat{D})|H_0} - (a_3+a_4)M}{\sqrt{V(\text{MUF}-\hat{D})|H_1}} \qquad \text{(eq. 4.5.18)}$$

Basis

The basis for the first approach is developed in the Method 4.3 basis. The key equation, (eq. 4.5.16), follows directly from (eq. 4.4.52) with t_β replaced by t_5 and $C_0{}^2$ replaced by $C_1{}^2$.

For the second approach, refer to the Basis for Method 4.4, replacing $\hat{D}_K$ by (MUF-$\hat{D}$) and a_1 by (a_3+a_4).

Examples

EXAMPLE 4.7 (a)

Consider the low enriched uranium fuel fabrication facility of example 4.3 (a), and treated again in example 4.5 (a). Find the probability that the (MUF-$\hat{D}$) test will detect a diversion of 400 kg U obscured by small data falsifications and one of 500 kg U diverted into MUF. Use $\alpha = 0.05$.

The first approach is used since the test is to be evaluated based on planned sample sizes. From example 4.5 (a), values were given for σ_s^2 and θ. Also, from example 3.4 (a), which pertains to this same facility,

$$V(MUF) = 45{,}010 \text{ kg}^2\text{U}$$

$$V_0 = 0$$

Then, from (eq. 4.4.23),

$$k_2 = 45{,}010/87{,}675 = 0.513$$

and

$$a = 0.487$$

On applying (eq. 4.5.16),

$$t_5 = \frac{1.645\sqrt{0.487 + 0.571} - 900/296}{\sqrt{0.487 + (2.351)(0.571)}} = -0.997$$

$$Q_5 = 0.159 \quad \text{, nondetection probability}$$

$$1\text{-}Q_5 = 0.841 \quad \text{, probability of detection}$$

EXAMPLE 4.7 (b)

In the example just concluded, suppose again that 900 kg U total were diverted through the same two diversion strategies, but that the split were different, with 100 kg U obscured by small data falsifications and 800 kg U diverted into MUF. All the calculations in the preceding example remain unchanged except that the value for C_1^2 in the denominator of the expression for t_5 differs. In the preceding example, C_1^2 was 2.351 since a_3m was 1.351 for a_3M = 400 kg. If a_3M = 100 kg U, then a_3m is 100/296 = 0.338 and C_1^2 is 1.338. Thus,

$$t_5 = \frac{1.645\sqrt{1.058} - 900/296}{\sqrt{0.487+(1.338)(0.571)}} = -1.206$$

$Q_5 = 0.114$, nondetection probability

$1-Q_5 = 0.886$, probability of detection

Since the diverter would like to maximize Q_5, or minimize $1-Q_5$, he would prefer the diversion strategy of example 4.7 (a) to that of 4.7 (b). This concept of best diversion strategies is fully treated in Section 4.6.

EXAMPLE 4.7 (c)

Consider the mixed oxide fuel fabrication facility of example 4.5 (c). Find the probability that the (MUF-$\hat{D}$) test will detect a total diversion of 6.5 kg Pu if 5 kg Pu is obscured by small data falsifications and 1.5 kg Pu is diverted into MUF. Use $\alpha = 0.025$.

The second approach of Method 4.7 is used since the test is to be evaluated based on actual sample sizes, rather than on planned sample sizes.

From the cited example, values were given for $V(\hat{D})|H_0$ and $V(\hat{D})|H_1$.

$$V(\hat{D})|H_0 = 27.303719 \text{ kg}^2\text{Pu}$$

$$V(\hat{D})|H_1 = 30.698816 \text{ kg}^2\text{Pu}$$

This value for $V(\hat{D})|H_1$ is the appropriate one to use since, in the cited example, $a_3M = 5$ kg Pu as in this example. Were the amount of diversion through small data falsifications different from 5 kg Pu, then $V(\hat{D})|H_1$ would have to be recalculated since it is a function of C_1^2, and hence, of a_3M.

To continue, values for V(MUF) and V_0 for this facility were given in example 3.13 (b).

$$V(\text{MUF}) = 10.729310 \text{ kg}^2\text{Pu}$$

$$V_0 = 2.230517 \text{ kg}^2\text{Pu}$$

Thus,

$$V(\text{MUF}-\hat{D})|H_0 = 27.303719 - 10.729310 + 2(2.230517)$$

$$= 21.035443 \text{ kg}^2\text{Pu}$$

$$V(\text{MUF}-\hat{D})|H_1 = 30.698816 - 10.729310 + 2(2.230517)$$

$$= 24.430540 \text{ kg}^2\text{Pu}$$

From (eq. 4.5.18),

$$t_5 = \frac{1.960\sqrt{21.035443} - 6.5}{\sqrt{24.430540}} = 0.504$$

$Q_5 = 0.693$, nondetection probability

$1-Q_5 = 0.307$, probability of detection

4.6 OVERALL PROBABILITY OF DETECTION OF GOAL AMOUNT

As was mentioned in Section 4.5.1, in evaluating his inspection plan, the inspector is primarily interested in determining how his plan, together with measurements performed by the facility, will react to diversion strategies used by the diverter. Although he may perform the evaluations presented in Sections 4.5.2 - 4.5.6 in this connection, either before the inspection or after the fact, this leaves unanswered the question as to how he can combine the resulting information to arrive at an overall assessment of the inspection plan. This section will address this question.

Two combinations of principal statistical tests are considered. First, in Section 4.6.1, test Combination 1 identified in Section 4.5.1 is considered. This includes the combination of attributes tests and of the test using (MUF-$\hat{D}$). In Section 4.6.2, test Combination 2 that employs the attributes tests, the test on $\hat{D}$, and the test on MUF, is discussed.

In evaluating the test combinations in each instance, the measure used is the overall probability of detecting the goal amount diverted. Detection means that a positive signal, or significant result, is returned with at least one of the tests. In application, it is simpler to calculate the probability of nondetection, the detection probability then being the complement of this quantity.

The detection probability is quite obviously a function of the diversion strategy, i.e., of how the goal amount is apportioned among the various diversion possibilities. Again responding to the assumption that the plan is to combat an intelligent adversary bent on diversion, the inspector is chiefly interested in calculating the probability of nondetection corresponding to the best adversary strategy. For test Combination 1, this probability is designated by Q_{max} while for test Combination 2, Q'_{max} denotes this probability. Clearly, the inspector will prefer to apply the combination of tests that yields the smallest value for the nondetection probability, i.e., the largest probability of detection. In making this decision, of course, he recognizes that should the adversary not employ his best strategy, the inspector may be better off using the other test combination; however, his probability of nondetection can be no larger than Q_{max} in the case of test Combination 1 or Q'_{max} in the case of Combination 2.

4.6.1 Attributes and (MUF-$\hat{D}$) Tests

In this section, the probability of nondetection is calculated for test Combination 1 as identified in Section 4.5.1. The discussions in Sections 4.5.2 and 4.5.6 form the basis for Method 4.8 to follow.

Method 4.8

Notation

See the notation in Section 4.5.2 and for Method 4.7 in Section 4.5.6. In addition,

Q = probability of nondetection for test Combination 1

Q_{max} = maximum value for Q; that value of Q corresponding to the best strategy of the diverter

Model

See the discussion in Section 4.5.2 and the model discussion for Method 4.7.

Results

The joint probability of nondetection is

$$Q = \beta^{a_2} Q_5 \qquad \text{(eq. 4.6.1)}$$

Q_5 is a function of a_3 and a_4. The quantities a_2, a_3, and a_4 sum to 1. Q_5 is given by (eq. 4.5.17) and (eq. 4.5.16) or (eq. 4.5.18) for given a_3 and a_4.

Use a trial and error approach to find Q_{max}. Vary a_2 in increments from 0 to 1 (steps of 0.1 are adequate). At a given value of a_2, calculate Q_5, where (a_3+a_4) is equal to $(1-a_2)$.

In calculating C_1^2, use the following rule to select the value for a_3:

$$\left.\begin{array}{l}\text{If } t_5 \text{ is negative, choose } a_3 = 1-a_2 \\ \text{If } t_5 \text{ is positive, choose } a_3 = 0\end{array}\right\} \qquad \text{(rule 4.6.1)}$$

Q_{max} is easily determined, either from the table of Q versus a_2 values, with a_2 rounded to the nearest tenth, or from a plot of the tabled data, which permits choosing a value of a_2 between tabled values.

Basis

Equation 4.6.1 follows from the fact that the tests on the attributes and the (MUF-$\hat{D}$) test are independent. Rule 4.6.1 applies because that value of a_3 should be chosen to give the largest value to Q_5. If t_5 is negative, Q_5 is maximized when t_5 is as close to zero as possible, i.e., when the denominator of t_5 is made as large as possible. This occurs when $a_3=1-a_2$. If t_5 is positive, Q_5 is maximized when t_5 is made as large as possible, i.e., when the denominator is made as small as possible. This occurs when $a_3=0$.

Examples

EXAMPLE 4.8 (a)

In following up on the low enriched uranium fuel fabrication facility of example 4.7 (a), set M = 1500 kg U and find the value of Q_{max}. Assume that β = 0.05 for the attributes tests and that α = 0.05 for the (MUF-$\hat{D}$) test.

From example 4.7 (a),

$$t_5 = \frac{1.6920 - 1500(a_3+a_4)/296}{\sqrt{0.487+0.571C_1^2}}$$

$$= \frac{1.6920 - 5.0676(a_3+a_4)}{\sqrt{0.487+0.571C_1^2}}$$

The table below gives the value for Q as a function of a_2. The column headed "NUM" is the sign of the numerator in the expression for t_5, which dictates the value for a_3.

a_2	β^{a_2}	NUM	a_3	C_1^2	t_5	Q_5	Q
0	1.0000	NEG	1	4	-2.028	0.0213	0.0213
0.1	0.7411	NEG	0.9	4	-1.723	0.0424	0.0314
0.2	0.5493	NEG	0.8	4	-1.419	0.0779	0.0428
0.3	0.4071	NEG	0.7	4	-1.115	0.1324	0.0539
0.4	0.3017	NEG	0.6	4	-0.810	0.2090	0.0631
0.5	0.2236	NEG	0.5	3.53	-0.532	0.2974	0.0665
0.6	0.1657	NEG	0.4	3.03	-0.225	0.4110	0.0681
0.7	0.1228	POS	0	1	0.167	0.5663	0.0695
0.8	0.0910	POS	0	1	0.660	0.7454	0.0678
0.9	0.0675	POS	0	1	1.152	0.8753	0.0591
1.0	0.0500	POS	0	1	1.645	0.9500	0.0475

For a_2 rounded to the nearest tenth, Q_{max} is 0.0695. Thus, the maximum probability of nondetection, 0.0695, occurs at $a_2 = 0.7$, $a_3 = 0$, $a_4 = 0.3$, or when 1050 kg U is diverted into large and medium data falsifications, and 450 kg U is diverted into MUF.

EXAMPLE 4.8 (b)

In following up on the mixed oxide fuel fabrication facility of example 4.7 (c), set M = 8 kg Pu and find the value of Q_{max}. Assume that $\beta = 0.20$ for the attributes tests and that $\alpha = 0.025$ for the (MUF-D) test. (Note: This large value for β is reasonable because of the inability of the variables measurements to detect the goal amount of M = 8 kg Pu. The value for Q_{max} will be independent of the β value for the attributes testing up to fairly large values of β.)

For example 4.7 (c) and the earlier example 4.5 (c),

$$t_5 = \frac{1.960\sqrt{21.035443} - 8(a_3+a_4)}{\sqrt{23.978453 - 10.729310 + 2(2.230517) + 3.325266C_1^2}}$$

$$= \frac{8.9894 - 8(a_3+a_4)}{\sqrt{17.7102 + 3.325266C_1^2}}$$

The table below gives Q values as a function of a .

a_2	β^{a_2}	NUM	a_3	C_1^2	t_5	Q_5	Q
0	1.0000	POS	0	1	0.216	0.5855	0.5855
0.1	0.8513	POS	0	1	0.390	0.6517	0.5548
0.2	0.7248	POS	0	1	0.565	0.7140	0.5175
0.3	0.6170	POS	0	1	0.730	0.7673	0.4734
0.4	0.5253	POS	0	1	0.913	0.8194	0.4304
0.5	0.4472	POS	0	1	1.088	0.8617	0.3854
0.6	0.3807	POS	0	1	1.262	0.8965	0.3413
0.7	0.3241	POS	0	1	1.437	0.9246	0.2997
0.8	0.2759	POS	0	1	1.611	0.9464	0.2611
0.9	0.2349	POS	0	1	1.786	0.9630	0.2262
1.0	0.200	POS	0	1	1.960	0.9750	0.1950

Q_{max} is 0.5855, occurring at $a_4 = 0$, i.e., when all 8 kg Pu is diverted into MUF(a_4=1). Clearly, the adversary would be unwise were he to divert into large and medium data falsifications in this example since the inspector can control his β error in the attributes inspection. The ability of the inspector to detect the goal amount is limited by systematic errors of measurement that affect the $\hat{D}$ and MUF statistics, and hence, the (MUF-$\hat{D}$) statistic.

Note that at a_2 = 0.1, Q would be larger than the value at a_2 = 0 if

$$0.6517\,\beta^{0.1} > 0.5855$$

This inequality holds if $\beta > 0.343$, which means that the β value for attributes inspection could be increased to 0.343, with the attendent reduced inspection sample sizes, before the intelligent adversary would choose to divert even a portion of the goal amount into large and/or medium data falsifications. Stated from a different perspective, Q_{max} will be 0.5855 for all values of the attributes inspection, β, less than 0.343.

4.6.2 Attributes, $\hat{D}$, and MUF Tests

In this section, the probability of nondetection is calculated for the test Combination 2 as identified in Section 4.5.1. With reference to Sections 4.5.2, 4.5.4, and 4.5.5, the probability of nondetection for the attributes, $\hat{D}$, and MUF tests is not $\beta^{a_2}Q_3Q_4$ as might be expected because $\hat{D}$ and MUF are correlated. This correlation must be taken into account when performing the calculations. This is not simply an academic point; failure to take into account the correlation may lend to totally erroneous conclusions.

The procedures for calculating the probability of nondetection using test Combination 2 are now given by Method 4.9.

Method 4.9

See the notation in Section 4.5.2 and for Methods 4.5, 4.6, and 4.7. In addition,

Q' = probability of nondetection for test Combination 2

Q'_{max} = maximum value for Q'; that value of Q' corresponding to the best strategy of the diverter

ρ = correlation coefficient between $\hat{D}'$ and MUF

Model

See the discussion in Section 4.5.2 and the model discussions for Methods 4.5 and 4.6. Further, it is assumed that $\hat{D}'$ and MUF are jointly distributed as the bivariate normal with correlation coefficient ρ.

Results

The probability of nondetection is

$$Q' = \beta^{a2} L(t_3, t_4, \rho) \qquad \text{(eq. 4.6.2)}$$

where

$$\rho = \frac{-(k_2 + k_3)}{\sqrt{(1+C_1{}^2\theta)(k_2+2k_3)}} \qquad \text{(eq. 4.6.3)}$$

where $k_3 = V_0/\sigma_s^2$ (eq. 4.6.4)

and where k_2 is given by (eq. 4.4.23)

The expression for ρ given by (eq. 4.6.3) applies if the evaluation is being performed based on sample sizes developed during the planning stage. If implementation sample sizes are used, then the more general expression for ρ applies.

$$\rho = \frac{-V(MUF) + V_0}{\sqrt{[V(\hat{D})|H_1][V(MUF)]}} \qquad \text{(eq. 4.6.5)}$$

To continue with (eq. 4.6.2), $L(t_3,t_4,\rho)$ is the probability that two random variables, distributed as a bivariate normal distribution, are jointly less than t_3 and t_4 respectively, where ρ is the correlation coefficient. This probability may be found by table look-up or by computer calculations. Both approaches are considered.

Before proceeding further, it is necessary to remark on the choices for t_α used in the equations for t_3, (eq. 4.5.10) and for t_4, (eq. 4.5.13). If one were to assume independence between MUF and $\hat{D}$, and if α were the overall significance level, then α_3 and α_4 would satisfy the relationship

$$1-\alpha = (1-\alpha_3)(1-\alpha_4) \qquad \text{(eq. 4.6.6)}$$

Further, if $\alpha_3 = \alpha_4, = \alpha_0$, then α_0 is the solution of the equation

$$(1-\alpha_0)^2 = 1-\alpha \quad ,$$

the solution being

$$\alpha_0 = 1 - \sqrt{1-\alpha} \qquad \text{(eq. 4.6.7)}$$

Thus, for α = 0.05, α_0 would be 0.0253 and $t_3 = t_4$ = 1.955.

However, the assumptions leading to the result (eq. 4.6.7) are not valid; MUF and $\hat{D}$ are correlated because both statistics are calculated from operator data. Further, there is no real basis for equating α_3 to α_4. The problem of selecting values for α_3 and α_4 in this situation was considered by Avenhaus and Beedgen [4.10]. In a minimax sense, optimal values for α_3 and α_4 were calculated for the special but common case in which α = 0.05, and as a function of ρ. Minimax means that the inspector chooses values for α_3 and α_4 that minimize the probability of nondetection given that the diverter had first chosen the strategy that maximizes this probability. The results are given as Figure 1 in [4.10] and reproduced here as Annex 4.4. (In the Annex 4.4 plot, the notation of [4.10] is altered to correspond to the present notation.) Thus, one may use the Annex 4.4 figure to select the values for α_3 and α_4. Graphical interpolation is adequate for purposes of evaluating inspection plans.

For values of α other than 0.05, the Annex 4.4 plot may also be used as a good approximation simply by a change of scale. Thus, for α = 0.025, simply divide α_3 and α_4 in Annex 4.4 by 2; for α = 0.01, divide by 5; etc. This is a good approximation because it is shown in [4.11] that the shapes of the curves in Annex 4.4 for α = 0.01 and for α = 0.05 are very nearly identical.

Approach 1: Table look-up

Approach 1 is based on reference [4.12]. The probability $L(t_3,t_4,\rho)$ is given by

$$L(t_3,t_4,\rho) = 0.5\ (Q_3 + Q_4) - T(t_3,\Delta_3) - T(t_4,\Delta_4) - H \qquad \text{(eq. 4.6.8)}$$

where

$$\Delta_3 = \frac{t_4 - \rho t_3}{t_3\sqrt{1-\rho^2}} \qquad \text{(eq. 4.6.9)}$$

$$\Delta_4 = \frac{t_3 - \rho t_4}{t_4\sqrt{1-\rho^2}} \qquad \text{(eq. 4.6.10)}$$

and

$$H = \left.\begin{array}{l} 0 \text{ if } t_3 t_4 > 0 \text{ or if} \\ t_3 t_4 = 0 \text{ and } (t_3+t_4) \geq 0 \\ = 0.5 \text{ otherwise} \end{array}\right\} \qquad \text{(eq. 4.6.11)}$$

The function $T(t_j,\Delta_j)$ for j = 3,4 is tabled as Annex 4.2 (a)-(d) for coarse groupings on Δ_j and fine groupings on t_j, and as Annex 4.3 (a)-(d) for fine groupings on Δ_j and coarse groupings on t_j. In the tables, make the identification $h = t_j$ and $a = \Delta_j$. Ordinary linear interpolation should be satisfactory in most applications; if refined calculations are to be made, detailed interpolation procedures are given in [4.12].

The T-function is tabulated only for $0 \le \Delta_j \le 1$ and ∞; for values of Δ_j between 1 and ∞, use the following equation

$$T(t_j,\Delta_j) = 0.5[Q(t_j)+Q(t_j\Delta_j)]-Q(t_j)Q(t_j\Delta_j)-T(t_j\Delta_j,1/\Delta_j) \qquad \text{(eq. 4.6.12)}$$

where Q(t) is the area under the standardized normal curve from $-\infty$ to t. To account for negative values of t_j or Δ_j, use

$$T(t_j,-\Delta_j) = -T(t_j,\Delta_j), \text{ and} \qquad \text{(eq. 4.6.13)}$$

$$T(-t_j,\Delta_j) = T(t_j,\Delta_j) \qquad \text{(eq. 4.6.14)}$$

Approach 2: Computer Calculations

Evaluation of $L(t_3,t_4,\rho)$ by table look-up involves a fair amount of effort. It is far simpler to use an existing computer subroutine that computes $L(t_3,t_4,\rho)$ directly, and with greater precision. Subroutine MDBWOR from the IMSL library of programs (International Mathematical and Statistical Libraries), is one example of a computer subroutine that is easily applied.

Having calculated Q', trial and error calculations in which a_2 and a_3 (and hence a_4) are varied must be performed to find Q'_{max}. This exercise is not as straightforward as for determing Q_{max} because in that instance, the optimum value of a_3 (from a diverter's viewpoint) was known for given a_2; this is not true for determining Q'_{max}. Thus, two parameters must be varied, and not just the one.

Basis

For approach 1, the basis for (eq. 4.6.6) - (eq. 4.6.12) is given in pages 184-186 of [4.12].

Examples

EXAMPLE 4.9 (a)

For example 4.8 (a), Q_{max} was found to be 0.0695 occurring at $a_2 = 0.7$, $a_3 = 0$, and $a_4 = 0.3$. Find Q' for these values of a_2, a_3, and a_4. In order for the comparison to be more valid from a false alarm viewpoint, initially set

$$\alpha = 1- \sqrt{0.95} = 0.0253$$

for the $\hat{D}$ and MUF tests, since α was 0.05 for the (MUF-$\hat{D}$) test.

From example 4.3 (a),

$$\theta = (1.75)^{-1} = 0.5714$$

Also, $\quad m = 1500/296 = 5.0676$

and $\quad t_\alpha = 1.955$

so that from (eq. 4.5.10),

$$t_3 = \frac{1.955\ \sqrt{1.5714}\ \ -5.0676\ a_3}{\sqrt{1+0.5714\ C_1^2}}$$

From (eq. 4.5.13) and the data of example 4.7 (a),

$$t_4 = 1.955 - 1500\ a_4/\ \sqrt{45{,}010}$$

$$= 1.955 - 7.0703\ a_4$$

From (eq. 4.4.23), (eq. 4.6.3), and (eq. 4.6.4), and the data of example 4.7 (a),

$$k_2 = 45{,}010/87{,}675 = 0.513\quad ;\ k_3 = 0$$

$$\rho = \frac{-0.513}{\sqrt{(1+0.5714C_1^2)(0.513)}} = \frac{-0.716}{\sqrt{(1+0.5714C_1^2)}}$$

In the expressions for t_3 and ρ,

$$C_1^2 = \min\ (4,\ 1+5.0676\ a_3)$$

First, use approach 1, (table look-up), to compute Q´ for $a_2 = 0.7$, $a_3 = 0$, and $a_4 = 0.3$.

$$t_3 = 1.955$$

$$t_4 = -0.166$$

$$\rho = -0.571$$

From (eq. 4.5.12) and (eq. 4.5.14),

$$Q_3 = 0.9747\quad ,\quad Q_4 = 0.4341$$

From (eq. 4.6.9) and (eq. 4.6.10),

$$\Delta_3 = 0.592\quad ,\quad \Delta_4 = -13.65$$

From Annex 4.3 (b) and 4.3 (c),

$$T(1.955,0.592) = 0.0106$$

To evaluate T(-0.166,-13.65), use (eq. 4.6.13) and (eq. 4.6.14) to eliminate the negative values.

$$T(-0.166,\ -13.65) = T(0.166,-13.65) = -T(0.166,13.65)$$

Then use (eq. 4.6.12) since $\Delta_4 > 1$.

$$T(0.166, 13.65) = 0.5\ (0.5659 + 0.9883) - (0.5659)(0.9883) - 0.0009$$
$$= 0.2169$$

Finally, H of (eq. 4.6.8) is 0.5 so that, from (eq. 4.6.8)

$$L(1.955, -0.166, -0.571) = 0.5\ (0.9747 + 0.4341) - 0.0106 + 0.2169 - 0.5$$
$$= 0.4107$$

Then, from (eq. 4.6.2),

$$Q' = (0.05)^{0.7}(0.4107) = 0.0504$$

Using the MDBNOR subroutine, $L(t_3,t_4,\rho)$ is calculated to be 0.4109, in good agreement with the table look-up value.

Note that at this combination of values of a_2, a_3, and a_4, Q' is smaller than Q; this does not mean, however, that Q'_{max} is smaller than Q_{max} since Q and Q' may not reach their respective maximum values at the same combination of values for a_2, a_3, and a_4. Further calculations are needed to determine Q' max.

Before proceeding with these additional calculations, the optimal choices for α_3 and α_4 are used in place of $\alpha_3 = \alpha_4 = 0.0253$, where these optimal choices are given by Annex 4.4. In using Annex 4.4, the value of ρ is, of course, that calculated under the null hypothesis, i.e., corresponding to $C_1{}^2 = 1$. This is

$$\rho = -0.716/\sqrt{1.5714} = -0.571$$

Thus, from Annex 4.4, by graphical interpolation, chose

$$\alpha_3 = 0.035 \qquad \alpha_4 = 0.021$$
$$t_{\alpha_3} = 1.811 \qquad t_{\alpha 4} = 2.034$$

The example calculation is now performed with these values used instead of 1.955 in the expressions for t_3 and t_4. At $a_2 = 0.7$, $a_3 = 0$, and $a_4 = 0.3$,

$$t_3 = 1.811 \qquad t_4 = -0.087 \qquad \rho = -0.571$$

From (eq. 4.5.12) and (eq. 4.5.14),

$$Q_3 = 0.965 \qquad Q_4 = 0.465$$

From (eq. 4.6.9) and (eq. 4.6.10),

$$\Delta_3 = 0.637 \qquad \Delta_4 = -24.66$$

From Annex 4.3 (c),

$$T(1.811, 0.637) = 0.0148$$

To evaluate $T(-0.087, -24.66)$, use (eq. 4.6.13) and (eq. 4.6.14) to eliminate the negative values.

$T(-0.087, -24.66) = T(0.087, -24.66) = -T(0.087, 24.66)$

Then, since $\Delta_4 > 1$, use (eq. 4.6.12),

$$T(0.087, 24.66) = 0.5\ (0.535 + 0.984) - (0.535)(0.984) - 0.0007$$
$$= 0.2324$$

Finally, since H of (eq. 4.6.8) is 0.5, the result is

$$L(1.811, -0.087, -0.571) = 0.5\ (0.965 + 0.465) - 0.0148 + 0.2324 - 0.5$$
$$= 0.4327$$

This compares with the value of 0.4107 for the case $\alpha_3 = \alpha_4 = 0.0253$. The Q' value is $(0.05)^{0.7}(0.4327) = 0.0531$, compared with the earlier value, 0.0504.

The further calculations are now made to investigate the relationship between Q' and diversion strategy, and also determine an approximate value for Q'_{max}. The table gives values of Q' for various sets of values of a_2, a_3, and a_4. The $L(t_3,t_4,\rho)$ values are calculated using the MDBNOR subroutine. Two sets of values for $L(t_3,t_4,\rho)$ and Q' are given. Those with subscript 1 are based on $\alpha_3 = \alpha_4 = 0.0253$ while for $\alpha_3 = 0.035$ and $\alpha_4 = 0.021$, the subscript 2 values are calculated.

a_2	a_3	a_4	$C_1{}^2$	β^{a2}	$L_1(t_3,t_4,\rho)$	$L_2(t_3,t_4,\rho)$	Q_1'	Q_2'
0	0	1	1	1	0.0000	0.0000	0.0000	0.0000
0	0.25	0.75	2.267	1	0.0001	0.0001	0.0001	0.0001
0	0.50	0.50	3.534	1	0.0098	0.0101	0.0098	0.0101
0	0.75	0.25	4	1	0.0822	0.0757	0.0822	0.0757
0	1	0	4	1	0.0671	0.0559	0.0671	0.0559
0.25	0	0.75	1	0.4729	0.0002	0.0002	0.0001	0.0001
0.25	0.25	0.50	2.267	0.4729	0.0243	0.0259	0.0115	0.0122
0.25	0.50	0.25	3.534	0.4729	0.2097	0.2010	0.0992	0.0951
0.25	0.75	0	4	0.4729	0.2136	0.1879	0.1010	0.0889
0.50	0	0.50	1	0.2236	0.0474	0.0531	0.0106	0.0119
0.50	0.25	0.25	2.267	0.2236	0.3963	0.3946	0.0886	0.0882
0.50	0.50	0	3.534	0.2236	0.4596	0.4224	0.1028	0.0944
0.75	0	0.25	1	0.1057	0.5498	0.5710	0.0581	0.0604
0.75	0.25	0	2.267	0.1057	0.7577	0.7255	0.0801	0.0767
1	0	0	1	0.0500	0.9494	0.9440	0.0475	0.0472
0.333	0.333	0.333	2.688	0.3684	0.1718	0.1733	0.0633	0.0638

Note from this table that the Q_1' and the Q_2' values are quite comparable. It is recommended that in application the Q_2' values be used, i.e., the nondetection probabilities be based on the optimal assignment of values for α_3 and α_4. It is very simple to determine this optimal assignment using Annex 4.4.

For the values in this table, Q_2' max is 0.0951, considerably larger than the Q_{max} value of 0.0695. (The actual value of $Q_2'{}_{max}$ over the entire space will be greater than 0.0951 by some amount which could be determined by performing additional runs in the region of the maximum.) The $Q_2'{}_{max}$ value of 0.0951 occurs when

375 kg U is diverted into large and medium data falsifications, 750 kg U is diverted into small falsifications, and 375 kg U is diverted into MUF. Note that this region is quite far removed from the region of Q_{max} illustrating that the two test combinations react differently to different diversion strategies.

EXAMPLE 4.9 (b)

In example 4.8 (b), Q was found to be 0.3413 at $a_2 = 0.6$, $a_3 = 0$, and $a_4 = 0.4$. Find the corresponding value of Q'. In order for the comparison to be more valid, initially set $\alpha = 0.0126$ for the $\hat{D}$ and MUF tests, where $\alpha = 1- \sqrt{0.975}$.

Since the sample sizes in this example are implementation sample sizes rather than planning sample sizes, t_3, t_4, and ρ are expressed in terms of $V(\hat{D}')|H_0$, $V(\hat{D}')|H_1$, V(MUF), and V_0 rather than k_2, k_3, and θ.

From (eq. 4.5.6) and the data of example 4.7 (c),

$$C_1^2 = \min\left(4,\ 1 + 8a_3/\sqrt{23.978453}\right)$$

$$= \min\left(4,\ 1 + 1.634a_3\right)$$

where, in applying (eq. 4.5.6), a_1 is replaced by a_3 and $\hat{D}_k$ by $\hat{D}'$. From (eq. 4.5.7), again applied to $\hat{D}'$ rather than to $\hat{D}_k$,

$$t_3 = \frac{2.238\sqrt{27.303719} - 8a_3}{\sqrt{23.978453 + 3.325266\ C_1^2}}$$

$$= \frac{11.6942 - 8a_3}{\sqrt{23.978453 + 3.325266\ C_1^2}}$$

From (eq. 4.5.13),

$$t_4 = 2.238 - 8a_4/\sqrt{10.729310}$$

$$= 2.238 - 2.4423a_4$$

From (eq. 4.6.5),

$$\rho = \frac{-10.729310 + 2.230517}{\sqrt{(23.978453 + 3.325266C_1^2)(10.729310)}}$$

$$= \frac{-2.5946}{\sqrt{(23.978453 + 3.325266C_1^2)}}$$

First, use the table look-up approach 1 to compute Q' for $a_2 = 0.6$, $a_3 = 0$, $a_4 = 0.4$. Since $a_3 = 0$, $C_1{}^2 = 1$ so

$$t_3 = 2.238 \qquad t_4 = 1.261 \qquad \rho = -0.497$$

From (eq. 4.5.12) and (eq. 4.5.14),

$$Q_3 = 0.9874 \qquad Q_4 = 0.8964$$

From (eq. 4.6.9) and (eq. 4.6.10),

$$\Delta_3 = 1.222 \qquad \Delta_4 = 2.618$$

From Annex 4.3 (d),

$$T(2.230,\ 1.222) = 0.0063$$

To evaluate T(1.261, 2.618), apply (eq. 4.6.12),

$$T(1.261, 2.618) = 0.5\ (0.8964+0.9995)-(0.8964)(0.9995) - 0.0002$$

$$= 0.0518$$

Finally, since H of (eq. 4.6.8) is 0,

$$L(2.238, 1.261, -0.497) = 0.5\ (0.9874+0.8964) - 0.0063 - 0.0518$$

$$= 0.8838$$

Then, from (eq. 4.6.2),

$$Q' = (0.20)^{0.6}(0.8838) = 0.3365$$

Using the MDBNOR subroutine, $L(t_3,\ t_4, \rho)$ is calculated to be 0.8838, in agreement with the table look-up value.

Before performing further calculations at other combinations of diversion strategies, use the optimal values of α_3 and α_4 in place of $\alpha_3 = \alpha_4 = 0.0126$. These are found from Annex 4.4 where the values for α_3 and α_4 read from the graph must be divided by 2 since $\alpha = 0.025$ instead of 0.05. The value for ρ at $C_1{}^2 = 1$ is $\rho = -0.497$. Then, from graphical interpolation in Annex 4.4,

$$\alpha_3 = 0.5(0.036) = 0.018 \qquad \alpha_4 = 0.5(0.018) = 0.009$$

$$t_{\alpha_3} = 2.098 \qquad t_{\alpha_4} = 2.365$$

The example calculations are now performed with these values used instead of 2.238 in the expressions for t_3 and t_4. At $a_2 = 0.6$, $a_3 = 0$, and $a_4 = 0.4$,

$$t_3 = 2.098 \qquad t_4 = 1.388 \qquad \rho = -0.497$$

From (eq. 4.5.12) and (eq. 4.5.14),

$$Q_3 = 0.982 \qquad Q_4 = 0.917$$

From (eq. 4.6.9) and (eq. 4.6.10),

$\Delta_3 = 1.335 \qquad \Delta_4 = 2.315$

From Annex 4.3 (d),

$T(2.098,\ 1.335) = 0.0093$

To evaluate T(1.388, 2.315), apply (eq. 4.6.12),

$$T(1.388,\ 2.315) = 0.5(0.917+0.999)-(0.917)(0.999) - 0.0003$$
$$= 0.0416$$

Finally, since H of (eq. 4.6.8) is 0,

$$L(2.098,\ 1.388,\ -0.497) = 0.5\ (0.982+0.917)-0.0093-0.0416$$
$$= 0.8986$$

This compares with the value of 0.8838 for the case $\alpha_3 = \alpha_4 = 0.0126$. The Q' value is $(0.20)^{0.6}(0.8986) = 0.3421$, compared with the earlier value, 0.3365.

The further calculations are now made to investigate the relationship between Q' and diversion strategy, and also to determine an approximate value for Q'_{max}. The table gives values of Q' for various combinations of values of a_2, a_3, and a_4. The $L(t_3,\ t_4,\ \rho)$ values are calculated using the MDBNOR subroutine. Two sets of values for $L(t_3,t_4,\rho)$ and Q' are given. Those with subscript 1 are based on $\alpha_3 = \alpha_4 = 0.0126$ while for $\alpha_3 = 0.018$ and $\alpha_4 = 0.009$, the subscript 2 values are calculated.

a_2	a_3	a_4	C_1^2	β^{a_2}	$L_1(t_3,t_4,\rho)$	$L_2(t_3,t_4,\rho)$	Q_1'	Q_2'
0	0	1	1	1	0.4080	0.4531	0.4080	0.4531
0	0.25	0.75	1.409	1	0.6243	0.6579	0.6243	0.6579
0	0.50	0.50	1.817	1	0.7666	0.7733	0.7666	0.7733
0	0.75	0.25	2.226	1	0.7941	0.7732	0.7941	0.7732
0	1	0	2.634	1	0.7285	0.6890	0.7285	0.6890
0.25	0	0.75	1	0.6687	0.6454	0.6854	0.4316	0.4583
0.25	0.25	0.50	1.409	0.6687	0.8107	0.8270	0.5421	0.5530
0.25	0.50	0.25	1.817	0.6687	0.8682	0.8586	0.5806	0.5741
0.25	0.75	0	2.226	0.6687	0.8327	0.8033	0.5568	0.5372
0.50	0	0.50	1	0.4472	0.8329	0.8558	0.3725	0.3827
0.50	0.25	0.25	1.409	0.4472	0.9131	0.9133	0.4083	0.4084
0.50	0.50	0	1.817	0.4472	0.9072	0.8891	0.4057	0.3976
0.75	0	0.25	1	0.2991	0.9355	0.9423	0.2798	0.2818
0.75	0.25	0	1.409	0.2991	0.9523	0.9439	0.2848	0.2823
1	0	0	1	0.2000	0.9748	0.9730	0.1950	0.1946
0.333	0.333	0.333	1.545	0.5848	0.8758	0.8776	0.5122	0.5132

For the values in this table, Q_2' max is 0.7733. The Q max value was 0.5855 from example 4.8 (b). As in the prior example, Q_2' max is much larger than Qmax.

4.7 REFERENCES

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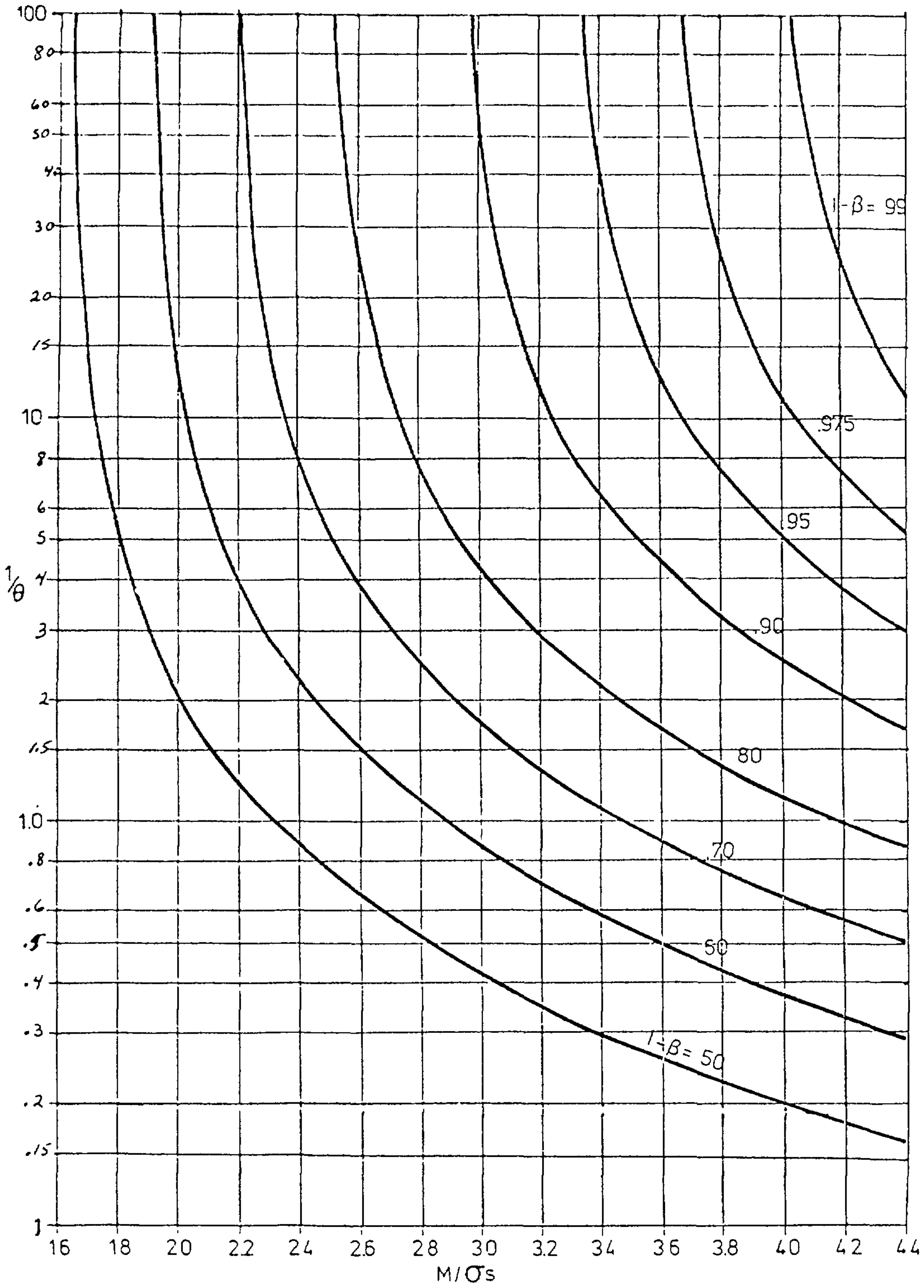
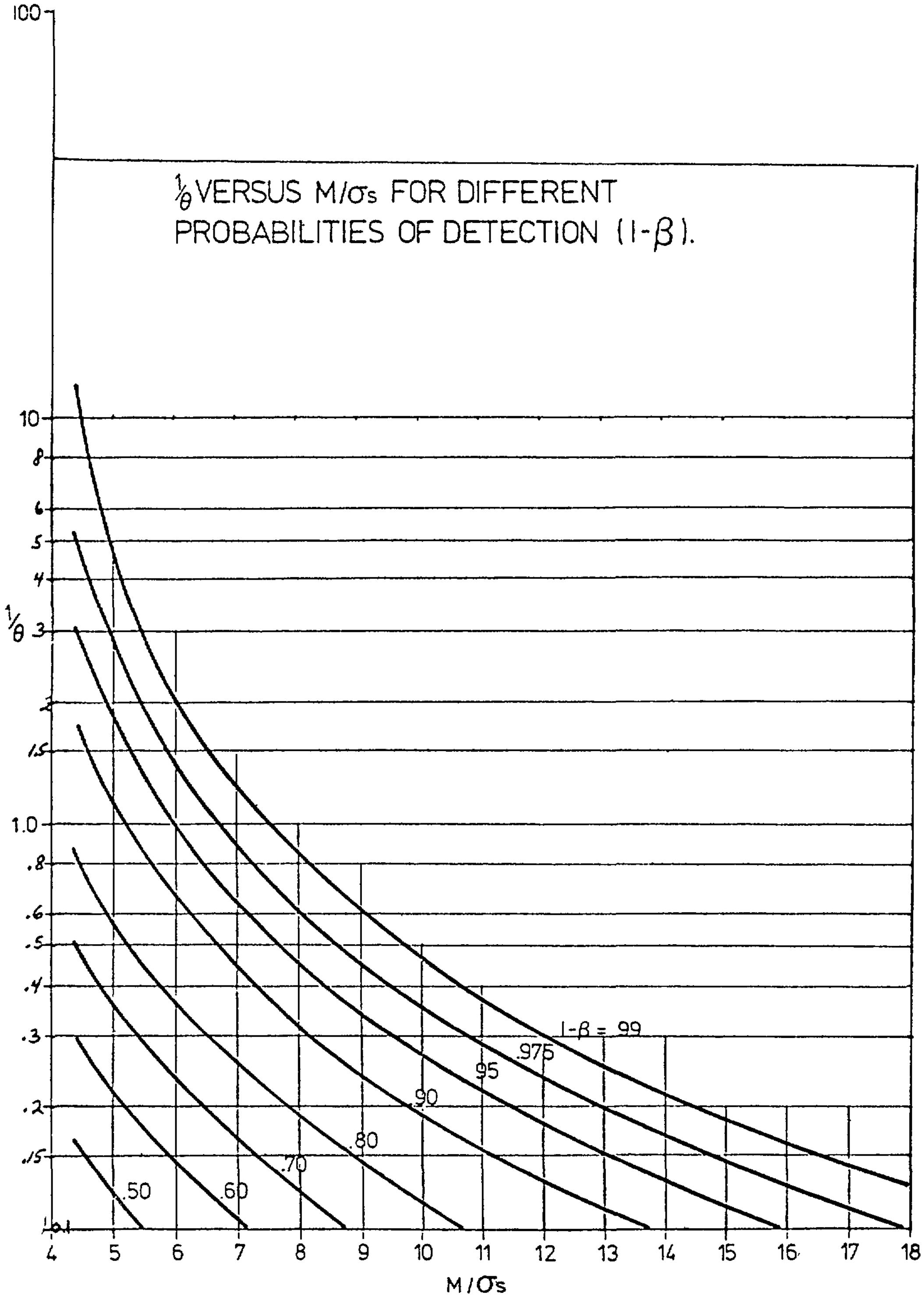
Annex 4.1 (a)



$1/\theta$ VERSUS M/σ_s FOR DIFFERENT PROBABILITIES OF DETECTION $(1-\beta)$

Annex 4.1 (b)


1/θ VERSUS M/σs FOR DIFFERENT PROBABILITIES OF DETECTION (1-β).
1/θ
M/σs
1-β = .99
.975
.95
.90
.80
.70
.60
.50

Annex 4.2 (a)

A Function for Computing Bivariate Normal Probabilities [162]

Table A of $T(h, a)$

$h \diagdown a$	0.25	0.50	0.75	1.00
0.00	0.038990	0.073792	0.102416	0.125000
0.01	0.038988	0.073788	0.102410	0.124992
0.02	0.038982	0.073776	0.102393	0.124968
0.03	0.038972	0.073756	0.102363	0.124928
0.04	0.038958	0.073728	0.102321	0.124873
0.05	0.038940	0.073692	0.102267	0.124801
0.06	0.038918	0.073649	0.102202	0.124714
0.07	0.038892	0.073597	0.102124	0.124611
0.08	0.038862	0.073538	0.102035	0.124492
0.09	0.038829	0.073470	0.101934	0.124357
0.10	0.038791	0.073395	0.101821	0.124207
0.11	0.038750	0.073312	0.101697	0.124041
0.12	0.038704	0.073221	0.101561	0.123860
0.13	0.038655	0.073122	0.101413	0.123663
0.14	0.038602	0.073016	0.101253	0.123450
0.15	0.038545	0.072902	0.101082	0.123223
0.16	0.038484	0.072780	0.100900	0.122980
0.17	0.038419	0.072651	0.100706	0.122722
0.18	0.038350	0.072514	0.100501	0.122449
0.19	0.038278	0.072369	0.100285	0.122162
0.20	0.038202	0.072217	0.100057	0.121859
0.21	0.038122	0.072058	0.099819	0.121542
0.22	0.038038	0.071891	0.099569	0.121210
0.23	0.037951	0.071717	0.099308	0.120864
0.24	0.037860	0.071535	0.099037	0.120503
0.25	0.037766	0.071347	0.098755	0.120129
0.26	0.037668	0.071151	0.098462	0.119740
0.27	0.037566	0.070948	0.098158	0.119337
0.28	0.037461	0.070738	0.097844	0.118921
0.29	0.037352	0.070521	0.097520	0.118492
0.30	0.037240	0.070297	0.097186	0.118048
0.31	0.037124	0.070066	0.096841	0.117592
0.32	0.037005	0.069828	0.096487	0.117123
0.33	0.036882	0.069584	0.096122	0.116641
0.34	0.036756	0.069333	0.095748	0.116146
0.35	0.036627	0.069076	0.095365	0.115639
0.36	0.036495	0.068812	0.094971	0.115119
0.37	0.036359	0.068542	0.094569	0.114587
0.38	0.036220	0.068265	0.094157	0.114044
0.39	0.036078	0.067983	0.093736	0.113489
0.40	0.035933	0.067694	0.093306	0.112922
0.41	0.035785	0.067399	0.092868	0.112344
0.42	0.035634	0.067098	0.092421	0.111755
0.43	0.035479	0.066791	0.091965	0.111155
0.44	0.035322	0.066479	0.091501	0.110545

Annex 4.2 (b)

Table A of $T(h, a)$

h \ a	0.25	0.50	0.75	1.00
0.45	0.035162	0.066161	0.091028	0.109924
0.46	0.034999	0.065837	0.090548	0.109293
0.47	0.034834	0.065508	0.090060	0.108652
0.48	0.034665	0.065173	0.089564	0.108001
0.49	0.034494	0.064834	0.089060	0.107341
0.50	0.034320	0.064489	0.088549	0.106671
0.51	0.034144	0.064139	0.088031	0.105993
0.52	0.033965	0.063784	0.087506	0.105305
0.53	0.033783	0.063424	0.086974	0.104609
0.54	0.033599	0.063059	0.086435	0.103905
0.55	0.033413	0.062690	0.085889	0.103193
0.56	0.033224	0.062316	0.085337	0.102473
0.57	0.033033	0.061938	0.084779	0.101745
0.58	0.032840	0.061555	0.084215	0.101010
0.59	0.032645	0.061168	0.083645	0.100268
0.60	0.032447	0.060778	0.083069	0.099519
0.61	0.032247	0.060383	0.082487	0.098764
0.62	0.032046	0.059984	0.081901	0.098002
0.63	0.031842	0.059581	0.081309	0.097234
0.64	0.031636	0.059175	0.080712	0.096460
0.65	0.031429	0.058765	0.080110	0.095681
0.66	0.031219	0.058352	0.079504	0.094896
0.67	0.031008	0.057936	0.078893	0.094106
0.68	0.030795	0.057516	0.078278	0.093312
0.69	0.030581	0.057093	0.077658	0.092512
0.70	0.030365	0.056667	0.077035	0.091709
0.71	0.030147	0.056239	0.076408	0.090901
0.72	0.029928	0.055807	0.075777	0.090089
0.73	0.029707	0.055373	0.075143	0.089274
0.74	0.029485	0.054937	0.074506	0.088455
0.75	0.029262	0.054498	0.073866	0.087634
0.76	0.029038	0.054056	0.073223	0.086809
0.77	0.028812	0.053613	0.072577	0.085982
0.78	0.028585	0.053167	0.071928	0.085152
0.79	0.028357	0.052720	0.071278	0.084320
0.80	0.028128	0.052270	0.070625	0.083486
0.81	0.027898	0.051819	0.069970	0.082651
0.82	0.027667	0.051367	0.069313	0.081814
0.83	0.027435	0.050912	0.068655	0.080975
0.84	0.027202	0.050457	0.067995	0.080136
0.85	0.026968	0.050000	0.067333	0.079296
0.86	0.026734	0.049542	0.066671	0.078455
0.87	0.026499	0.049083	0.066007	0.077614
0.88	0.026264	0.048622	0.065343	0.076773
0.89	0.026027	0.048161	0.064678	0.075932

Table A of $T(h, a)$

h \ a	0.25	0.50	0.75	1.00
0.90	0.025791	0.047700	0.064013	0.075091
0.91	0.025554	0.047237	0.063347	0.074251
0.92	0.025316	0.046775	0.062681	0.073411
0.93	0.025079	0.046311	0.062015	0.072572
0.94	0.024840	0.045848	0.061349	0.071734
0.95	0.024602	0.045384	0.060684	0.070898
0.96	0.024363	0.044920	0.060016	0.070063
0.97	0.024125	0.044456	0.059354	0.069230
0.98	0.023886	0.043992	0.058690	0.068398
0.99	0.023647	0.043528	0.058027	0.067569
1.00	0.023408	0.043065	0.057365	0.066742
1.01	0.023169	0.042602	0.056704	0.065917
1.02	0.022931	0.042139	0.056044	0.065095
1.03	0.022692	0.041677	0.055386	0.064276
1.04	0.022454	0.041216	0.054729	0.063459
1.05	0.022216	0.040755	0.054075	0.062646
1.06	0.021978	0.040295	0.053421	0.061836
1.07	0.021740	0.039836	0.052770	0.061029
1.08	0.021503	0.039378	0.052121	0.060226
1.09	0.021266	0.038922	0.051474	0.059426
1.10	0.021030	0.038466	0.050830	0.058630
1.11	0.020794	0.038012	0.050188	0.057839
1.12	0.020559	0.037559	0.049548	0.057051
1.13	0.020325	0.037107	0.048911	0.056268
1.14	0.020091	0.036657	0.048277	0.055489
1.15	0.019857	0.036209	0.047646	0.054714
1.16	0.019625	0.035762	0.047018	0.053945
1.17	0.019393	0.035317	0.046393	0.053180
1.18	0.019162	0.034874	0.045771	0.052420
1.19	0.018931	0.034432	0.045152	0.051664
1.20	0.018702	0.033993	0.044537	0.050914
1.21	0.018473	0.033556	0.043925	0.050169
1.22	0.018246	0.033120	0.043317	0.049430
1.23	0.018019	0.032687	0.042712	0.048696
1.24	0.017794	0.032256	0.042112	0.047967
1.25	0.017569	0.031828	0.041515	0.047244
1.26	0.017345	0.031402	0.040922	0.046527
1.27	0.017123	0.030978	0.040333	0.045815
1.28	0.016902	0.030556	0.039746	0.045109
1.29	0.016682	0.030138	0.039167	0.044409
1.30	0.016463	0.029721	0.038590	0.043715
1.31	0.016245	0.029308	0.038018	0.043027
1.32	0.016028	0.028897	0.037450	0.042345
1.33	0.015813	0.028488	0.036886	0.041670
1.34	0.015599	0.028083	0.036327	0.041000

Annex 4.2 (c)

A Function for Computing Bivariate Normal Probabilities (*cont.*)

Table A of $T(h, a)$

$h \backslash a$	0.25	0.50	0.75	1.00
1.35	0.015387	0.027680	0.035773	0.040337
1.36	0.015176	0.027281	0.035223	0.039680
1.37	0.014966	0.026884	0.034578	0.039030
1.38	0.014757	0.026490	0.034137	0.038386
1.39	0.014550	0.026099	0.033601	0.037749
1.40	0.014345	0.025711	0.033070	0.037118
1.41	0.014140	0.025326	0.032543	0.036493
1.42	0.013938	0.024944	0.032022	0.035875
1.43	0.013737	0.024566	0.031505	0.035264
1.44	0.013537	0.024190	0.030994	0.034659
1.45	0.013339	0.023818	0.030487	0.034061
1.46	0.013142	0.023449	0.029985	0.033470
1.47	0.012947	0.023084	0.029489	0.032885
1.48	0.012754	0.022721	0.028997	0.032308
1.49	0.012562	0.022362	0.028511	0.031736
1.50	0.012372	0.022006	0.028029	0.031172
1.51	0.012184	0.021654	0.027553	0.030614
1.52	0.011997	0.021305	0.027082	0.030063
1.53	0.011812	0.020959	0.026616	0.029519
1.54	0.011628	0.020617	0.026155	0.028982
1.55	0.011446	0.020279	0.025700	0.028451
1.56	0.011266	0.019943	0.025249	0.027927
1.57	0.011088	0.019611	0.024804	0.027410
1.58	0.010911	0.019283	0.024364	0.026899
1.59	0.010736	0.018958	0.023929	0.026395
1.60	0.010563	0.018637	0.023500	0.025898
1.61	0.010391	0.018319	0.023075	0.025408
1.62	0.010221	0.018005	0.022656	0.024924
1.63	0.010053	0.017694	0.022242	0.024447
1.64	0.009887	0.017387	0.021833	0.023976
1.65	0.009723	0.017083	0.021430	0.023512
1.66	0.009560	0.016783	0.021032	0.023055
1.67	0.009399	0.016486	0.020638	0.022604
1.68	0.009240	0.016193	0.020250	0.022159
1.69	0.009082	0.015903	0.019868	0.021721
1.70	0.008927	0.015617	0.019490	0.021290
1.71	0.008773	0.015334	0.019117	0.020865
1.72	0.008621	0.015055	0.018750	0.020446
1.73	0.008470	0.014780	0.018388	0.020033
1.74	0.008322	0.014508	0.018030	0.019627
1.75	0.008175	0.014239	0.017678	0.019227
1.76	0.008030	0.013974	0.017331	0.018833
1.77	0.007887	0.013713	0.016989	0.018446
1.78	0.007745	0.013454	0.016651	0.018064
1.79	0.007605	0.013200	0.016319	0.017689

A Function for Computing Bivariate Normal Probabilities (*cont.*)

Table A of $T(h, a)$

$h \backslash a$	0.25	0.50	0.75	1.00
1.80	0.007467	0.012949	0.015992	0.017320
1.81	0.007331	0.012701	0.015669	0.016956
1.82	0.007197	0.012457	0.015352	0.016599
1.83	0.007064	0.012216	0.015039	0.016247
1.84	0.006933	0.011978	0.014731	0.015901
1.85	0.006804	0.011744	0.014428	0.015561
1.86	0.006676	0.011514	0.014130	0.015227
1.87	0.006550	0.011288	0.013836	0.014898
1.88	0.006426	0.011062	0.013547	0.014575
1.89	0.006304	0.010841	0.013263	0.014258
1.90	0.006183	0.010624	0.012983	0.013946
1.91	0.006064	0.010410	0.012706	0.013639
1.92	0.005947	0.010199	0.012437	0.013338
1.93	0.005831	0.009991	0.012171	0.013042
1.94	0.005717	0.009786	0.011909	0.012752
1.95	0.005605	0.009585	0.011652	0.012467
1.96	0.005495	0.009387	0.011399	0.012187
1.97	0.005386	0.009192	0.011150	0.011911
1.98	0.005278	0.009000	0.010905	0.011641
1.99	0.005172	0.008811	0.010665	0.011376
2.00	0.005068	0.008625	0.010429	0.011116
2.01	0.004966	0.008442	0.010197	0.010861
2.02	0.004865	0.008263	0.009970	0.010611
2.03	0.004765	0.008086	0.009746	0.010365
2.04	0.004667	0.007912	0.009527	0.010124
2.05	0.004571	0.007741	0.009311	0.009887
2.06	0.004476	0.007573	0.009099	0.009656
2.07	0.004383	0.007408	0.008891	0.009428
2.08	0.004291	0.007245	0.008687	0.009205
2.09	0.004201	0.007086	0.008487	0.008987
2.10	0.004112	0.006929	0.008291	0.008773
2.11	0.004025	0.006775	0.008098	0.008563
2.12	0.003939	0.006624	0.007909	0.008357
2.13	0.003854	0.006476	0.007724	0.008155
2.14	0.003771	0.006330	0.007542	0.007958
2.15	0.003690	0.006187	0.007363	0.007764
2.16	0.003610	0.006046	0.007188	0.007575
2.17	0.003531	0.005905	0.007017	0.007389
2.18	0.003453	0.005772	0.006849	0.007207
2.19	0.003377	0.005639	0.006684	0.007029
2.20	0.003303	0.005509	0.006523	0.006855
2.21	0.003229	0.005381	0.006364	0.006684
2.22	0.003157	0.005255	0.006209	0.006517
2.23	0.003086	0.005132	0.006058	0.006354
2.24	0.003017	0.005011	0.005909	0.006194

A Function for Computing Bivariate Normal Probabilities (*cont.*)

Table A of $T(h, a)$

h \ a	0.25	0.50	0.75	1.00
2.25	0.002948	0.004893	0.005763	0.006038
2.26	0.002881	0.004777	0.005620	0.005884
2.27	0.002816	0.004663	0.005481	0.005735
2.28	0.002751	0.004551	0.005344	0.005588
2.29	0.002688	0.004442	0.005210	0.005445
2.30	0.002625	0.004334	0.005079	0.005305
2.31	0.002565	0.004229	0.004951	0.005167
2.32	0.002505	0.004126	0.004825	0.005034
2.33	0.002446	0.004025	0.004702	0.004903
2.34	0.002388	0.003926	0.004582	0.004774
2.35	0.002332	0.003829	0.004465	0.004649
2.36	0.002277	0.003735	0.004350	0.004527
2.37	0.002222	0.003642	0.004237	0.004407
2.38	0.002169	0.003551	0.004127	0.004291
2.39	0.002117	0.003462	0.004020	0.004177
2.40	0.002066	0.003375	0.003915	0.004065
2.41	0.002016	0.003289	0.003812	0.003956
2.42	0.001967	0.003206	0.003711	0.003850
2.43	0.001919	0.003124	0.003613	0.003746
2.44	0.001672	0.003044	0.003517	0.003645
2.45	0.001826	0.002966	0.003424	0.003546
2.46	0.001781	0.002890	0.003332	0.003449
2.47	0.001737	0.002815	0.003243	0.003355
2.48	0.001693	0.002742	0.003155	0.003263
2.49	0.001651	0.002670	0.003070	0.003173
2.50	0.001609	0.002600	0.002987	0.003086
2.51	0.001569	0.002532	0.002905	0.003000
2.52	0.001529	0.002465	0.002826	0.002917
2.53	0.001490	0.002400	0.002748	0.002835
2.54	0.001452	0.002336	0.002673	0.002756
2.55	0.001415	0.002274	0.002599	0.002679
2.56	0.001379	0.002213	0.002527	0.002603
2.57	0.001343	0.002153	0.002457	0.002530
2.58	0.001308	0.002095	0.002388	0.002458
2.59	0.001274	0.002039	0.002321	0.002388
2.60	0.001241	0.001983	0.002256	0.002320
2.61	0.001209	0.001929	0.002192	0.002253
2.62	0.001177	0.001876	0.002130	0.002189
2.63	0.001146	0.001825	0.002070	0.002126
2.64	0.001115	0.001774	0.002011	0.002064
2.65	0.001086	0.001725	0.001953	0.002004
2.66	0.001057	0.001677	0.001897	0.001946
2.67	0.001028	0.001630	0.001843	0.001889
2.68	0.001001	0.001585	0.001789	0.001834
2.69	0.000974	0.001540	0.001737	0.001780

A Function for Computing Bivariate Normal Probabilities (*cont.*)

Table A of $T(h, a)$

h \ a	0.25	0.50	0.75	1.00
2.70	0.000947	0.001497	0.001687	0.001727
2.71	0.000922	0.001454	0.001636	0.001676
2.72	0.000896	0.001413	0.001590	0.001627
2.73	0.000872	0.001373	0.001543	0.001578
2.74	0.000848	0.001334	0.001498	0.001531
2.75	0.000824	0.001295	0.001453	0.001485
2.76	0.000802	0.001258	0.001410	0.001441
2.77	0.000779	0.001222	0.001368	0.001397
2.78	0.000758	0.001186	0.001327	0.001355
2.79	0.000736	0.001152	0.001288	0.001314
2.80	0.000716	0.001118	0.001249	0.001274
2.81	0.000696	0.001085	0.001211	0.001235
2.82	0.000676	0.001054	0.001175	0.001198
2.83	0.000657	0.001022	0.001139	0.001161
2.84	0.000638	0.000992	0.001104	0.001125
2.85	0.000620	0.000963	0.001071	0.001091
2.86	0.000602	0.000934	0.001038	0.001057
2.87	0.000585	0.000906	0.001006	0.001024
2.88	0.000568	0.000879	0.000975	0.000992
2.89	0.000551	0.000853	0.000945	0.000961
2.90	0.000535	0.000827	0.000916	0.000931
2.91	0.000520	0.000802	0.000887	0.000902
2.92	0.000504	0.000777	0.000859	0.000874
2.93	0.000490	0.000754	0.000833	0.000846
2.94	0.000475	0.000731	0.000806	0.000819
2.95	0.000461	0.000708	0.000781	0.000793
2.96	0.000448	0.000686	0.000756	0.000768
2.97	0.000434	0.000665	0.000732	0.000743
2.98	0.000421	0.000645	0.000709	0.000720
2.99	0.000409	0.000625	0.000687	0.000696
3.00	0.000396	0.000605	0.000665	0.000674
3.01	0.000384	0.000586	0.000643	0.000652
3.02	0.000373	0.000568	0.000623	0.000631
3.03	0.000361	0.000550	0.000603	0.000611
3.04	0.000350	0.000533	0.000583	0.000591
3.05	0.000340	0.000516	0.000564	0.000571
3.06	0.000329	0.000499	0.000546	0.000553
3.07	0.000319	0.000483	0.000528	0.000535
3.08	0.000309	0.000468	0.000511	0.000517
3.09	0.000300	0.000453	0.000494	0.000500
3.10	0.000290	0.000438	0.000478	0.000483
3.11	0.000281	0.000424	0.000462	0.000467
3.12	0.000273	0.000410	0.000447	0.000452
3.13	0.000264	0.000397	0.000432	0.000437
3.14	0.000256	0.000384	0.000417	0.000422

A Function for Computing Bivariate Normal Probabilities (*cont.*)

Table B of $T(h, a)$

h \ a	0.00	0.01	0.02	0.03	0.04
0.00	0.000000	0.001591	0.003183	0.004773	0.006363
0.25	0.000000	0.001543	0.003085	0.004626	0.006167
0.50	0.000000	0.001404	0.002809	0.004212	0.005615
0.75	0.000000	0.001201	0.002402	0.003603	0.004802
1.00	0.000000	0.000965	0.001930	0.002895	0.003858
1.25	0.000000	0.000729	0.001457	0.002185	0.002912
1.50	0.000000	0.000517	0.001033	0.001549	0.002064
1.75	0.000000	0.000344	0.000688	0.001032	0.001375
2.00	0.000000	0.000215	0.000431	0.000646	0.000860
2.25	0.000000	0.000127	0.000253	0.000379	0.000506
2.50	0.000000	0.000070	0.000140	0.000210	0.000279
2.75	0.000000	0.000036	0.000073	0.000109	0.000145
3.00	0.000000	0.000018	0.000035	0.000053	0.000071
3.25	0.000000	0.000008	0.000016	0.000024	0.000032

h \ a	0.05	0.06	0.07	0.08	0.09
0.00	0.007951	0.009538	0.011123	0.012705	0.014285
0.25	0.007706	0.009244	0.010780	0.012314	0.013845
0.50	0.007016	0.008416	0.009814	0.011209	0.012603
0.75	0.006000	0.007197	0.008392	0.009585	0.010775
1.00	0.004821	0.005782	0.006741	0.007698	0.008653
1.25	0.003638	0.004363	0.005086	0.005807	0.006527
1.50	0.002579	0.003092	0.003604	0.004115	0.004624
1.75	0.001717	0.002059	0.002399	0.002739	0.003077
2.00	0.001074	0.001288	0.001500	0.001712	0.001923
2.25	0.000631	0.000757	0.000881	0.001005	0.001129
2.50	0.000348	0.000417	0.000486	0.000555	0.000622
2.75	0.000181	0.000216	0.000252	0.000287	0.000322
3.00	0.000088	0.000105	0.000123	0.000140	0.000157
3.25	0.000040	0.000048	0.000056	0.000064	0.000072

h \ a	0.10	0.11	0.12	0.13	0.14
0.00	0.015863	0.017437	0.019008	0.020575	0.022138
0.25	0.015373	0.016898	0.018420	0.019938	0.021452
0.50	0.013993	0.015380	0.016764	0.018144	0.019521
0.75	0.011963	0.013147	0.014328	0.015506	0.016680
1.00	0.009605	0.010555	0.011501	0.012444	0.013384
1.25	0.007244	0.007958	0.008670	0.009379	0.010084
1.50	0.005131	0.005635	0.006138	0.006638	0.007135
1.75	0.003413	0.003748	0.004081	0.004412	0.004740
2.00	0.002133	0.002341	0.002548	0.002754	0.002958
2.25	0.001251	0.001373	0.001494	0.001614	0.001733
2.50	0.000690	0.000757	0.000823	0.000888	0.000953
2.75	0.000357	0.000391	0.000426	0.000459	0.000492
3.00	0.000174	0.000190	0.000207	0.000223	0.000239
3.25	0.000079	0.000087	0.000094	0.000102	0.000109

A Function for Computing Bivariate Normal Probabilities (*cont.*)

Table B of $T(h, a)$

h \ a	0.15	0.16	0.17	0.18	0.19
0.00	0.023697	0.025251	0.026800	0.028344	0.029883
0.25	0.022962	0.024467	0.025968	0.027463	0.028953
0.50	0.020893	0.022260	0.023623	0.024980	0.026333
0.75	0.017850	0.019015	0.020176	0.021331	0.022482
1.00	0.014319	0.015251	0.016178	0.017100	0.018018
1.25	0.010786	0.011485	0.012179	0.012869	0.013555
1.50	0.007629	0.008120	0.008608	0.009092	0.009573
1.75	0.005067	0.005391	0.005712	0.006031	0.006347
2.00	0.003160	0.003360	0.003559	0.003755	0.003950
2.25	0.001850	0.001967	0.002082	0.002195	0.002308
2.50	0.001017	0.001081	0.001143	0.001205	0.001266
2.75	0.000525	0.000558	0.000590	0.000621	0.000652
3.00	0.000255	0.000270	0.000285	0.000300	0.000315
3.25	0.000116	0.000123	0.000130	0.000136	0.000143

h \ a	0.20	0.21	0.22	0.23	0.24
0.00	0.031416	0.032944	0.034465	0.035980	0.037488
0.25	0.030437	0.031916	0.033388	0.034854	0.036313
0.50	0.027679	0.029020	0.030355	0.031683	0.033005
0.75	0.023627	0.024766	0.025899	0.027027	0.028148
1.00	0.018930	0.019837	0.020739	0.021635	0.022525
1.25	0.014237	0.014913	0.015585	0.016252	0.016913
1.50	0.010050	0.010523	0.010992	0.011456	0.011917
1.75	0.006659	0.006969	0.007276	0.007579	0.007879
2.00	0.004142	0.004332	0.004520	0.004705	0.004888
2.25	0.002418	0.002528	0.002635	0.002741	0.002846
2.50	0.001325	0.001384	0.001442	0.001499	0.001555
2.75	0.000682	0.000712	0.000741	0.000769	0.000797
3.00	0.000329	0.000343	0.000357	0.000371	0.000384
3.25	0.000149	0.000156	0.000162	0.000167	0.000173

h \ a	0.25	0.26	0.27	0.28	0.29
0.00	0.038990	0.040484	0.041971	0.043451	0.044923
0.25	0.037766	0.039211	0.040649	0.042080	0.043503
0.50	0.034320	0.035628	0.036929	0.038223	0.039509
0.75	0.029262	0.030370	0.031470	0.032564	0.033650
1.00	0.023408	0.024286	0.025156	0.026021	0.026878
1.25	0.017569	0.018219	0.018864	0.019502	0.020135
1.50	0.012372	0.012823	0.013269	0.013710	0.014147
1.75	0.008175	0.008467	0.008756	0.009041	0.009322
2.00	0.005068	0.005246	0.005421	0.005593	0.005762
2.25	0.002948	0.003049	0.003148	0.003245	0.003341
2.50	0.001609	0.001663	0.001716	0.001767	0.001817
2.75	0.000824	0.000851	0.000877	0.000902	0.000927
3.00	0.000396	0.000409	0.000421	0.000432	0.000444
3.25	0.000179	0.000184	0.000189	0.000194	0.000199

A Function for Computing Bivariate Normal Probabilities (*cont.*)

Table B of $T(h, a)$

$h \backslash a$	0.30	0.31	0.32	0.33	0.34
0.00	0.046387	0.047843	0.049291	0.050730	0.052161
0.25	0.044918	0.046326	0.047725	0.049115	0.050497
0.50	0.040787	0.042057	0.043319	0.044573	0.045818
0.75	0.034728	0.035799	0.036862	0.037916	0.038963
1.00	0.027728	0.028571	0.029407	0.030235	0.031056
1.25	0.020761	0.021381	0.021994	0.022601	0.023201
1.50	0.014578	0.015003	0.015424	0.015839	0.016248
1.75	0.009599	0.009873	0.010142	0.010406	0.010667
2.00	0.005929	0.006092	0.006253	0.006411	0.006565
2.25	0.003434	0.003526	0.003615	0.003703	0.003789
2.50	0.001866	0.001914	0.001961	0.002006	0.002050
2.75	0.000951	0.000974	0.000997	0.001019	0.001040
3.00	0.000455	0.000465	0.000476	0.000486	0.000495
3.25	0.000204	0.000209	0.000213	0.000217	0.000221

$h \backslash a$	0.35	0.36	0.37	0.38	0.39
0.00	0.053583	0.054997	0.056401	0.057797	0.059183
0.25	0.051871	0.053235	0.054591	0.055937	0.057275
0.50	0.047054	0.048282	0.049501	0.050711	0.051912
0.75	0.040001	0.041031	0.042052	0.043065	0.044068
1.00	0.031868	0.032673	0.033470	0.034259	0.035040
1.25	0.023795	0.024381	0.024961	0.025533	0.026098
1.50	0.016652	0.017050	0.017443	0.017829	0.018210
1.75	0.010923	0.011175	0.011423	0.011666	0.011905
2.00	0.006717	0.006866	0.007011	0.007154	0.007293
2.25	0.003872	0.003954	0.004034	0.004112	0.004188
2.50	0.002093	0.002135	0.002176	0.002215	0.002254
2.75	0.001061	0.001081	0.001101	0.001119	0.001137
3.00	0.000505	0.000513	0.000522	0.000530	0.000538
3.25	0.000225	0.000229	0.000232	0.000236	0.000239

$h \backslash a$	0.40	0.41	0.42	0.43	0.44
0.00	0.060559	0.061927	0.063284	0.064633	0.065971
0.25	0.058602	0.059921	0.061230	0.062529	0.063818
0.50	0.053103	0.054285	0.055458	0.056621	0.057775
0.75	0.045063	0.046048	0.047025	0.047992	0.048950
1.00	0.035812	0.036576	0.037332	0.038079	0.038817
1.25	0.026656	0.027207	0.027750	0.028286	0.028815
1.50	0.018585	0.018954	0.019318	0.019675	0.020026
1.75	0.012140	0.012370	0.012595	0.012816	0.013033
2.00	0.007430	0.007563	0.007693	0.007820	0.007944
2.25	0.004261	0.004333	0.004403	0.004471	0.004537
2.50	0.002291	0.002327	0.002362	0.002395	0.002428
2.75	0.001155	0.001171	0.001188	0.001203	0.001218
3.00	0.000546	0.000553	0.000560	0.000567	0.000573
3.25	0.000242	0.000245	0.000248	0.000250	0.000253

A Function for Computing Bivariate Normal Probabilities (*cont.*)

Table B of $T(h, a)$

$h \backslash a$	0.45	0.46	0.47	0.48	0.49
0.00	0.067299	0.068618	0.069926	0.071225	0.072513
0.25	0.065098	0.066368	0.067628	0.068877	0.070117
0.50	0.058918	0.060052	0.061176	0.062290	0.063395
0.75	0.049898	0.050837	0.051767	0.052687	0.053597
1.00	0.039547	0.040268	0.040980	0.041684	0.042379
1.25	0.029336	0.029849	0.030355	0.030854	0.031345
1.50	0.020371	0.020710	0.021043	0.021370	0.021691
1.75	0.013245	0.013453	0.013656	0.013855	0.014049
2.00	0.008065	0.008183	0.008298	0.008410	0.008519
2.25	0.004601	0.004663	0.004723	0.004782	0.004838
2.50	0.002459	0.002490	0.002519	0.002547	0.002574
2.75	0.001232	0.001246	0.001259	0.001272	0.001284
3.00	0.000579	0.000585	0.000590	0.000595	0.000600
3.25	0.000255	0.000257	0.000259	0.000262	0.000263

$h \backslash a$	0.50	0.51	0.52	0.53	0.54
0.00	0.073792	0.075060	0.076318	0.077566	0.078803
0.25	0.071347	0.072566	0.073775	0.074974	0.076163
0.50	0.064489	0.065573	0.066647	0.067710	0.068764
0.75	0.054498	0.055389	0.056270	0.057141	0.058003
1.00	0.043065	0.043742	0.044410	0.045069	0.045720
1.25	0.031828	0.032304	0.032772	0.033232	0.033685
1.50	0.022006	0.022315	0.022619	0.022916	0.023207
1.75	0.014239	0.014425	0.014606	0.014783	0.014956
2.00	0.008625	0.008728	0.008828	0.008926	0.009020
2.25	0.004893	0.004946	0.004997	0.005046	0.005094
2.50	0.002600	0.002626	0.002650	0.002673	0.002695
2.75	0.001295	0.001306	0.001317	0.001327	0.001337
3.00	0.000605	0.000610	0.000614	0.000618	0.000622
3.25	0.000265	0.000267	0.000268	0.000270	0.000271

$h \backslash a$	0.55	0.56	0.57	0.58	0.59
0.00	0.080030	0.081247	0.082453	0.083649	0.084835
0.25	0.077341	0.078509	0.079667	0.080814	0.081951
0.50	0.069807	0.070841	0.071864	0.072876	0.073879
0.75	0.058855	0.059697	0.060530	0.061353	0.062165
1.00	0.046361	0.046994	0.047618	0.048233	0.048839
1.25	0.034131	0.034569	0.034999	0.035422	0.035838
1.50	0.023492	0.023771	0.024045	0.024312	0.024574
1.75	0.015124	0.015288	0.015448	0.015604	0.015756
2.00	0.009112	0.009201	0.009287	0.009370	0.009451
2.25	0.005140	0.005184	0.005227	0.005268	0.005308
2.50	0.002716	0.002737	0.002756	0.002775	0.002793
2.75	0.001346	0.001354	0.001363	0.001370	0.001378
3.00	0.000625	0.000629	0.000632	0.000635	0.000638
3.25	0.000273	0.000274	0.000275	0.000276	0.000277

A Function for Computing Bivariate Normal Probabilities (*cont.*)

Table B of *T*(*h*, *a*)

h \ a	0.60	0.61	0.62	0.63	0.64
0.00	0.086010	0.087176	0.088330	0.089475	0.090609
0.25	0.083078	0.084194	0.085300	0.086396	0.087482
0.50	0.074671	0.075854	0.076626	0.077788	0.078739
0.75	0.062969	0.063762	0.064546	0.065320	0.066084
1.00	0.049436	0.050024	0.050604	0.051175	0.051738
1.25	0.036246	0.036646	0.037040	0.037426	0.037805
1.50	0.024831	0.025081	0.025326	0.025566	0.025800
1.75	0.015904	0.016048	0.016188	0.016324	0.016456
2.00	0.009530	0.009606	0.009679	0.009750	0.009818
2.25	0.005346	0.005383	0.005418	0.005452	0.005485
2.50	0.002810	0.002826	0.002842	0.002857	0.002871
2.75	0.001385	0.001391	0.001398	0.001404	0.001409
3.00	0.000640	0.000643	0.000645	0.000647	0.000649
3.25	0.000278	0.000279	0.000280	0.000280	0.000281
	0.65	**0.66**	**0.67**	**0.68**	**0.69**
0.00	0.091733	0.092847	0.093950	0.095044	0.096127
0.25	0.088557	0.089622	0.090677	0.091722	0.092756
0.50	0.079681	0.080612	0.081534	0.082445	0.083347
0.75	0.066839	0.067584	0.068320	0.069046	0.069762
1.00	0.052291	0.052836	0.053373	0.053901	0.054421
1.25	0.038177	0.038541	0.038899	0.039250	0.039594
1.50	0.026028	0.026251	0.026469	0.026682	0.026889
1.75	0.016585	0.016709	0.016831	0.016948	0.017063
2.00	0.009885	0.009949	0.010010	0.010070	0.010127
2.25	0.005516	0.005546	0.005574	0.005602	0.005628
2.50	0.002884	0.002897	0.002909	0.002921	0.002932
2.75	0.001415	0.001420	0.001424	0.001429	0.001433
3.00	0.000651	0.000653	0.000655	0.000656	0.000658
3.25	0.000282	0.000282	0.000283	0.000283	0.000284
	0.70	**0.71**	**0.72**	**0.73**	**0.74**
0.00	0.097200	0.098263	0.099316	0.100360	0.101393
0.25	0.093781	0.094796	0.095800	0.096795	0.097780
0.50	0.084239	0.085120	0.085992	0.086854	0.087707
0.75	0.070469	0.071167	0.071856	0.072535	0.073205
1.00	0.054932	0.055435	0.055929	0.056416	0.056894
1.25	0.039930	0.040261	0.040584	0.040901	0.041211
1.50	0.027091	0.027289	0.027481	0.027669	0.027851
1.75	0.017173	0.017281	0.017385	0.017486	0.017583
2.00	0.010182	0.010236	0.010287	0.010336	0.010384
2.25	0.005653	0.005677	0.005700	0.005722	0.005743
2.50	0.002942	0.002952	0.002961	0.002970	0.002979
2.75	0.001437	0.001441	0.001444	0.001447	0.001450
3.00	0.000659	0.000660	0.000662	0.000663	0.000664
3.25	0.000284	0.000285	0.000285	0.000285	0.000286

A Function for Computing Bivariate Normal Probabilities (*cont.*)

Table B of *T*(*h*, *a*)

h \ a	0.75	0.76	0.77	0.78	0.79
0.00	0.102416	0.103430	0.104434	0.105428	0.106413
0.25	0.098755	0.099720	0.100675	0.101621	0.102557
0.50	0.088549	0.089382	0.090206	0.091020	0.091824
0.75	0.073866	0.074518	0.075161	0.075794	0.076420
1.00	0.057365	0.057827	0.058282	0.058728	0.059167
1.25	0.041515	0.041812	0.042103	0.042388	0.042666
1.50	0.028029	0.028203	0.028371	0.028536	0.028695
1.75	0.017678	0.017770	0.017858	0.017944	0.018027
2.00	0.010429	0.010473	0.010515	0.010556	0.010595
2.25	0.005763	0.005782	0.005800	0.005818	0.005834
2.50	0.002987	0.002994	0.003001	0.003008	0.003014
2.75	0.001453	0.001456	0.001458	0.001461	0.001463
3.00	0.000665	0.000665	0.000666	0.000667	0.000668
3.25	0.000286	0.000286	0.000286	0.000287	0.000287
	0.80	**0.81**	**0.82**	**0.83**	**0.84**
0.00	0.107388	0.108354	0.109310	0.110257	0.111195
0.25	0.103484	0.104401	0.105309	0.106208	0.107097
0.50	0.092620	0.093406	0.094182	0.094950	0.095708
0.75	0.077036	0.077643	0.078242	0.078832	0.079414
1.00	0.059598	0.060022	0.060438	0.060847	0.061248
1.25	0.042939	0.043205	0.043466	0.043721	0.043970
1.50	0.028851	0.029002	0.029149	0.029292	0.029431
1.75	0.018107	0.018184	0.018259	0.018331	0.018401
2.00	0.010632	0.010668	0.010702	0.010735	0.010766
2.25	0.005850	0.005865	0.005879	0.005892	0.005905
2.50	0.003020	0.003026	0.003031	0.003036	0.003041
2.75	0.001465	0.001467	0.001469	0.001470	0.001472
3.00	0.000668	0.000669	0.000669	0.000670	0.000670
3.25	0.000287	0.000287	0.000287	0.000287	0.000287
	0.85	**0.86**	**0.87**	**0.88**	**0.89**
0.00	0.112124	0.113043	0.113954	0.114855	0.115747
0.25	0.107977	0.108848	0.109710	0.110563	0.111407
0.50	0.096458	0.097198	0.097930	0.098653	0.099367
0.75	0.079988	0.080553	0.081109	0.081658	0.082199
1.00	0.061642	0.062029	0.062409	0.062782	0.063148
1.25	0.044213	0.044451	0.044693	0.044910	0.045132
1.50	0.029566	0.029697	0.029825	0.029948	0.030068
1.75	0.018469	0.018533	0.018596	0.018656	0.018715
2.00	0.010796	0.010825	0.010853	0.010879	0.010905
2.25	0.005917	0.005929	0.005940	0.005950	0.005960
2.50	0.003045	0.003049	0.003053	0.003057	0.003060
2.75	0.001473	0.001475	0.001476	0.001477	0.001478
3.00	0.000671	0.000671	0.000672	0.000672	0.000672
3.25	0.000288	0.000288	0.000285	0.000288	0.000288

Annex 4.3 (d)

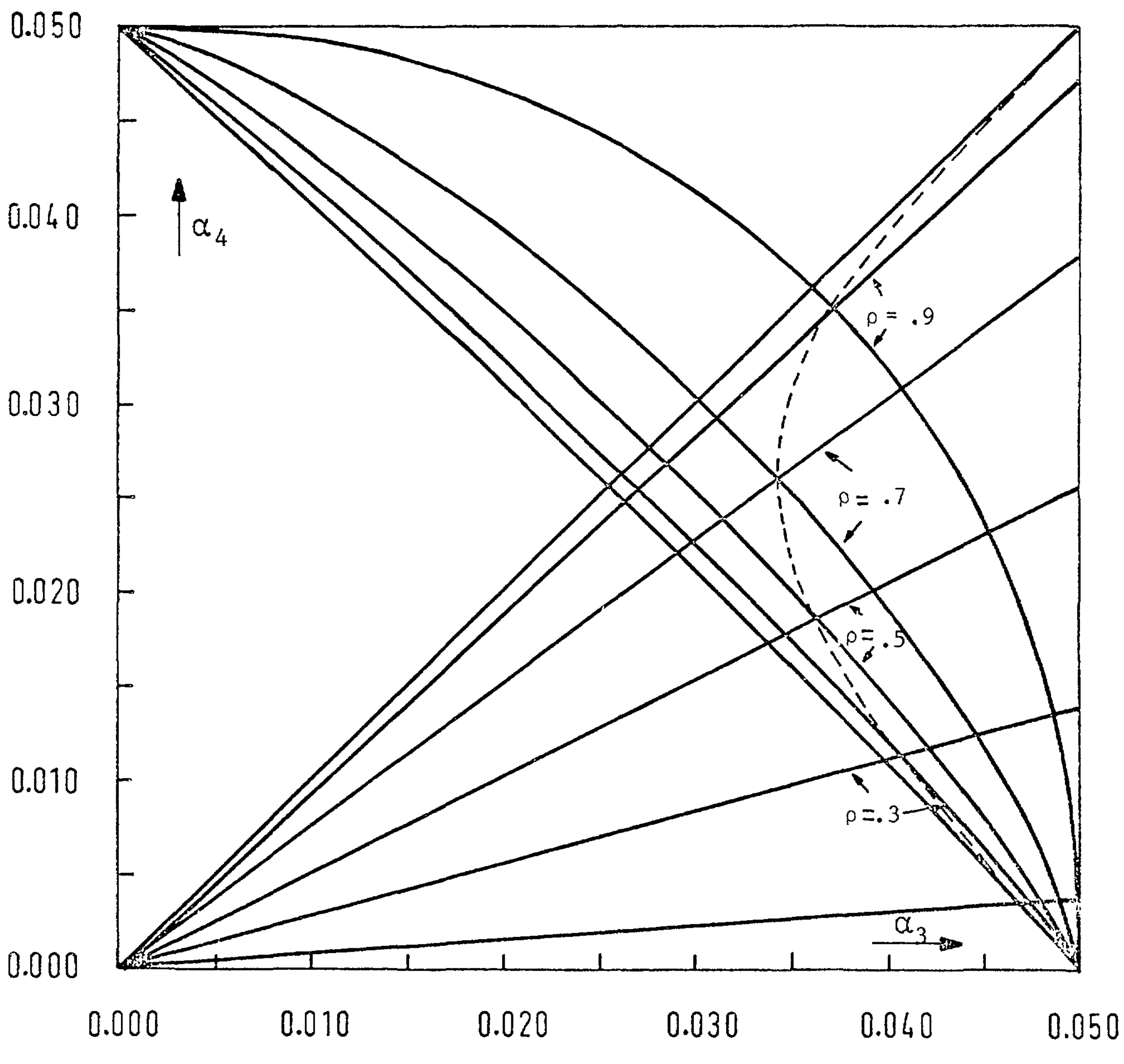
A Function for Computing Bivariate Normal Probabilities *(cont.)*

Table B of *T*(*h*, *a*)

h \ a	0.90	0.91	0.92	0.93	0.94
0.00	0.116631	0.117506	0.118372	0.119230	0.120079
0.25	0.112243	0.113069	0.113887	0.114696	0.115497
0.50	0.100073	0.100770	0.101458	0.102138	0.102810
0.75	0.082731	0.083256	0.083772	0.084281	0.084783
1.00	0.063507	0.063859	0.064205	0.064544	0.064877
1.25	0.045348	0.045559	0.045766	0.045967	0.046163
1.50	0.030185	0.030298	0.030408	0.030514	0.030617
1.75	0.018771	0.018825	0.018877	0.018927	0.018975
2.00	0.010929	0.010952	0.010974	0.010995	0.011015
2.25	0.005969	0.005978	0.005986	0.005994	0.006001
2.50	0.003063	0.003066	0.003069	0.003072	0.003074
2.75	0.001479	0.001480	0.001481	0.001482	0.001482
3.00	0.000672	0.000673	0.000673	0.000673	0.000673
3.25	0.000288	0.000288	0.000288	0.000288	0.000288
	0.95	0.96	0.97	0.98	0.99
0.00	0.120920	0.121752	0.122576	0.123392	0.124200
0.25	0.116290	0.117074	0.117850	0.118617	0.119377
0.50	0.103474	0.104129	0.104777	0.105416	0.106047
0.75	0.085276	0.085762	0.086241	0.086713	0.087177
1.00	0.065203	0.065523	0.065837	0.066145	0.066446
1.25	0.046355	0.046542	0.046724	0.046902	0.047075
1.50	0.030717	0.030814	0.030908	0.030999	0.031087
1.75	0.019021	0.019066	0.019109	0.019150	0.019189
2.00	0.011034	0.011052	0.011069	0.011086	0.011101
2.25	0.006008	0.006015	0.006021	0.006027	0.006032
2.50	0.003076	0.003078	0.003080	0.003082	0.003084
2.75	0.001483	0.001483	0.001484	0.001485	0.001485
3.00	0.000673	0.000674	0.000674	0.000674	0.000674
3.25	0.000288	0.000288	0.000288	0.000288	0.000288
	1.00	1.25	1.50	2.00	∞
0.00	0.125000	0.142612	0.156416	0.176208	0.250000
0.25	0.120129	0.136540	0.149156	0.166613	0.200647
0.50	0.106671	0.119952	0.129584	0.141581	0.154269
0.75	0.087634	0.096973	0.103119	0.109570	0.113314
1.00	0.066742	0.072452	0.075735	0.078468	0.079328
1.25	0.047244	0.050283	0.051753	0.052673	0.052825
1.50	0.031172	0.032582	0.033134	0.033383	0.033404
1.75	0.019227	0.019798	0.019973	0.020028	0.020030
2.00	0.011116	0.011318	0.011365	0.011375	0.011375
2.25	0.006038	0.006100	0.006111	0.006112	0.006112
2.50	0.003086	0.003103	0.003105	0.003105	0.003105
2.75	0.001485	0.001490	0.001490	0.001490	0.001490
3.00	0.000674	0.000675	0.000675	0.000675	0.000675
3.25	0.000288	0.000289	0.000289	0.000289	0.000289

Annex 4.4

Optional Values for Significance Levels in $\hat{D}$ and MUF Tests



Chapter 5

IMPLEMENTING INSPECTION PLANS

5.1 ON-SITE ACTIVITIES

The activities that are statistical in nature and that are performed on site, i.e., at the facility, are those activities that relate to attributes inspection, using both the attributes and variables tester, and to variables inspection. A detailed description of the kinds of activities involved is given in Section 4.2. The discussion in Section 4.2 does not address specific problems in implementation however. Such problems include the procedure for drawing a random sample, and the means of establishing defect criteria. These topics are now addressed in Sections 5.1.1 and 5.1.2 respectively. A third topic dealing with on-site inspection activities is the construction of confidence intervals based on data derived from the attributes inspection. This topic is covered in Section 5.1.3.

5.1.1 Drawing a Random Sample

It is important that items selected for measurement during an inspection be selected in a random fashion. Clearly, it is unacceptable to select only those items that are easily accessible, for then the diverter would falsify only the items that are not readily accessible. Nor should one rely on his own ability to select a random sample; this is difficult to do without introducing some non-random features into the selection process. In order to combat the possibility that the diverter might anticipate which items are likely to be inspected, the sample should be selected by some random process. Some means for doing this are set forth in the next three sections. In each case, it is necessary that the items in the population be numbered serially from 1 to N, and the problem is to randomly select n from these N items.

5.1.1.1 Random Number Table

Random number tables may be used to supply random numbers. Annex 5.1 (a)-(e) gives 5,000 four digit random numbers taken from [5.1]. More extensive tables are available [5.2].

Random number tables are simple to use. The procedure for using the Annex 5.1 tables is given as Method 5.1. Clearly, Method 5.1 may be applied to any set of such random numbers.

Method 5.1

Notation

N = number of items in population to be sampled

n = number of sampled items, with $n<N$

Model

In Annex 5.1, each four digit number from 0000 to 9999 is equally likely to occur, i.e., will occur with probability 0.0001 at each entry. This probability is independent of entries preceding the entry in question so that movement in the table may be in any direction (down, up, left, right). A similar statement applies if one uses only the first or last three, two, or one digits.

Results

Arbitrarily select a starting point in the random number tables of Annex 5.1, and decide upon a method of moving in the tables to select successive entries.

Determine the number of digits to use. If $1 \leq N \leq 9$, use either the first, second, third, or fourth digit of each number; if $10 \leq N \leq 99$, use either the first two, middle two, or last two digits; if $100 \leq N \leq 999$, use either the first three or last three digits. For $N > 9999$, use all 4 digits plus the first digit in the next column, etc.

Examine each tabled value in turn and write down all numbers between 1 and N until the n items are identified. Do not duplicate numbers already listed and ignore any numbers in the table that exceed N.

Basis

The numbers tabled in Annex 5.1 meet the criteria for randomness suggested by the model. Numbers selected from this table in the fashion indicated will likewise meet these criteria.

Examples

EXAMPLE 5.1 (a)

From example 4.1 (b), 48 items are to be inspected from the total of 360 items in the stratum. For n=48 and N=360, use Method 5.1 to select the items to be inspected.

Arbitrarily start with the four digit number, 3501, in column 4 row 12 of Annex 5.1 (b). Select the numbers by moving downward in the table. Arbitrarily use the last three digits of each four digit number, since N=360 is a three digit number. The 48 three digit numbers between 001 and 360 are as follows.

183	177	216	170	256	68	9	66
29	341	53	160	100	~~183~~	178	339
279	125	93	8	237	22	148	
202	126	82	131	107	144	60	
166	354	156	~~53~~	1	293	297	
270	33	12	274	~~156~~	116	123	
347	38	284	295	340	306	223	

Note that 51 numbers had to be selected to obtain the 48 distinct numbers since the numbers 53, 156, and 183 were duplicates, as indicated.

5.1.1.2 Pocket Calculator

Many of the pocket calculators available to the inspector either have built-in random number generators or else they can be programmed very easily to generate random numbers. Further, the problem illustrated in Example 5.1 (a) where so many numbers had to be ignored because they exceeded N=360 can be circumvented quite easily so that virtually all of the numbers generated are useable. (A modification may be made to Method 5.1 to accomplish this also, but it is usually simpler when using random number tables to simply ignore the unwanted numbers.)

Method 5.2 indicates how a pocket calculator may be used to generate random numbers.

Method 5.2

Notation

N and n are defined as in Method 5.1.

d = number of digits in N

Model

The numbers generated by this method are uniformly distributed on the interval 0 to 1. By using the first d digits and ignoring the decimal point, random numbers from 1 to N are created.

Results

Assume initially that the pocket calculator is not programmable. Then, the random numbers from 1 to N may be generated as follows.

Calculate $m = 10^d/N$, rounded down to integer (eq. 5.1.1)

Select an arbitrary "seed number" between 0 and 1 and containing d digits. Call this number f_0, and enter it in the calculator.

Calculate $g_1 = 997\ f_0$ (eq. 5.1.2)

Define f_1 = fractional part of g_1

Calculate $h_1 = f_1/m$ (eq. 5.1.3)

Ignore the decimal in h_1. The first d digits is the first random number from 1 to N. If this number should exceed N, ignore it. To find the next number, repeat the above steps, replacing f_0 by f_1, f_1 by f_2, g_1 by g_2, and h_1 by h_2.

If the pocket calculator is programmable, the above procedure may be programmed very easily to generate random numbers. This is illustrated for the HP-67 pocket calculator. The specific programming steps will depend on which calculator is available.

Programming Steps:

```
RCL 2
RCL 0
  x
g frac
STO 2
RCL 1
  ÷
h rtn
```

To execute the program, make the following entries.

```
DSP d
997 STO 0
 m  STO 1
 f0 STO 2  (The arbitrary seed)
```

Then, repeatedly initiate the program to generate the numbers, ignoring the decimal. If a given number exceeds N, ignore it. Also, as with Method 5.1, numbers that are duplicates of numbers previously generated must be ignored.

Basis

The algorithm used for generating the random numbers is adapted from the algorithm of Reference [5.3]. The adaptation consists of dividing by m to reduce the number of generated numbers that exceed N. As stated in [5.3], the generator in question passes the chi-square frequency test for uniformity, and the serial and run tests for randomness. Of course, with each application, a different seed number, f_0, must be used. This is true also if the calculator in question has a built-in random number generator.

Examples

EXAMPLE 5.2 (a)

Redo Example 5.1 (a) using Method 5.2. The parameter values are

$N = 360 \qquad d = 3$

$m = \text{integral part of } 10^3/360 = 2$

Choose the arbitrary seed number

$f_0 = 0.298$ (contains d=3 digits)

In applying Method 5.2, all numbers generated by the algorithm are listed below. Those that are deleted either because they exceed 360 or because they are duplicated are crossed out until the n=48 useable numbers are generated.

53	233	213	~~493~~	73	~~453~~	133	113	~~393~~
341	301	~~361~~	21	281	141	101	161	321
~~477~~	97	~~417~~	~~437~~	157	77	197	17	37
69	209	249	189	29	269	~~409~~	~~449~~	
293	~~373~~	253	~~433~~	~~413~~	193	273	153	
121	~~381~~	241	201	261	~~421~~	181	41	
137	357	277	~~397~~	217	237	~~457~~	~~377~~	
89	~~429~~	169	309	349	289	129	~~369~~	

5.1.1.3 Computer

The Agency has computer programs available that facilitate the drawing of random samples. These are especially useful if the inspector has good knowledge of his inspection parameters, N and n, prior to leaving for the facility. In this event, he can request that he be supplied with a listing of randomly selected items in advance.

Available computer programs are of two types. In the first type, one inputs values for n and N, and the n items to be inspected are printed out. These are listed in numerical order to facilitate inspection. An example of computer output for n=65 and N=1200 is shown

			18	31	39	70	91	97	98
100	104	129	139	146	155	161	237	262	280
281	285	297	306	309	330	335	336	339	349
362	365	383	385	392	461	488	515	533	597
645	689	696	700	709	729	740	757	781	815
829	835	865	902	911	920	923	944	970	978
1011	1032	1048	1057	1073	1080	1162	1179		

In some inspection situations, a different kind of sample selection process may be simpler to implement. For example, if the population items are stacked in some sort of array, a skip-sampling procedure may be used conveniently. With skip sampling, the sampling scheme indicates how many items are skipped, then how many are inspected, then how many are skipped, etc. A computer program for this time of inspection sampling is also available and may be used, again if the inspector has advance information on the inspection parameters, N and n.

An example of computer output for skip sampling with N=1200 and n=65 is shown. If this specific plan were implemented, the inspector would decide on how he would order the array of items. Then, he would skip 2 items, sample 1, skip 12, sample 1, skip 16, sample 1, etc.

For either type of random sample selection, once a set of computer-generated random numbers have been used on an inspection, they should not be reused at the same facility. A new computer request should be made for each application.

Sample Size: 65 Population: 1200

Sample	Skip	Sample	Skip	Sample	Skip
	B = 2	C = 1	D = 25	E = 1	F = 2
A = 1	B = 12	C = 1	D = 16	E = 1	F = 23
A = 1	B = 16	C = 1	D = 81	E = 1	F = 6
A = 1	B = 15	C = 1	D = 8	E = 1	F = 15
A = 1	B = 18	C = 1	D = 1	E = 1	F = 32
A = 1	B = 33	C = 1	D = 23	E = 1	F = 8
A = 1	B = 8	C = 1	D = 48	E = 1	F = 23
A = 1	B = 13	C = 1	D = 33	E = 1	F = 23
A = 1	B = 16	C = 1	D = 7	E = 1	F = 2
A = 1	B = 19	C = 1	D = 24	E = 1	F = 35
A = 1	B = 2	C = 1	D = 24	E = 1	F = 9
A = 1	B = 8	C = 1	D = 3	E = 1	F = 74
A = 1	B = 60	C = 1	D = 24	E = 1	F = 5
A = 1	B = 11	C = 1	D = 52	E = 1	F = 2
A = 1	B = 25	C = 1	D = 7	E = 1	F = 1
A = 1	B = 19	C = 1	D = 3	E = 1	F = 10
A = 1	B = 23	C = 1	D = 4	E = 1	F = 15
A = 1	B = 7	C = 1	D = 5	E = 1	F = 34
A = 1	B = 5	C = 1	D = 15	E = 1	F = 10
A = 1	B = 12	C = 1	D = 27	E = 1	F = 20
A = 2	B = 3	C = 1	D = 8	E = 1	F = 2
A = 1	B = 2	C = 1	D = 17		

5.1.2. Attributes Inspection Defect Criteria

In Section 4.2, the concept of attributes inspection was discussed, and it was pointed out there that in attributes inspection, each item inspected is classified as being either acceptable or a defect. Attributes inspection activities were described in Section 4.2.1. Three inspection activities described there, activities 3, 5 and 6 require that a defect be defined, i.e., that defect criteria be established. Activity 3 deals with inspection to detect recording and/or calculational mistakes, and the problem of defining defect criteria for this activity and the effect of such mistakes on MUF are considered in Section 5.1.2.1. Section 5.1.3.1 considers the confidence interval for the number of mistakes. The effect of defects observed in connection with activity 5 and 6 of Section 4.2.1 are treated in exactly the same way in Sections 5.1.2.1 and 5.1.3.1, except, the basis for the defect criteria are different as described in Section 5.1.2.2 for attributes inspection and Section 5.1.2.3 for variables inspection.

5.1.2.1 Effects of Mistakes (Defects) on MUF

Defects observed in checking source data for recording and/or calculation mistakes and the large or medium defects observed during attributes and variables inspection are treated in this section in order to determine their effect on MUF. It may be impossible to detect and eliminate all mistakes in recording or measurement when there are thousands of entries in a material balance period. One hundred percent inspection may also be impossible. Therefore it is desirable to require that the net effect of mistakes and defects on the facility MUF be evaluated.

Method 5.3 to follow provides a procedure to determine the quantitative effect of medium and large defects on MUF. Small mistakes in records are also considered by this method, however Method 5.3 will tolerate many more of the small defects than it will medium or large defects. When a mistake is found, the records for that item should be corrected. However, since only a sample of items are checked, the likelihood (or at least the possibility) exists that other mistakes still exist that will affect the facility MUF. In Method 5.3, the random variable Z is introduced. This is the adjusted sum of defects as they affect MUF in their combined effects. The mean and variance of Z are calculated under specific assumptions about the probability distribution of defects. A confidence interval may then be constructed about the true mean of Z to assess the overall impact of the defects on MUF in a probabilistic sense. The evaluation of defects is made to combat the diversion strategy in which "mistakes" are made to conceal diversion.

Method 5.3

Notation

N_k = number of items in stratum k

n_{ak} = sample size for attributes inspection in stratum k

n_{dk} = observed number of defects in stratum k

d_{ki} = facility value minus inspector value for the i-th defect in stratum k

U_k = largest value for d_{ki} in stratum k

L_k = smallest value for d_{ki} in stratum k

a_k = a constant, equal to +1 if stratum k is an input or beginning inventory stratum and equal to -1 otherwise.

Model

Under the assumption that mistakes are unintentional, it is assumed that d_{ki} is uniformly distributed in the interval from $x_{ok}-\theta_k$ to $x_{ok}+\theta_k$. Estimates are found for x_{ok} and for θ_k for each stratum in which defects are found in the inspection.

Results

Estimate x_{ok} and θ_k by the following equations.

$$x_{ok} = (L_k + U_k)/2 \qquad \text{(eq. 5.1.4)}$$

$$\theta_k = (U_k - L_k)/2(0.05)^{1/n_{dk}} \qquad \text{(eq. 5.1.5)}$$

These equations apply for $n_{dk} \geq 2$. For $n_{dk}=1$, set $x_{ok}= d_{k1}$ and θ_k is given by (eq. 5.1.5) with (U_k-L_k) replaced by $|d_{k1}|$. For $n_{dk}=0$, the stratum is not included in further calculations.

The quantity Z is given by

$$Z = \sum_{k=1}^{K} a_k \; h_k \; n_{dk} \; x_{ok} \qquad \text{(eq. 5.1.6)}$$

where $$h_k = (N_k - n_{ak})/n_{ak} \qquad \text{(eq. 5.1.7)}$$

if the n_{dk} mistakes are corrected for, or

$$h_k = N_k/n_{ak} \qquad \text{(eq. 5.1.7a)}$$

if the n_{dk} mistakes are not corrected for.

The variance of Z is

$$V(Z) = 1/3 \sum_{k=1}^{K} h_k^2 \; \theta_k^2 \; n_{dk} \qquad \text{(eq. 5.1.8)}$$

A two-standard deviation confidence interval on Z is given by

$$Z \pm 2\sqrt{V(Z)}$$

If Z is approximately normally distributed, which will occur in a limiting sense as a result of the central limit theorem, then the above confidence interval is an approximate 95 % interval. The interpretation of this interval is very similar to that for MUF or D as described in Section 5.2.3.1. In fact, the intervals for MUF, Z and D should be compared with each other. Z represents the result of the Agency's attribute verification of MUF and $\hat{D}$ is the result of the variables verification. If Z is much larger than MUF or D then one might conclude that the effect of mistakes (defects) is much greater than normal measurement errors. Under such circumstances the evaluation of MUF may not be meaningful until the frequency and magnitude of such defects can be brought under control at the facility level by improvements in accounting practices and measurement control programs. Of course, the Agency must also make sure that the observed mistakes (defects) are not caused by misapplication of their own verification procedures.

Basis

First, consider the basis for (eq. 5.1.4) and (eq. 5.1.5). Dropping the k subscript for simplicity, it is assumed that the size of a mistake is uniformly distributed between $x_o - \theta$ and $x_o + \theta$. The mean of this distribution is x_o, and it is estimated by the mid-range, (eq. 5.1.4). To estimate θ, equate the probability that all n_d observations will fall between L and U, given β=0.05, and solve for θ. The quantity θ is then the solution of the equation:

$$\left(\frac{U-L}{2\theta}\right)^{n_d} = 0.05 \qquad \text{(eq. 5.1.9)}$$

The solution is given by (eq. 5.1.5).

Given x_o, which represents the average difference per mistake in the stratum, the average difference per inspected item is

$$n_d x_o / n_a$$

The total number of uninspected items is $(N-n_a)$, so the extrapolated net effect on MUF of the mistakes in the stratum is given by

$$a\ (N-n_a) n_d x_o / n_a$$

if the observed mistakes n_d were corrected for.

Upon replacing $(N-n_a)/n_a$ by h and summing over the strata, the result is (eq. 5.1.6). However, h should be replaced by N/n_a if the observed mistakes n_d were not corrected for.

To compute the variance of Z, use the fact that the variance of d_i is, from reference [5.4].

$$V(d_i) = (2\theta)^2/12 = \theta^2/3$$

The variance of n_d such values is simply n_d times $V(d_i)$, and (eq. 5.1.8) follows immediately.

Examples

EXAMPLE 5.3 (a)

In the low enriched uranium fuel fabrication facility of Example 3.3 (a), say that the attribute sample sizes for the 7 strata are 470, 470, 3, 15, 8, 15, and 8 respectively. Review the following calculational mistakes detected during the inspection and make a judgement as to their acceptability.

Stratum	Facility	Inspector	d_{ki}
1	20.116	20.143	-0.027
	19.853	21.356	-1.503
	20.231	20.328	-0.097
	21.021	22.021	-1.000
	19.899	19.987	-0.088
	20.072	19.514	0.558
	19.847	19.184	0.663
	20.128	20.170	-0.042
	20.019	20.103	-0.084
2	5.015	4.599	0.416
	5.133	5.142	-0.009
	4.978	4.993	-0.015
	5.024	4.712	0.312
	4.961	4.950	0.011
4	4.120	4.348	-0.228
7	5.013	5.103	-0.090

The following values are found for L_k and U_k

$$L_1 = -1.503 \qquad U_1 = 0.663$$

$$L_2 = -0.015 \qquad U_2 = 0.416$$

From (eq. 5.1.4) and (eq. 5.1.5),

$$x_{01} = -0.420 \qquad \theta_1 = 1.511$$

$$x_{02} = \ 0.201 \qquad \theta_2 = 0.392$$

For strata 4 and 7,

$$x_{04} = -0.228 \qquad x_{07} = -0.090$$

$$\theta_4 = \ 2.280 \qquad \theta_7 = \ 0.900$$

Then, by (eq. 5.1.7),

$$h_1 = (12{,}000 - 470)/470 = 24.532$$

$$h_2 = (47{,}760 - 470)/470 = 100.617$$

$$h_4 = (1800 - 15)/15 = 119.000$$

$$h_7 = (800 - 8)/8 = 99.000$$

Z is calculated by (eq. 5.1.6)

$$Z = (24.532)(9)(-0.420) + (100.617)(5)(0.201) + (119.000)(1)(-0.228) + \\ + (99.000)(1)(-0.090) = -212 \text{ kg U}$$

(Note that Z is negative, a direction favorable to a diverter).

The variance of Z is calculated from (eq. 5.1.8)

$$V(Z) = {}^1/_3[(24.532)^2(1.511)^2(9) + \ldots + (99.000)^2(0.900)^2(1)]$$

$$= 101{,}698/3 = 33{,}899 \text{ kg}^2 \text{ U}$$

The two-standard deviation confidence interval is

$$-212 \pm 2\sqrt{33{,}899} = -212 \pm 368 \text{ kg U}$$

or $(-580, \ 156)$

Note that:

(1) The confidence interval embraces zero, i.e., there is no reason to believe that the net effect on MUF due to these mistakes differs from zero. Note, however, that $\sqrt{V(Z)}$ is larger than $\sqrt{V(MUF)}$ = 212 kg U from Example 3.5(a).

(2) The lower limit (-580), is smaller than the goal quantity for this type facility. Thus, these small mistakes in recording do not have a large effect on MUF.

5.1.2.2 Large Defects--Attributes Tester

For planning purposes, a gross or large defect was defined in Section 4.3.1 to be of size equal to the average element weight in the stratum, this weight being denoted by $\bar{x}_k$. However, for many attributes testers, a discrepancy need not be as large as $\bar{x}_k$ to be detected, since detection simply implies that the tester has determined that the facility value associated with the item in question is incorrect, with high probability. As an illustration, if the attributes test involves tipping a container to see if it has the listed amount of material in it, then quite clearly the defect need not be of size $\bar{x}_k$ (i.e., empty container) to be detected. Quite likely, if the weight were off by, say, 50% or more, this would be detected as a significant discrepancy, i.e., it would be classified as a defect.

The foregoing discussion leads to the following definition of a large defect.

Definition: In attributes inspection with the attributes tester, a defect is a discrepancy between the facility and the inspector values that is larger than $\gamma_k \bar{x}_k$ in absolute value.

To further explain this definition, if the tester in question has an associated error of measurement, then γ_k may be a multiple of the measurement error standard deviation. On a relative basis, let δ_k be the standard deviation of the attributes tester. Then, $\gamma_k=4\delta_k$ would be a reasonable value for γ_k since any discrepancy larger than 4 measurement error standard deviations is almost certainly a real discrepancy, one that cannot be explained as being due to measurement errors. This relationship between δ_k and γ_k assumes that the facility error of measurement is negligible in size compared with the inspector measurement error for the attributes tester. If this is not the case, then δ_k would be the standard deviation of the difference between the facility and the inspector value, i.e., would be the square root of the sum of the squared standard deviations for both parties.

This discussion may be summarized as Method 5.4, the procedure for defining a defect in attributes inspection with the attributes tester.

Method 5.4

Notation

$\bar{x}_k$ = average element weight for an item in stratum k

γ_k = a fractional value which, when multiplied by $\bar{x}_k$, defines a dispancy that is larger in absolute value than can be explained by errors of measurement

δ_{k_1} = relative random error standard deviation for facility measurement

δ_{k_2} = relative random error standard deviation for inspector measurement

Note: The definition for $\bar{x}_k$ implies that the characteristic being measured is the element weight. More generally, $\bar{x}_k$ relates to any measured characteristic being checked by the inspector, e.g., the enrichment being verified by a stabilized assay meter (SAM).

Model

When relating γ_k to δ_{k1} and δ_{k2}, it is implicitly assumed that a discrepancy due to measurement errors is normally distributed.

However, this is not an important assumption since the factor of 4 used in (eq. 5.1.10) has significance for distribution functions other than the normal distribution, i.e., it is quite unlikely that an observation will differ from the mean by more than 4 standard deviations even for non-normal populations that might occur.

Results

Assuming that the attributes tester in question has an associated measurement error, calculate

$$\gamma_k = 4(\delta_{k1}^2 + \delta_{k2}^2)^{1/2} \qquad \text{(eq. 5.1.10)}$$

Calculate the product $\gamma_k\bar{x}_k$. If a discrepancy exceeds this product in absolute value, it is labeled a defect. If a zero-acceptance sampling plan is being utilized, then discovery of at least one defect leads to "rejection" of the stratum in question, and whatever action such rejection implies (see Section 4.3.1). Procedures for evaluation of the effect of discrepancies on the material balance are given in Section 5.1.3, Confidence Interval for Defects and Section 5.1.2.1, Effect of Mistakes (Defects) on MUF.

If the particular attributes tester cannot be characterized as having an error of measurement in the strict sense of the word, assign a value to γ_k such that there is little question but that a discrepancy larger in absolute value than $\gamma_k\bar{x}_k$ is unexplainable as being due to errors in measurement.

Basis

For γ_k defined by (eq. 5.1.10), and assuming measurement errors to be normally distributed, a discrepancy as large as $\gamma_k\bar{x}_k$ would occur due to chance alone less than 7 times out of 100,000. Thus, a discrepancy larger in absolute value is correctly labeled a defect with very high probability.

Examples

EXAMPLE 5.4 (a)

With reference to example 4.1 (a), a total of 470 items (containers of UO_2 powder) are to be inspected. Suppose that the attributes inspection consists of verifying the enrichment with a stabilized assay meter (SAM), which has a relative error standard deviation of 0.025 (2.5% relative). The problem is to define a defect.

In applying (eq. 5.1.10), $\delta_{k2} = 0.025$. The value for δ_{k1} is negligibly small and may be ignored since the facility value is based on a mass spectrometer measurement. Therefore,

$$\gamma_k = (4)(0.025) = 0.10$$

To assign a value to $\bar{x}_k$, note from Example 3.6 (a) that the UO_2 powder is at 3 enrichments: 3.25%, 2.67%, and 1.52%. Therefore, 3 different defect criteria are established.

For the 3.25% containers, $\gamma_k \bar{x}_k$ = 0.325% U-235

For the 2.67% containers, $\gamma_k \bar{x}_k$ = 0.267% U-235

For the 1.52% containers, $\gamma_k \bar{x}_k$ = 0.152% U-235

5.1.2.3 Medium Defects--Variables Tester

The discussion in this section perhaps more logically belongs in Section 5.2 dealing with post-inspection activities as opposed to on-site activities because the final data analysis will probably have to depend on awaiting the results from the laboratory. However, the statistical aspects of this problem are closely related to those discussed in Section 5.1.2.2, and on that basis, it is reasonable to consider the subject at this time. Further, variables measurements may, in fact, be performed nondestructively so that in some cases, the evaluation discussed here could be performed on site.

As was the case with gross defects, there is a distinction between a medium defect as defined for inspection planning purposes and a medium defect defined in implementation. In planning, the inspector wants assurance that his sample size is large enough to combat the diverter's best strategy, that of falsifying given containers as much as feasible to escape detection with the attributes tester. From this viewpoint, defect size was related to the measurement error of the <u>attributes</u> tester. On the other hand, in implementation, any discrepancy that cannot be explained as being due to the measurement errors of the <u>variables</u> tester, and of the facility measurement, is labeled a defect.

Previous results may be used in defining a defect specifically for a given stratum. The computational formulas given by Methods 3.8, 3.9, and 3.10 are adapted to apply to a single item. Method 5.5 gives the procedure for defining a defect when the variables tester is used in the attributes mode.

Method 5.5

Notation

The notation is given in Sections 3.4.3.2 and 3.5.3.2. Further notation is as follows:

d_{ki} = facility value minus inspector value for element weight of item i in stratum k

x_{ikqpt} = facility value for element weight of item i in stratum k based on facility measurement method q for the bulk measurement, p for sampling, and t for analytical

y_{ikqpt} = defined as x_{ikqpt}, except it is an inspector value

Model

Under the hypothesis that discrepancies are due to measurement errors, it is implicitly assumed that a given discrepancy is normally distributed inasmuch as a discrepancy larger than four standard deviations is concluded to occur with very small probability due to chance alone. If the assumption is valid, then this probability can be calculated exactly. There is little reason to question the validity of this assumption in most applications but, in any event, it is not an important assumption. For any reasonable probability distribution likely to be encountered in this application, a four standard deviation discrepancy will occur with very low probability if measurement error is the only cause of the discrepancy. The actual value of this "low" probability is not really important.

Results

Calculate the following quantities:

$$V_{rx}(d_{ki}) = x^2_{ikqpt}\,(\delta^2_{rq\cdot\cdot x} + \delta^2_{r\cdot p\cdot x}/r_k + \delta^2_{r\cdot\cdot tx}/c_k r_k) \qquad \text{(eq. 5.1.11)}$$

$$V_{ry}(d_{ki}) = y^2_{ikqpt}\,(\delta^2_{rq\cdot\cdot y} + \delta^2_{r\cdot p\cdot y}/v_k + \delta^2_{r\cdot\cdot ty}/a_k v_k) \qquad \text{(eq. 5.1.12)}$$

$$V_r(d_{ki}) = V_{rx}(d_{ki}) + V_{ry}(d_{ki}) \qquad \text{(eq. 5.1.13)}$$

$$V_{gx}(d_{ki}) = x^2_{ikqpt}\,(\delta^2_{gq\cdot\cdot x} + \delta^2_{g\cdot\cdot tx}) \qquad \text{(eq. 5.1.14)}$$

$$V_{gy}(d_{ki}) = y^2_{ikqpt}\,(\delta^2_{gq\cdot\cdot y} + \delta^2_{g\cdot\cdot ty}) \qquad \text{(eq. 5.1.15)}$$

$$V_g(d_{ki}) = V_{gx}(d_{ki}) + V_{gy}(d_{ki}) \qquad \text{(eq. 5.1.16)}$$

$$V_{sx}(d_{ki}) = x^2_{ikqpt}\,(\delta^2_{sq\cdot\cdot x} + \delta^2_{s\cdot\cdot tx}) \qquad \text{(eq. 5.1.17)}$$

$$V_{sy}(d_{ki}) = y^2_{ikqpt}\,(\delta^2_{sq\cdot\cdot y} + \delta^2_{s\cdot\cdot ty}) \qquad \text{(eq. 5.1.18)}$$

$$V_s(d_{ki}) = V_{sx}(d_{ki}) + V_{sy}(d_{ki}) \qquad \text{(eq. 5.1.19)}$$

$$V(d_{ki}) = V_r(d_{ki}) + V_g(d_{ki}) + V_s(d_{ki}) \qquad \text{(eq. 5.1.20)}$$

If a discrepancy, d_{ki}, exceeds $4\sqrt{V(d_{ki})}$ in absolute value, it is labeled a defect. (Physically, $\sqrt{V(d_{ki})}$ is the standard deviation of a given difference. In view of the dominance of certain type errors, it may not be necessary to calculate all the above components; some may be ignored in practice). Procedures for the evaluation of the effect of discrepancies on the material balance are given in Section 5.1.3, Confidence Intervals for Defects and Section 5.1.2.1, Effect of Mistakes (Defects) on MUF.

Basis

If the assumption of normality is valid, then a discrepancy as large as $4\sqrt{V(d_{ki})}$ would occur due to chance alone less than 7 times out of 100,000. Thus, a discrepancy larger in absolute value is correctly labeled a defect with very high probability.

It is noted that the results were given on the basis that the item characteristic in question is the element weight. It is simple to modify the key equations should another characteristic be under investigation. Only those measurement operations that affect the difference statistic are, of course, included in the calculations. It may be that additional terms would be required, e.g., if the item characteristic is isotope weight rather than element weight. The results in Section 3.5.4, appropriately modified to apply to the single item, are pertinent in this event.

The calculations indicated by the equations seem quite involved. However, they may be applied quite easily by noting that: (1) for error propagation purposes, x_{ikqpt} and y_{ikqpt} are essentially identical so that various pairs of equations may be combined; (2) some error variances are zero and will hence not appear in the equations; and (3) the calculations for $V_r(d_{ki})$, $V_g(d_{ki})$, and $V_s(d_{ki})$ need not actually be performed; they are intermediate values leading up to the final (5.1.20).

Examples

EXAMPLE 5.5 (a)

The low enriched uranium fuel fabrication facility of example 3.3 (a) is inspected. In the UO_2 powder stratum, 36 measurements are made of item element weight, using the data of example 3.8 (a) (12 batches sampled with 3 items weighed per batch). Determine the defect criterion for the variables tester in the attributes mode. Use the nominal value of 20 kg U for x_{ikqpt} and y_{ikqpt} for all items.

The following parameter values are given in prior example. From example 3.3 (a), for the facility:

$$q = 1 \qquad p = 1 \qquad t = 1 \qquad k = 1$$

$$r_1 = 5 \qquad c_1 = 1$$

$$\delta_{r_1 \cdot \cdot x} = 0.000658 \qquad \delta_{r \cdot 1 \cdot x} = 0.000531 \qquad \delta_{r \cdot \cdot 1x} = 0.000433$$

From example 3.5 (a)

$$\delta_{s_1 \cdot \cdot x} = 0.000439 \qquad \delta_{s \cdot \cdot 1x} = 0.000571$$

From example 3.8 (a), for the inspector

$$q = 1 \qquad p = 1 \qquad t = 1$$

$$v_1 = 2 \qquad a_1 = 2$$

$$\delta_{r1 \cdot \cdot y} = 0.000658 \qquad \delta_{r \cdot 1 \cdot y} = 0.000531 \qquad \delta_{r \cdot \cdot 1y} = 0.000433$$

From example 3.9 (a),

$$\delta_{g\cdot\cdot 1y} = 0.000544$$

From example 3.10 (a),

$$\delta_{s1\cdot\cdot y} = 0.000439 \qquad \delta_{s\cdot\cdot 1y} = 0.000172$$

The equations, (eq. 5.1.11) - (eq. 5.1.20) are now applied to give the following results.

$$V_{rx}(d_{1i}) = 0.000211 \qquad V_{ry}(d_{1i}) = 0.000248 \qquad V_{r}(d_{1i}) = 0.000459$$

$$V_{gx}(d_{1i}) = 0 \qquad V_{gy}(d_{1i}) = 0.000118 \qquad V_{g}(d_{1i}) = 0.000118$$

$$V_{sx}(d_{1i}) = 0.000208 \qquad V_{sy}(d_{1i}) = 0.000089 \qquad V_{s}(d_{1i}) = 0.000297$$

$$V(d_{1i}) = 0.000874$$

$$4\sqrt{V(d_{1i})} = 0.118 \text{ kg U, defect criterion}$$

5.1.3 Confidence Intervals for Defects

5.1.3.1 Numbers of Defects

Upon completion of the attributes inspection, a confidence interval may be calculated for the number of defects in the population, (or, in the case of mistakes, the number of mistakes). This interval may be helpful in reaching a decision as to the need for further inspection in a given stratum. A procedure for estimating the effect of the defects on MUF is given in Section 5.1.2.1.

The methods for constructing a confidence interval are given. Method 5.6 applies when zero defects are observed. Method 5.7 is applicable for ≥ 1 observed defects.

Method 5.6

Notation

N = number of items in population

n = sample size

$1-\beta$ = confidence coefficient

U = upper limit of confidence interval

Model

The random variable is the number of defects found in a sample of n items selected at random from a population of N items containing a given number of defects. It is well known that this random variable follows a hypergeometric density function [5.5].

Results

With 100(1-β)% confidence, the upper limit on the number of defects in the population, given that no defects are observed in the sample, is

$$U = 0.5(1-\beta^{1/n})(2N-n+1) \qquad \text{(eq. 5.1.21)}$$

Basis

If U is the number of defects in the population of N items, find that value for U such that the probability of observing zero defects in the sample of size n is β. This defines the 100(1-β)% upper confidence limit on the number of defects.

The appropriate value for U is the solution of the equation:

$$\frac{\binom{U}{0}\binom{N-U}{n}}{\binom{N}{n}} = \beta \qquad \text{(eq. 5.1.22)}$$

The left hand side is

$$\frac{(N-U)!}{(N-U-n)!}\frac{(N-n)!}{N!} = \frac{(N-U)(N-U-1)\cdots(N-U-n+1)}{N(N-1)\cdots(N-n+1)}$$

$$= (1-\frac{U}{N})(1-\frac{U}{N-1})\cdots(1-\frac{U}{N-n+1})$$

These are n factors in this expression. The "middle" factor is

$$(1-\frac{2U}{2N-n+1})$$

so that the approximate value for β is

$$\beta \approx (1-\frac{2U}{2N-n+1})^n \qquad \text{(eq. 5.1.23)}$$

Solving this for U gives (eq. 5.1.21).

Examples

EXAMPLE 5.6 (a)

With reference to example 4.1 (b), in the mixed oxide powder stratum of the mixed oxide fuel fabrication facility, 48 items out of the 360 total items were inspected in attributes sampling. No defects were found. What is the 95% upper confidence limit on the number of defects in the remaining 312 items?

The parameter values are:

$N = 360 \qquad n = 48 \qquad \beta = 0.05$

By (eq. 5.1.21), the limit is

$$U = 0.5\ (1-0.05^{1/48})(720-48+1)$$
$$= 20.4, \text{ or } 21 \text{ items}$$

Method 5.7 is applicable when ≥ 1 defects are observed in the sample.

Method 5.7

Notation

N, n, $(1-\beta)$, and U as in Method 5.6

d = number of defects observed in the sample

Model

See Method 5.6

Results

Define $t_{1-\beta}$ by

$$\frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{t_{1-\beta}} e^{-z^2/2} dz = 1-\beta \qquad \text{(eq. 5.1.24)}$$

That is, the area under the standardized normal curve from $-\infty$ to $t_{1-\beta}$ is $(1-\beta)$.

Calculate the quantities z, x, A, B, C, and p using the following equations.

$$z = d+0.5 \qquad \text{(eq. 5.1.25)}$$

$$x = 1- (n-1)/(N-1) \qquad \text{(eq. 5.1.26)}$$

$$A = n^2 + t^2_{1-\beta}nx \qquad \text{(eq. 5.1.27)}$$

$$B = nz + 0.5t^2_{1-\beta}nx \qquad \text{(eq. 5.1.28)}$$

$$C = z^2 \qquad \text{(eq. 5.1.29)}$$

$$p = (B + \sqrt{B^2-AC})/A \qquad \text{(eq. 5.1.30)}$$

Then, the upper limit, U, is given by

$$U = [pN + 0.5]-1 \qquad \text{(eq. 5.1.31)}$$

where the brackets indicate that the largest integer contained in (pN+0.5) should be used. Note that since U is the limit on the number of defects in the lot, then (U-d) is the limit for the (N-n) uninspected items.

Basis

The normal approximation to the hypergeometric distribution forms the basis for this result. See reference [5.6].

Examples

EXAMPLE 5.7 (a)

With reference to example 5.6 (a), suppose that 2 defects were found in the sample of 48 items. What is the 95% upper limit on the number of defects in the remaining 312 items?

$N = 360$ $\quad n = 48$ $\quad \beta = 0.05$ $\quad d = 2$

$t_{1-\beta} = 1.645$ $\quad z = 2.5$ $\quad x = 0.8691$

$A = 2416.8867$

$B = 176.4434$

$C = 6.25$

$p = 0.1254$

$U = 45.14 -1 = 44$ defects, including those found in the sample.

$U-d = 42$, required limit

5.2 POST-INSPECTION ACTIVITIES

Generally speaking, the statistical activities discussed in Section 5.1 are those that would be performed on-site during the course of an inspection. In Section 5.2, further statistical analyses are described, those that would normally be performed after the physical inspection has been completed and the appropriate inspection data accumulated.

Not all of the statistical techniques described need be performed on any given inspection. The collection of techniques are those available to the inspector for his use in evaluating the data, and include those techniques most likely to be utilized, if not in connection with each phase of each inspection, certainly frequently enough to warrant inclusion here.

In Section 5.2.1, supplemental statistical tests of hypotheses are described (see Section 4.5.1 for the distinction between supplemental and primary tests). These tests include the test for normality, tests for randomness, variance tests, and the test for the significance of $\hat{D}_k$, the difference statistic in stratum k. The principal statistical tests are described in Section 5.2.2. These include the test on $\hat{D}$, the test on MUF, and the test on (MUF-$\hat{D}$). Finally, the construction of confidence intervals for the facility MUF and for the inspector's estimate of the facility MUF, (MUF-$\hat{D}$), are treated in Section 5.2.3.

5.2.1 Supplemental Tests of Hypotheses

5.2.1.1 Normality Tests

Various statistical procedures to be applied in the analyses of variables data are based on the assumption of normality. The effect of a failure in this assumption on the validity of the procedure depends on the nature and degree of non-normality and on the specific statistical procedure involved. Tests for normality are given in Methods 5.8 and 5.9. One of these tests may be applied whenever there is a question on the validity of this assumption. If the test results indicate that the assumption is of questionable validity, then guidance should be sought from a statistical expert to determine the effects of the non-normality on the statistical procedures to be applied. Guidance can also be given on the need to modify the procedures to account for the non-normality. Further, non-normality of the data in question may be indicative of unindentified variables affecting the results, and may therefore provide valuable insight into the structure of the data. For example, data falsifications introduced by a diverter, if not carefully and thoughtfully introduced, could perhaps be detected by a test of normality. Of course, it doesn't necessarily follow that detection of non-normality in the data signals diversion; it merely signals the need to perform a more careful investigation of the data in order to uncover the reasons for the departure from normality.

There are a number of statistical tests of normality that have been suggested. Some require large sample sizes and are therefore generally inappropriate for most situations likely to be encountered by the inspector. In the absence of knowledge about the alternative hypothesis, i.e., about the particular type of non-normality likely to exist, it is advisable to apply a statistical test that is generally sensitive to all kinds of non-normality. The so called W-test for normality is such a test. It is applicable for sample sizes up to 50, and is described in Method 5.8. For sample sizes larger than 50, a related test referred to as the D′ test may be applied. Method 5.9 describes this test.

Method 5.8

Notation

$x_1, x_2, \cdots, x_n$ = sample values for some random variable, ordered such that $x_1 < x_2 < \cdots < x_n$

Model

Under the null hypothesis, the data are normally distributed. The distribution under the alternative hypothesis is not specified.

Results

Using the coefficients a_{n-i+1} given in Annex 5.2, calculate

$$b = a_n(x_n - x_1) + a_{n-1}(x_{n-1} - x_2) + \cdots + a_{n-k+1}(x_{n-k+1} - x_k) \qquad \text{(eq. 5.2.1)}$$

where $k = n/2$ for even n

$= (n-1)/2$ for odd n

Calculate

$$S^2 = \sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2 / n \qquad \text{(eq. 5.2.2)}$$

(Note that S^2 is simply $(n-1)$ times the calculated variance of the x_i values.)

Calculate

$$w = b^2/S^2 \qquad \text{(eq. 5.2.3)}$$

A small value for W is indicative of non-normality. If W is less than the critical value given in Annex 5.3 for a given significance level, α, conclude that the data are not normally distributed.

Basis

The W-test for normality is covered in [5.7].

Examples

EXAMPLE 5.8 (a)

In the low enriched uranium fuel fabrication plant treated in previous examples, 19 UO_2 powder cans in stratum 1 are measured for total uranium by the inspector. His results are compared item by item with the facility values by noting the difference in uranium weights. These facility-minus-inspector values are given in grams uranium. Test the hypothesis that the differences are normally distributed, using a significance level α of 0.01.

-32	32	-31	-4
-10	-71	-20	-1
-72	40	42	0
8	33	12	-22
40	25	13	

The first step is to arrange the differences in descending order, i.e., from the largest to the smallest.

42	25	-1	-31
40	13	-4	-32
40	12	-10	-71
33	8	-20	-72
32	0	-22	

The a_{n-i-1} values are taken from Annex 5.2, and (eq. 5.2.1) is applied.

$$b = 0.4808(114) + 0.3232(111) + \cdots + 0.0303(9)$$
$$= 140.28$$

S^2 is then calculated using (eq. 5.2.2).

$$\sum_{i=1}^{19} x_i = -18 \qquad \sum_{i=1}^{19} x_i^2 = 21290$$

$$S^2 = 21272.95$$

From (eq. 5.2.3),

$$W = 0.925$$

From Annex 5.3, for $n = 19$ and $\alpha = 0.01$, the critical value is 0.863. Since 0.925 is greater than 0.863, do not reject the hypothesis that the differences are normally distributed.

For sample sizes greater than 50, the D´ test of Method 5.9 may be applied.

Method 5.9

Notation

The x_i are defined as for Method 5.8.

Model

See Method 5.8.

Results

Calculate

$$T = \sum_{i=1}^{n} [i-0.5(n+1)]\, x_i \qquad \text{(eq. 5.2.4)}$$

Calculate S^2 from (eq. 5.2.2)

Compute

$$D' = T/S \qquad \text{(eq. 5.2.5)}$$

If D′ lies between the critical values given in Annex 5.4, do not reject the hypothesis that the data have been sampled from a normal population. For an α - level of significance, the two critical values are in columns $P = \alpha/2$ and $P = 1-\alpha/2$.

Basis

The D′ test for normality is given in reference [5.8].

Examples

EXAMPLE 5.9 (a)

In the same context as in example 5.8 (a), 59 differences in amounts of uranium are recorded. The values, ordered from smallest to largest, are given below. Apply the D′ test for normality with the significance level $\alpha = 0.01$.

-100	-29	18	69	142	186
-95	-26	29	71	146	190
-76	-22	31	86	147	215
-69	-12	34	95	155	218
-68	-5	39	100	156	220
-62	-1	42	103	159	268
-50	2	45	107	163	298
-48	4	50	108	169	301
-39	5	50	116	180	344
-30	14	60	128	182	

From (eq. 5.2.4),

$$T = -29(-100) -28(-95) - \cdots + 29(344)$$

$$= 104{,}398$$

From (eq. 5.2.2),

$$\sum_{i=1}^{59} x_i = 4513 \qquad \sum_{i=1}^{59} x_i^2 = 1{,}004{,}439$$

$$S^2 = 659{,}232.75$$

$$S = 811.93$$

From (eq. 5.2.5),

$$D' = 104{,}398/811.93 = 128.56$$

For $\alpha = 0.01$, and $n = 59$, the lower critical value in the $P = 0.005$ column is 121.0 by interpolation. In the $P = 0.995$ column, the upper critical value is 130.1. Since D' falls between these two critical values, do not reject the hypothesis that the data have been sampled from a normal population.

5.2.1.2 Randomness Tests

Testing for randomness of a set of data is difficult because randomness is a characteristic that is difficult to define. It is simpler to approach the problem from the alternative viewpoint, i.e., to consider different types of non-randomness that may characterize the data. In the context of inspection, certain kinds of diversion strategies that include data falsification could create some types of non-randomness, e.g., the creation of sub-populations of falsified and non-falsified items. Further, tests for randomness could uncover assignable causes in the data, i.e., could lead to a better understanding of the data structure that would aid the statistical analysis.

The tests for normality discussed in Section 5.2.1.1 may be regarded as tests of randomness in the specific instance in which randomness is equated to normality. This equivalence between randomness and normality is often present with measurements data, and so the normality tests could have wide application here.

As another special case especially relevant to inspection, inspection of a flow stratum may take place over a period of time. Bias shifts in either the facility's or inspector's data could create non-randomness in the data. A similar statement applies to shipper-receiver data. Thus, a particular type of non-randomness that the inspector should be aware of is shifts in the population parameter values when data are ordered in time. This is the particular type of non-randomness addressed in this section.

Two methods are given to address this type of non-randomness. The first method is simple to apply and is very effective in providing a good visual impression of the data. It detects bias shifts very effectively but is partly subjective in that a quantitative measure of non-randomness is not produced. Often, this first method, involving a CUSUM (cumulative sum) plot is sufficient in detecting non-randomness due to bias shifts. If a more objective measure of non-randomness is needed, then the second method, involving an analysis of variance, may be applied. Collectively, these two methods plus the test for normality provide the tests needed to test for the most important types of non-randomness likely to characterize inspection data.

Method 5.10

Notation

x_i = observed value for the random variable, ordered in time or with respect to some other factor

Model

For the CUSUM plot as presented here, the inferences are made on the basis of a visual impression from the plot. This impression is generally adequate in making conclusions about the presence of parameter shifts. There are objective means for making inferences from CUSUM plots. These involve the use of V-masks and are not considered further here [5.9]. Such decision-making tools assume a certain structure in the data, commonly, that the observations are drawn at random from a normal distribution. When the plot is made just to create a visual impression, this assumption about the data structure is not required.

Results

From an inspection of the data, as based on some other considerations, select some central value for the data. In the case of difference data, this central value would be zero. The purpose of this central value selection is to force the plot to be nearly horizontal in the absence of shifts in the parameter value. Call the central value μ.

Calculate the difference values:

$$d_i = (x_i - \mu) \qquad \text{(eq. 5.2.6)}$$

Calculate the CUSUM values

$$s_i = s_{i-1} + d_i \qquad \text{(eq. 5.2.7)}$$

where $s_0 = 0$. Thus,

$$s_1 = d_1$$

$$s_2 = s_1 + d_2$$

$$s_3 = s_2 + d_3 \quad \text{, etc.}$$

Using an equal spacing along the abscissa, plot the s_i values. Observe the plot to determine if more than one line segment is needed to connect the plotted points. If so, conclude that there is non-randomness exhibited by the data in the sense that the parameter describing central tendency is not constant (bias shift).

Basis

As long as the mean of the x_i values remains constant, all the plotted points may be connected visually by a straight line segment. If there is a shift in the mean, then this straight line will change direction, or, stated alternately, more than one line segment will be needed. If the central value, μ, is wisely chosen, and if there are no shifts in the parameter value (i.e., in the mean of the x_i values), the slope will be zero.

Examples

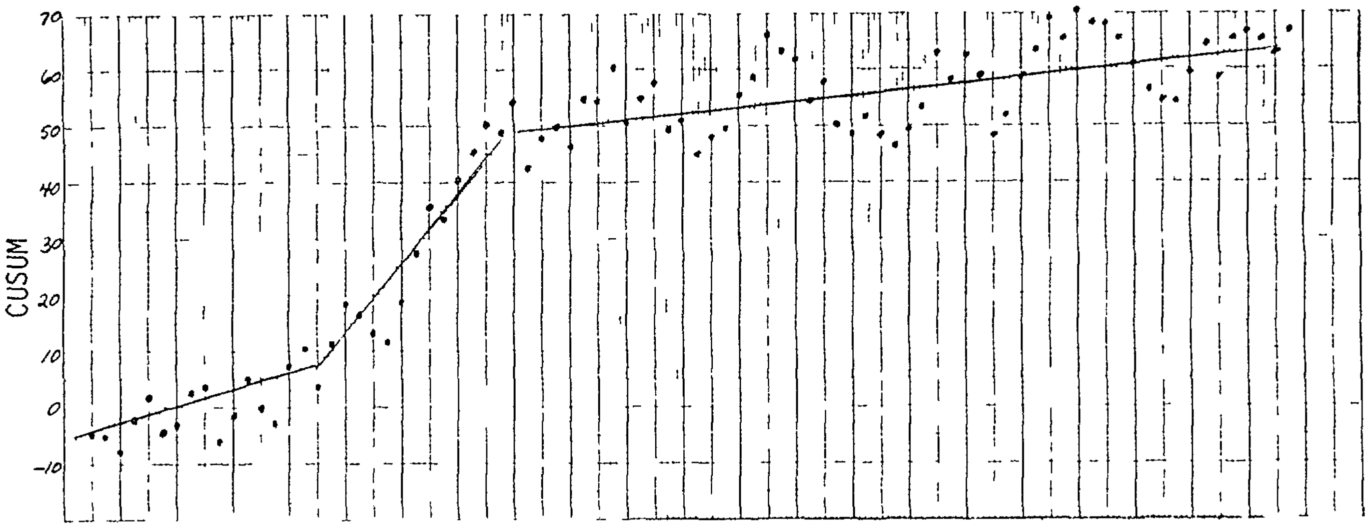
EXAMPLE 5.10 (a)

Percents uranium in cans of dirty UO_2 scrap powder are listed below in container number order. Construct a CUSUM plot for these data. (Data ordered by columns.)

73.33	85.41	81.10	84.41
77.63	86.31	79.62	83.26
75.18	86.24	83.76	83.33
83.55	76.03	81.36	74.79
81.93	85.45	85.64	82.96
71.93	82.08	74.95	75.87
79.49	82.92	76.44	77.45
83.52	76.76	70.76	75.45
78.92	83.25	81.39	73.32
68.24	66.24	70.35	73.35
82.42	83.31	76.26	76.40
84.84	79.92	81.40	77.65
72.73	74.46	74.28	83.06
75.47	87.02	76.14	83.23
88.17	77.44	80.98	72.07
81.04	84.02	82.32	84.64
71.26	67.99	87.64	79.46
85.70	82.30	73.04	76.38
85.25	80.69	82.26	75.39
75.84	69.91	74.49	82.73
74.74	79.07	67.08	
76.45	72.34	81.96	

Upon inspection of the data, a reasonable central value, μ, is 78. The table below lists the d_i values for the first few observations using (eq. 5.2.6), and all the s_i values, using (eq. 5.2.7)

x_i	d_i	s_i	s_i	s_i	s_i
73.33	-4.67	-4.67	19.04	47.89	58.42
77.63	-.37	-5.04	27.35	49.51	63.68
75.18	-2.82	-7.86	35.59	55.27	69.01
83.55	5.55	-2.31	33.62	58.63	65.80
81.93	3.93	1.62	41.07	66.27	70.76
		-4.45	45.15	63.22	68.63
		-2.96	50.07	61.66	68.08
		2.56	48.83	54.42	65.53
		3.48	54.08	57.81	60.85
		-6.28	42.32	50.16	56.20
		-1.86	47.63	48.42	54.60
		4.98	49.55	51.82	54.25
		-0.29	46.01	48.10	59.31
		-2.82	55.03	46.24	64.54
		7.35	54.47	49.22	58.61
		10.39	60.49	53.54	65.25
		3.65	50.48	63.18	66.71
		11.35	54.78	58.22	65.09
		18.60	57.47	62.48	62.48
		16.44	49.38	58.97	67.21
		13.18	50.45	48.05	
		11.63	44.79	52.01	



The data plot suggests quite strongly that the data divide into at least three groups. There is also some indication of cyclic behavior in the plotted points within the last grouping, but this is not as evident; objective tests would be required to determine if the non-randomness that appears to exist within the last group is real in a statistical sense.

Having detected this apparent non-randomness in the data, one would try to find an explanation for it. Perhaps the samples are taken and then stored for a brief time prior to performing the analyses on a campaign basis. Any analytical shifts could explain the non-randomness detected. It is the results of the investigation triggered by the CUSUM plot that are of primary interest, and not merely the fact that such non-randomness was detected. The aim, of course, is to determine if data falsification is the explanation, or if there are more innocent assignable causes that should be identified and corrected if possible.

When the data are grouped according to some criteria external to the data themselves, the analysis of variance previously given as Method 2.4 may be applied to determine if there are significant differences among the group means. In Method 2.4, the analysis of variance was presented as a method of estimating measurement error variance components. In the current problem situation, the analysis of variance is given from point of view of testing for significant differences among group means. The problem solution is presented as Method 5.11.

Method 5.11

Notation

The notation is consistent with that of Method 2.4 but is repeated here in part and redefined to meet the present problem situation.

x_{ij} = value for j-th observation in group i; i=1, 2, $\cdots$, m

n_i = number of observations in group i

Model

x_{ij} is normally distributed about μ_i, the mean for group i, and with variance σ^2, constant for all groups. Under the hypothesis, μ_i is the same for all i.

Results

Following Method 2.4, calculate the quantities:

T_i, T, S_0, S_1, S_2, M_B, and M_W

Form the F ratio:

$$F = M_B/M_W \qquad \text{(eq. 5.2.8)}$$

Select a value for α, the significance level of the test. For $\alpha = 0.05$, 0.025, and 0.01, enter the table in Annex 5.5 (a), 5.5 (b), or 5.5 (c) respectively. If F exceeds the tabled value for degrees of freedom for the numerator, (m-1), and degrees of freedom for the denominator, (n-m), conclude that $\mu_i \neq \mu$ for all i, i.e., reject the hypothesis that the group means are the same.

Basis

The statistical technique that forms the basis for this method is the one way analysis of variance which is described in many texts. See, for example, [5.10].

Examples

EXAMPLE 5.11 (a)

In Example 5.10 (a), say that the percent uranium values were determined in four analytical compaigns, i.e., during four distinct periods of operation in the laboratory. The data are divided into groups as follows:

Group 1: 73.33 - 71.26 ($n_1 = 17$)

Group 2: 85.70 - 83.25 ($n_2 = 14$)

Group 3: 66.24 - 81.96 ($n_3 = 35$)

Group 4: 84.41 - 82.73 ($n_4 = 20$)

Following Method 2.4:

$T_1 = 1329.65$	$S_0 = 533761.2853$
$T_2 = 1142.43$	$S_1 = 533902.8742$
$T_3 = 2727.93$	$S_2 = 536132.4325$
$T_4 = 1575.20$	$M_B = 47.1963$
$T = 6775.21$	$M_W = 27.1897$

From (eq. 5.2.8),

$$F = 47.1963/27.1897 = 1.736$$

Choose $\alpha = 0.05$ and enter the table of Annex 5.5 (a). The tabled value for 3 degrees of freedom in the numerator and 82 degrees of freedom in the denominator is 2.74, by interpolation. Since the F value of 1.736 does not exceed this tabled value, do not reject the hypothesis that the group means are equal.

Some further comments are in order because this conclusion seems to conflict with that reached in Example 5.10 (a) on the basis of the CUSUM plot. First, it was pointed out that the CUSUM plot provide a visual impression of the data, and in this particular example, is quite suggestive of a real difference in group means. However, the statistical significance of that conclusion is not stated. On the basis of this current example, it appears that the conclusion of a significant difference is not warranted at the $\alpha = 0.05$ significance level; the actual value for α, it turns out, would be slightly larger than $\alpha = 0.10$. The up-and-down pattern of the data plot is an indication of a large random error variance or in this case, of a rapidly shifting bias, which tends to obscure the differences in the group means.

Further, from the data plot, in observing the slopes of the line segments, it is apparent that group means 1, 3, and 4 are nearly equal whereas group mean 2 is relatively high. The analysis of variance tests the global hypothesis that all group means are equal; if three are in fact equal while one differs somewhat, the analysis of variance may not detect this fact so readily. In fact, had groups 1, 3, and 4 been combined and tested against group 2, then M_B and M_W would have been

$$M_B = 133.0415 \qquad M_W = 26.6441$$

and F with 1 and 84 degrees of freedom is 4.993, a significant result at $\alpha = 0.05$. Here is an example in which the CUSUM plot was perhaps more sensitive in detecting the <u>particular alternative hypothesis</u> that apparently exists; with the analysis of variance, the alternative hypothesis is not specified.

By way of summary, the four group means are

$$\bar{x}_1 = 78.21 \qquad \bar{x}_3 = 77.94$$

$$\bar{x}_2 = 81.60 \qquad \bar{x}_4 = 78.76$$

5.2.1.3 Variance Tests

In advance of an inspection, the facility provides values for its random and systematic error variances. These values are used in planning the inspection and also in the data analysis if, in fact, the inspection data substantiate the stated values. This section addresses the problem of verifying that the facility's stated values for the random error variances are valid. The method makes use of the facility minus inspection data.

Method 5.12

Notation

d_i = difference between facility and inspector values for specified characteristic and stratum

n = number of d_i values

σ_0^2 = facility's stated random error variance for given characteristic and stratum

σ_1^2 = corresponding inspector's value

Model

The quantity d_i is normally distributed with arbitrary mean and variance equal to $(\sigma_0^2+\sigma_1^2)$ under the null hypothesis that σ_0^2 and σ_1^2 are correctly stated.

Results

Calculate

$$S_d^2 = \sum_{i=1}^{n} d_i^2 - \left(\sum_{i=1}^{n} d_i\right)^2/n \qquad \text{(eq. 5.2.9)}$$

Calculate the chi-square statistic

$$\chi^2_{n-1} = S_d^2/(\sigma_0^2 + \sigma_1^2) \qquad \text{(eq. 5.2.10)}$$

Select a value for α, the significance level of the test. Enter the table in Annex 5.6 under the columns headed $\chi^2_{\alpha/2}$ and $\chi^2_{1-\alpha/2}$, and for degrees of freedom (df) equal to (n-1). If χ^2_{n-1} of (eq. 5.2.10)falls between the two tabled values, do not reject the hypothesis, i.e., conclude that the sum, $(\sigma_0^2+\sigma_1^2)$, is correct.

In the event the hypothesis is rejected, conclude that σ_0^2 misrepresents the actual variance for the facility. This conclusion assumes that the inspector can properly assign the σ_1^2 value.

Basis

The statistical technique that forms the basis for this method is the chi-square test on the variance which is described in many texts (see, for example, [5.11]).

Of underlying importance to this test is the validity of the assumption that the inspector's value, σ_1^2, is correct. This should pose no problem because of the wide base of experience upon which the σ_1^2 value will generally be based. However, the problem can be circumvented under some circumstances, i.e., it is possible to test the hypothesis about the validity of σ_0^2 irrespective of the value for σ_1^2.

The procedure for making this test, and the conditions under which the alternate test is preferred, are given in reference [5.12].

Example

EXAMPLE 5.12 (a)

Consider the data of Example 5.8 (a). These data represent differences between facility and inspector measurements of total uranium in containers of low enriched UO_2 powder.

From example 3.3 (a),

$$\sigma_0^2 = [(.000950)(20000)]^2 = 361.00 \ g^2U$$

From example 3.8 (a),

$$\sigma_1^2 = 361.00 \ g^2U$$

Then, again from example 5.8 (a),

$$s_d^2 = 21272.95$$

From (eq. 5.2.10), the chi-square value is

$$\chi_{18}^2 = 21272.95/722.00 = 29.46$$

At $\alpha = 0.05$, from Annex 5.6, the tabled critical values are 8.23 and 31.53. Since 29.46 lies between these two numbers, do not reject the hypothesis that $(\sigma_0^2+\sigma_1^2) = 722.00$.

5.2.1.4 Test on $\hat{D}_k$ and on Shipper/Receiver Differences

The final supplemental test to be considered is the test on $\hat{D}_k$, the difference statistic in stratum k. Although primary emphasis in the analysis of the variable inspection data is on the difference statistic appropriately summed over all strata, there are occasions when one might wish to test for the significance of $\hat{D}_k$ in a given stratum. As a prime example of this, it was pointed out in Section 4.5.3 that a shipper-receiver difference test is formally equivalent to a test on $\hat{D}_k$, and one is interested in routinely performing such tests on shipper-receiver differences.

The $\hat{D}_k$ test was considered in Section 4.5.3 from point of view of its probability of detection. In this current section, emphasis is on how the significance test is performed.

Method 5.13

Notation

The notation is consistent with that given in Methods 3.8 - 3.10.

Model

The random variable, $\hat{D}_k$, is assumed to be normally distributed with variance $V(\hat{D}_k)$ and with mean zero under the null hypothesis.

Results

Calculate $V(\hat{D}_k)$ as the sum of three variance components

$$V(\hat{D}_k) = V_r(\hat{D}_k) + V_g(\hat{D}_k) + V_s(\hat{D}_k) \qquad \text{(eq. 5.2.11)}$$

where $V_r(\hat{D}_k)$ is given by (eq. 3.5.5), and where $V_g(\hat{D}_k)$ and $V_s(\hat{D}_k)$ are given respectively by (eq. 3.5.14) and (eq. 3.5.21), in each case the calculations being performed for only the single stratum.

Having observed $\hat{D}_k$, compute the statistic

$$t = \hat{D}_k / \sqrt{V(\hat{D}_k)} \qquad \text{(eq. 5.2.12)}$$

Select a value for α, the significance level of the test, and find the corresponding critical value, t_α, from Annex 5.7. Then,

(a) if stratum k is an input or beginning inventory stratum, reject the hypothesis that the mean of $\hat{D}_k$ is zero if $t<-t_\alpha$;

(b) if stratum k is an output or ending inventory stratum, reject the hypothesis if $t>t_\alpha$.

Basis

This is the standard test on the mean of a normally distributed random variable with known variance [5.13]. The test is constructed as a one-sided test because from a diversion viewpoint, only those values of $\hat{D}_k$ that differ from zero in one direction favor the diverter. Of course, a large value of $\hat{D}_k$ in the opposite direction would also be called to the attention of the facility as evidence of a malfunctioning measurement system.

Examples

EXAMPLE 5.13 (a)

In example 3.8 (b), the inspection sample sizes are given for the mixed oxide fuel fabrication facility. In the PuO_2 powder stratum, 16 of the 24 strata are randomly selected with 6 items weighed per batch and with 3 samples drawn per batch to determine percent plutonium. Duplicate analytical determinations are made. For the resulting 96 paired differences, say that

$\bar{d}_1 = -0.00468$ kg Pu, so that, from (eq. 3.5.1)

$\hat{D}_1 = -(768)(0.00468) = -3.594$

At a significance level, $\alpha = 0.010$, does this represent a significant difference?

From the cited example,

$$V_r(\hat{D}_1) = 0.296817 + 0.621036 = 0.917853 \text{ kg}^2\text{Pu}$$

From example 3.9 (b), the calculations for the short term systematic error variance are performed for stratum 1. Example 3.9 (b) made use of earlier calculations performed in Example 3.4 (a). In the expression for $V_g(\hat{D}_1)$ given below, the first two terms come from example 3.4 (a) and the last term from example 3.9 (b).

$$V_g(\hat{D}_1) = 0.012124 + 1.329070 + (0.0016)^2[2(768)^2]$$

$$= 4.361093 \text{ kg}^2\text{Pu}$$

From example 3.10 (b), the calculations for the long term systematic error variance are performed for stratum 1. Example 3.10 (b) made use of earlier results from example 3.5 (b). In the expression for $V_s(\hat{D}_1)$ given below, the first two terms come from example 3.5 (b) and the last two from example 3.10 (b).

$$V_s(\hat{D}_1) = 0.094372 + 1.156055 + (1656.6)^2(0.00030)^2+(1536)^2(0.0012)^2$$

$$= 4.894802 \text{ kg}^2\text{Pu}$$

The (eq. 5.2.11) may now be applied

$$V(\hat{D}_1) = 0.917853+4.361093+4.894802 = 10.173748 \text{ kg}^2\text{Pu}$$

From (eq. 5.2.12),

$$t = -3.594/\sqrt{10.173748} = -1.127$$

At $\alpha = 0.010$, $t_\alpha = 3.090$. Since -1.127 is not less than -3.090, conclude that the facility measurements of plutonium are not significantly smaller than the inspector measurements for this input stratum, i.e., there is no reason to conclude that the input values are understated.

5.2.1.5 International Standards of Accountancy [5.15]

In Section 3.4 the variance of MUF, V (MUF), is used to test the hypothesis that the true MUF is equal to some stated value (e.g., zero). V (MUF) may be calculated from the operator's design information data on his random and systematic random error variances. The inspector may verify these error estimates using inspection results, estimation methods described in Chapter 2 and the Chi-square test of Section 5.2.1.3. Depending on the outcome of these many possible tests the inspector may calculate his own estimate of V (MUF) in order to determine whether the operator's system of measurements conform to international standards or be equivalent in quality to such standards.

Standards do not exist for each measurement method and material combination. Instead the Agency has defined such standards for V (MUF) which are considered achievable in practice at bulk facilities of each identified type. For example, at uranium fabrication facilities the international standard of accountancy for σ_{MUF} is 0.3 %, expressed as a percentage of the larger of inventory or throughput. This represents a maximum expected value for σ_{MUF} that is easily achievable by the majority of facilities using well known measurement methods that are widely used in the industry.

EXAMPLE 5.13(a)

From example 3.3(a) we see that the outputs of 238800 + 1200 = 240000 kg U are much larger than beginning or ending inventory. In this case, the standard of accountancy would be

$$\sigma_{MUF} = (0.003)\ (240000) = 720 \text{ kg U}$$

and the calculated value of $\sqrt{V\ (MUF)}$ from Example 3.5(a) is

$$\sqrt{V\ (MUF)} = \sqrt{45010} = 212 \text{ kg U.}$$

Since 212 < 719 kg U we conclude that the operator's system of measurements is well within this international standard. In the event that $\sqrt{V\ (MUF)}$ is greater than the standard the test of MUF in Section 11.9 should be based on the standard value rather than the calculated value and the state should be informed that the system of measurements does not meet the maximum expected values for σ_{MUF}.

A similar approach can be used for testing the quality of the inspector's system of measurements as represented by V ($\hat{D}$). In fact this may be more important than checking V (MUF) since V ($\hat{D}$) is the primary measure of the Agency's verification accuracy. Safeguards effectiveness based on nuclear material accountancy will be limited by this verification accuracy as the size and throughput of bulk handling facilities increase.

5.2.2 Principal Tests of Hypotheses

In assessing the safeguards performance of a facility, the key measures are $\hat{D}$, MUF, and MUF-$\hat{D}$. The observed values of these quantities and their calculated variances form the basis for both the principal tests of hypotheses discussed in Section 4.5.1 and the estimation of the true MUF. This section describes the performance of significance tests, and the next section describes the construction of confidence intervals estimates.

5.2.2.1 Test on D

The $\hat{D}$ test was considered in Section 4.5.4 from point of view of its detection probability. In this current section, emphasis is on how the significant test is performed.

Method 5.14

Notation

The notation is that used in Methods 3.8 to 3.10.

Model

The random variable, $\hat{D}$, is assumed to be normally distributed with zero mean and with variance $V(\hat{D})$ under the null hypothesis.

Results

Calculate $V(\hat{D})$ as the sum of three variance components:

$$V(\hat{D}) = V_r(\hat{D}) + V_g(\hat{D}) + V_s(\hat{D}) \qquad \text{(eq. 5.2.13)}$$

where $V_r(\hat{D})$, $V_g(\hat{D})$, and $V_s(\hat{D})$ are given in (eq. 3.5.6), (eq. 3.5.14), and (eq. 3.5.21) respectively.

Compute

$$t = \hat{D}/\sqrt{V(\hat{D})} \qquad \text{(eq. 5.2.14)}$$

Select a value for α, the significance level of the test. If $t<-t_\alpha$, conclude that there is a significant difference between the facility and inspector results. The value for t_α comes from Annex 5.7.

Basis

The basis is the same as for the $\hat{D}_k$ test of Method 5.13. For $\hat{D}$, a large negative value favors the diverter and so the test is a one-sided test against that alternative.

Examples

EXAMPLE 5.14 (a)

Continue with the mixed oxide facility of example 5.13 (a). This relates to the inspection described in example 3.8 (b). At $\alpha = 0.025$, how large must $\hat{D}$ be in a negative direction in order to reject the hypothesis that its mean is zero?

From (eq. 5.2.14) and Annex 5.7, the hypothesis in question is rejected if

$$\hat{D}<-1.960\sqrt{V(\hat{D})}$$

The quantity $V(\hat{D})$ for this example was calculated in example 3.10 (b).

$$V(\hat{D}) = 27.303719 \text{ kg}^2\text{Pu}$$

Therefore, the hypothesis will be rejected if

$$\hat{D} < -1.960 \sqrt{27.303719} \text{ , or if}$$

$$\hat{D} < -10.242 \text{ kg Pu}$$

5.2.2.2 Test on MUF

The test on MUF was considered in Section 4.5.5 from point of view of its detection probability. In this current section, emphasis is on how the significance test is performed.

Method 5.15

Notation

The notation is given in Section 3.4.3.2.

Model

The random variable, MUF, is assumed to be normally distributed with zero mean and with variance V(MUF) under the null hypothesis.

Results

Calculate V(MUF) as the sum of three variance components.

$$V(MUF) = V_r(MUF) + V_g(MUF) + V_s(MUF) \qquad \text{(eq. 5.2.15)}$$

where V_r(MUF), V_g(MUF), and V_s(MUF) are given in (eq. 3.4.3), (eq. 3.4.7), and (eq. 3.4.11) respectively.

Compute

$$t = MUF/\sqrt{V(MUF)} \qquad \text{(eq. 5.2.16)}$$

Select a value for α, the significance level of the test. If t exceeds t_α from Annex 5.7, reject the hypothesis that the mean of MUF is zero.

Basis

The basis is the same as for the $\hat{D}_k$ test of Method 5.13. For MUF, a large positive value is an indication of diversion and so the test is one-sided against that alternative.

Examples

EXAMPLE 5.15 (a)

Consider the MUF for the mixed oxide fuel fabrication facility. Values for V_r(MUF), V_g(MUF), and V_s(MUF) were calculated in Examples 3.3 (b), 3.4 (a), and

3.5 (b) respectively. At $\alpha = 0.050$, how large must MUF be in order to reject the hypothesis that its mean is zero?

From the cited examples, as summarized in example 3.5 (b),

$$\sqrt{V(MUF)} = 3.276 \text{ kg Pu}$$

For $\alpha = 0.050$, $t_\alpha = 1.645$ so that the hypothesis is rejected if MUF exceeds

$$(1.645)(3.276) = 5.389 \text{ kg Pu}$$

5.2.2.3 Test on (MUF-$\hat{D}$)

The test on (MUF-$\hat{D}$) was considered in Section 4.5.6 from point of view of its probability of detection. In this current section, emphasis is on how the significance test is performed.

Method 5.16

Notation

The notation is that of Method 3.13.

Model

The random variable, (MUF-$\hat{D}$), is assumed to be normally distributed with zero mean and with variance V(MUF-$\hat{D}$) under the null hypothesis.

Results

Calculate V(MUF-$\hat{D}$) from (eq. 3.6.5). Compute

$$t = (MUF-\hat{D})/\sqrt{V(MUF-\hat{D})} \qquad \text{(eq. 5.2.17)}$$

Select a value for α, the significance level of the test. If t exceeds t_α from Annex 5.7, reject the hypothesis that the mean of (MUF-$\hat{D}$) is zero.

Basis

The basis is the same as for the $\hat{D}_K$ test of Method 5.13. For (MUF-$\hat{D}$), a large positive value is an indication of diversion and so the test is one-sided against that alternative.

Examples

EXAMPLE 5.16 (a)

Consider the low enriched uranium fuel fabrication facility of example 3.13 (a). At $\alpha = 0.050$, how large must (MUF-$\hat{D}$) be in order to reject the hypothesis that its mean is zero?

From the cited example,

$$V(MUF-\hat{D}) = 40{,}983 \text{ kg}^2\text{U}$$

For $\alpha = 0.050$, $t_\alpha = 1.645$ so that the hypothesis is rejected if $(MUF-\hat{D})$ exceeds

$$(1.645)\sqrt{40{,}983} = 333 \text{ kg U}$$

EXAMPLE 5.16 (b)

Suppose that in the previous example, $(MUF-\hat{D})$ were calculated for U-235 rather than for uranium. Then, since

$$V^*(MUF-\hat{D}) = 33.6843 \text{ kg}^2\text{U-235}$$

from Example 3.13 (a), the critical value is

$$1.645\sqrt{33.6843} = 9.547 \text{ kg U-235}$$

This example illustrates how Methods 5.13-5.16, written in terms of element weights, may easily be adapted to relate to isotope weights. The changes in calculations are obvious.

EXAMPLE 5.16 (c)

Consider the mixed oxide fuel fabrication facility of example 3.13 (b). At $\alpha = 0.050$, how large must $(MUF-\hat{D})$ be in order to reject the hypothesis that its mean is zero?

From the cited example,

$$V(MUF-\hat{D}) = 21.035443 \text{ kg}^2\text{Pu}$$

For $\alpha = 0.050$, $t_\alpha = 1.645$ so that the hypothesis is rejected if $(MUF-\hat{D})$ exceeds

$$1.645\sqrt{21.035443} = 7.545 \text{ kg Pu}$$

This completes the discussion on tests of hypothesis for the various statistics derived from the inspection data. The final topic on the analysis of inspection data is the construction of confidence intervals.

5.2.3 Confidence Intervals on Material Unaccounted For

The values of MUF and $MUF-\hat{D}$ are estimates of the true material unaccounted for calculated from facility data and facility data plus inspection data, respectively. These observed values and their calculated variances contain all the information about the magnitude of the true MUF available to the inspector. In addition to the specific tests of the hypothesis of no diversion (true MUF equal to zero) described in the preceding section,it is revealing to display the total information about material unaccounted for upon completion of the analysis of facility and inspection data. In particular, the estimated magnitude of a significant diversion relative to the goal quantity M used as a basis for inspection planning may be of interest.

A convenient way to convey this information is through the determination of a confidence interval estimate of the true MUF and a graphical presentation of this interval as it relates to both the absence of any diversion (true MUF equal 0) and the diversion of a significant quantity (true MUF equal M). In Section 5.2.3.1 the facility MUF forms the basis for construction of the confidence interval while in Section 5.2.3.2 the basis is the adjusted estimate MUF-$\hat{D}$ based on inspection data.

5.2.3.1 Confidence Interval Based on MUF

Method 5.17

Notation

The notation is given in Section 3.4.3.2. Further, M is the goal amount.

Model

The random variable, MUF, is assumed to be normally distributed with variance V(MUF). The observed MUF is assumed to be the estimate of the true unknown MUF in the sense that its expected value is the true MUF. (No small biases or data falsifications.)

Results

Calculate V(MUF) by (eq. 5.2.15). Choose a value for the confidence coefficient (1-α) and compute

$$L = MUF - t_{1-\alpha/2}\sqrt{V(MUF)} \qquad \text{(eq. 5.2.18)}$$

and

$$U = MUF + t_{1-\alpha/2}\sqrt{V(MUF)} \qquad \text{(eq. 5.2.19)}$$

where $t_{1-\alpha/2}$ is read from Annex 5.8 for given (1-α).

To display the results graphically, draw a horizontal line scaled in amounts of element. Let the smallest value on the left be min (0,L) and the largest value on the right be max (M,U). Indicate the values for 0, L, U, and M on this line by appropriately marking the line, and connect the L and U marks.

There are six possible ways in which the 0, L, U, and M marks may be ordered. (Note that L must be less than U and 0 is less than M.) These possibilities are listed below along with a narrative description of their interpretations. In the first three cases the test of the hypothesis of a true MUF less than or equal to zero would be rejected.

Case	Ordering	Description
1	0-L-U-M	The true MUF is greater than zero, but is less than the goal amount.
2	0-L-M-U	The true MUF is greater than zero but not greater than the goal amount.
3	0-M-L-U	The true MUF is greater than the goal amount.
4	L-0-M-U	The uncertainty in the estimate of MUF is large and little definitive can be said about the true MUF relative to zero or to the goal amount.
5	L-0-U-M	The true MUF is less than the goal amount and not greater than zero.
6	L-U-0-M	The true MUF is less than zero.

Basis

The method is based on the construction of confidence limits on the mean of a normally distributed random variable with known variance [5.14].

Examples

EXAMPLE 5.17 (a)

Suppose that for the mixed oxide fuel fabrication facility of example 5.15 (a), the MUF were 4.212 kg Pu and the goal quantity were 8 kg Pu. Then, from the cited example,

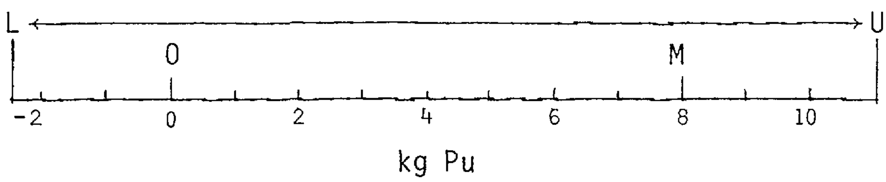
$$\sqrt{V(MUF)} = 3.276 \text{ kg Pu}$$

The limits, L and U, are calculated from (eq. 5.2.18) and (eq. 5.2.19) for $(1-\alpha) = 0.95$.

$$L = 4.212 - (1.960)(3.276) = -2.209 \text{ kg Pu}$$

$$U = 4.212 + (1.960)(3.276) = 10.633 \text{ kg Pu}$$

The horizontal line is drawn extending from -2.209 kg Pu to 10.633 kg Pu, and 0, L, U, and M are indicated on this line.



This is case 4; little definitive can be said about the true MUF relative to zero or to the goal amount.

5.2.3.2 Confidence Interval Based on $(MUF-\hat{D})$

Method 5.18

Notation

The notation is that of Method 3.13. M is the goal amount.

Model

The random variable $(MUF-\hat{D})$ is assumed to be normally distributed with variance $V(MUF-\hat{D})$. The observed $(MUF-\hat{D})$ is assumed to be the estimate of the true unknown MUF in the sense that its expected value is the true MUF (facility MUF corrected for biases or data falsifications).

Results

Calculate $V(MUF-\hat{D})$ from (eq. 3.6.5). Choose a value for the confidence coefficient $(1-\alpha)$ and compute the lower and upper confidence limits:

$$L = (MUF-\hat{D}) - t_{1-\alpha/2}\sqrt{V(MUF-\hat{D})} \qquad \text{(eq. 5.2.20)}$$

$$U = (MUF-\hat{D}) + t_{1-\alpha/2}\sqrt{V(MUF-\hat{D})} \qquad \text{(eq. 5.2.21)}$$

where $t_{1-\alpha/2}$ is read from Annex 5.8 for given $(1-\alpha)$.

Proceed as in Method 5.17 to display the results graphically.

Basis

The basis is the same as for Method 5.17.

Examples

EXAMPLE 5.18 (a)

Suppose that for the mixed oxide fuel fabrication facility of example 5.17 (a), D = -3.617 kg Pu. Then,

$$(MUF-\hat{D}) = 4.212 + 3.617 = 7.829 \text{ kg Pu}$$

For this facility, from example 5.16 (c),

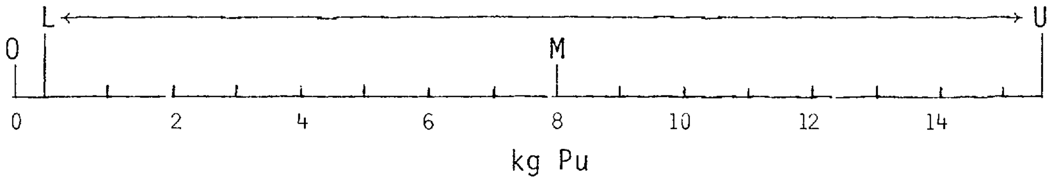
$$V(MUF-\hat{D}) = 21.035443 \text{ kg}^2\text{Pu}$$

From (eq. 5.2.20) and (eq. 5.2.21), for $(1-\alpha) = 0.90$,

$$L = 7.829 - 1.645\sqrt{21.035443} = 0.284 \text{ kg Pu}$$

$$U = 7.829 + 1.645\sqrt{21.035443} = 15.374 \text{ kg Pu}$$

The horizontal line is drawn extending from 0 to 15.374 kg Pu, and 0, L, U, and M are indicated on this line.



This is case 2. The true MUF is greater than zero but the uncertainty in the estimate of the MUF is very large and precludes making a definitive statement about the relationship of the true MUF to the goal amount.

EXAMPLE 5.18 (b)

In the low enriched uranium fuel fabrication facility of example 5.16 (b), the goal amount is 75 kg U-235. Method 5.18 is adapted to apply to isotope rather than element MUF.

Say that the value for MUF is 21.224 kg U-235 while $\hat{D}$ = -11.355 kg U-235. Then,

$$(\text{MUF}-\hat{D}) = 32.579 \text{ kg U-235}$$

From example 5.16 (b),

$$V^*(\text{MUF}-\hat{D}) = 33.6843 \text{ kg U-235}$$

Applying (eq. 5.2.20) and (eq. 5.2.21) for $(1-\alpha) = 0.95$,

$$L = 32.579 - 1.960\sqrt{33.6843} = 21.204 \text{ kg U-235}$$

$$U = 32.579 + 1.960\sqrt{33.6843} = 43.954 \text{ kg U-235}$$

The horizontal line is drawn extending from 0 to 75 kg U-235, and 0, L, U, and M are indicated on this line.

This is case 1. The true MUF is estimated quite precisely and is both greater than zero and considerably less than the goal amount.

5.3 REFERENCES

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[5.13] Bennett, C. A., and Franklin, N. L., op cit, 142-148.

[5.14] Bennett, C. A., and Franklin, N. L., op cit, 137-140.

[5.15] IAEA Safeguards Glossary, IAEA/SG/INF/1, Vienna, 1980, para. 100.

Random Numbers (*cont.*)

6063	2353	8531	8892	8977	5782	2283	1385	0699	5927
6305	1326	4551	2815	8913	2908	0698	5509	4303	9911
0143	0187	8127	2026	8313	8341	2479	4722	6602	2236
1031	0754	7989	4948	1804	3025	0997	9562	3674	7876
2022	3227	2147	5613	2857	8859	4941	7274	9412	0620
9149	0806	9751	8870	9677	9676	1854	8094	7658	7012
5563	0513	1402	3866	8696	9142	6063	2252	7818	2477
5724	0806	9644	8284	7010	0868	9076	4915	5751	9214
6753	4207	2958	5295	3175	3396	8117	5918	1037	4319
0962	1620	4690	0036	9654	4078	1918	8721	8454	7671
9394	2466	6427	5395	9393	0520	7074	0634	5578	4023
3220	3058	7787	7706	4094	5603	3303	8300	6185	8705
1491	3503	0584	7221	6176	0116	0309	1975	0910	3535
4368	5705	8579	5790	7244	6547	8495	7973	1805	7251
2325	4026	2919	8327	0267	2616	6572	8620	8245	6257
0591	1775	5134	8709	7373	3332	0507	5525	7640	2840
3471	1461	1149	6798	6070	9930	1862	3672	6718	3849
2600	9885	6219	3668	1005	5418	5832	0416	4220	4692
9572	7874	6034	4514	2628	1693	0628	2200	9006	3795
0822	2790	9386	5783	2689	2565	1565	0349	3410	5216
4329	3028	2549	2529	9434	3083	6800	8569	9290	8298
9289	5212	2355	9367	1297	1638	9282	3720	7178	2695
3932	9960	3399	1700	8253	1375	4594	6024	1223	5383
2282	0648	7561	7528	5870	7907	0713	8608	9682	8576
9933	3416	5957	2574	5553	5534	4707	3206	0963	2459
9015	6416	6603	2967	7591	5013	2878	8424	5452	4659
1539	0719	2637	9969	8450	4489	3528	3364	1459	9708
6549	5595	7969	2582	5627	1920	9772	8560	0892	6500
2523	7769	3536	9611	1079	1694	1254	4195	5799	5928
0701	7355	0587	8878	3446	1137	7690	0647	1407	6362
2163	8543	4594	6022	0496	8648	2999	1262	6702	0811
0327	5727	1070	5996	8660	9024	2135	9799	8414	9136
2169	3160	8707	6361	6339	4054	3251	7397	3480	5805
8393	8147	5360	4150	2990	3380	1789	7436	4781	0337
9726	9151	2064	0609	5878	9095	9737	2897	6510	8891
0515	2296	2636	9756	5313	7754	0916	6066	3905	1298
0549	8398	5614	0140	3155	2211	4988	3674	7663	0620
0026	9426	8005	8579	5774	7962	5092	5856	1626	0980
3422	0092	1626	1298	2475	1997	9796	7076	1541	1731
8191	1983	9164	1885	5468	8216	4327	8109	5880	9804
7408	0486	7654	4829	2711	6592	4785	5901	7147	9314
8261	9440	8118	6338	8157	9052	9093	8449	4066	4894
9274	8838	8342	3114	0455	6212	8862	6701	0099	0501
2699	0383	1400	3484	1492	4683	5369	3851	5870	0903
8740	0349	3502	3971	9960	6325	6727	4715	2945	9938
0247	2372	0424	0578	0036	1619	4479	7108	8520	1487
5136	9444	8343	1152	3615	1420	8923	7307	3978	5724
4844	8931	0964	2878	8212	9328	2656	1965	4805	0634
0205	8457	4333	2555	5353	9201	1606	2715	4014	1877
2517	5061	7642	3891	7713	7066	5435	1200	7455	5562

Random Numbers (*cont.*)

8529	7631	7050	2275	4383	0162	1937	0302	7109	9024
4272	3581	6632	5942	9513	6119	7721	7033	4632	6904
6009	1247	3898	2058	6466	8697	9562	3254	9644	8076
8714	0867	1189	9909	5113	7899	7558	2765	5076	2377
5862	5635	9795	6127	2872	5351	3380	2010	2836	4794
2834	0662	4423	2226	8886	4902	6038	3214	9241	1735
4537	0620	3480	6208	6784	4835	4567	5961	1202	8426
4090	3275	7424	3820	0106	5045	9145	7824	8365	5893
0827	6802	9469	4171	7282	4667	1838	5054	5145	5126
9104	5561	0764	6940	6960	2201	9636	0020	1845	1992
6395	0676	0014	3881	1247	3897	1951	2503	2864	6120
5860	4900	5410	2079	3125	7185	3593	3680	1846	9771
4869	7923	3859	1031	4559	2451	3592	3470	9728	3145
9482	9765	1065	6314	9731	7905	8245	6366	6283	7678
5106	6509	7110	1452	2693	2047	9311	9242	3977	5411
4932	9177	1521	8351	9323	6138	0147	6597	2757	2492
6979	7002	5668	8975	4593	5470	3210	7432	9644	8830
6392	0329	6775	9257	5190	0858	9602	5415	7922	3333
8115	4891	9133	8740	0349	3183	3662	0602	9763	6661
3962	2652	9426	8003	7631	7262	8127	1919	9353	9323
6760	6323	5492	6287	9442	4659	1136	7479	7883	1716
6216	2155	0832	7787	8022	2445	7512	2459	5158	6907
9684	9605	6838	9176	0684	3202	5595	8180	5352	9092
4656	5208	5255	1654	0591	2192	0675	3609	7364	7593
5664	6896	1992	4789	8101	8005	8682	6839	9601	5515
3987	2522	7665	1561	8164	2106	3559	9274	8626	2168
4786	1354	7318	0697	1381	6380	9580	7420	1848	1367
2430	6737	8440	8489	5691	9136	4691	0560	8603	4510
8616	5854	9854	0481	5877	8567	8239	8512	4199	7558
6225	3511	5336	8899	6965	8279	0529	9118	8078	4301
6654	3405	9538	4950	3170	5234	4458	2951	2062	4167
5303	4532	7187	8899	6975	0474	2187	7372	4547	9630
7486	2727	9740	2195	2112	8781	2406	8221	7598	7981
4999	3648	8313	8612	0078	1904	6313	9176	0997	9562
0680	1020	8141	3907	2033	7323	7096	7287	3041	4010
7401	8220	4851	3593	3678	1271	2563	4197	6717	3325
0066	2090	2961	0765	7437	5197	7929	9274	1368	9618
1882	4099	9294	4850	7082	8577	4762	8229	2863	5313
8231	3741	4893	9976	6201	2429	6105	5312	7335	4463
9402	9619	1779	3147	7140	1983	9160	0027	1270	1678
1637	8438	7649	1448	0273	5717	1825	7941	8345	2317
2103	2096	9248	9294	4958	3694	0085	6993	9408	3834
4465	4803	8562	8795	4919	9770	6864	8402	8091	5789
3852	2250	6978	6770	9076	4797	8036	1515	2497	1095
0793	5024	0470	8933	1793	2906	8995	0565	7505	7778
7311	5316	9218	3850	5241	4709	4709	4254	7349	6769
1779	2728	7148	9941	8777	0434	9693	5512	2251	7733
6105	5105	3599	7389	0183	2034	7642	3995	9766	4578
8698	2587	2087	7638	1872	0418	6340	9451	6937	4530
3937	3988	2038	1144	2640	1631	4008	1921	4546	9104

Random Numbers *(cont.)*

2271	2572	8665	3272	9033	8256	2822	3646	7599	0270
3025	0788	5311	7792	1837	4739	4552	3234	5572	9885
3382	6151	1011	3778	9951	7709	8060	2258	8536	2290
7870	5799	6032	9043	4526	8100	1957	9539	5370	0046
1697	0002	2340	6959	1915	1626	1297	1533	6572	3835
3395	3381	1862	3250	8614	5683	6757	5628	2551	6971
6081	6526	3028	2338	5702	8819	3679	4829	9909	4712
3470	9879	2935	1141	6398	6387	5634	9589	3212	7963
0432	8641	5020	6612	1038	1547	0948	4278	0020	6509
4995	5596	8286	8377	8567	8237	3520	8244	5694	3326
8246	6718	3851	5870	[illegible]	2107	1387	1621	5509	5772
7825	8727	2849	3501	3501	1001	0123	7873	5926	6078
6258	2450	2962	1183	3666	4156	4454	8239	4551	2920
3235	5783	2701	2378	7460	3398	1223	4688	3674	7872
2525	9008	5997	0885	1053	2340	7066	5328	6412	5054
5852	9739	1457	8999	2789	9068	9829	1336	3148	7875
0440	3769	7864	4029	4494	9829	1339	4910	1303	9161
0820	4641	2375	2542	4093	5364	1145	2848	2792	0431
7114	2842	8554	6881	6377	9427	8216	1193	8042	8449
6558	9301	9096	0577	8520	5923	4717	0188	8545	8745
0345	9937	5569	0279	8951	6183	7787	7808	5149	2185
7430	2074	9427	8422	4082	5629	2971	9456	0649	7981
8030	7345	3389	4739	5911	1022	9189	2565	1982	8577
6272	6718	3849	4715	3156	2823	4174	8733	5600	7702
4894	9847	5611	4763	8755	3388	5114	3274	6681	3657
2676	5984	6806	2692	4012	0934	2436	0869	9557	2490
9305	2074	9378	7670	8284	7431	7361	2912	2251	7395
5138	2461	7213	1905	7775	9881	8782	6272	0632	4418
2452	4200	8674	9202	0812	3986	1143	7343	2264	9072
8882	3033	8746	7390	8609	1144	2531	6944	8869	1570
1087	9336	8020	9166	4472	8293	2904	7949	3165	7400
5666	2841	8134	9588	2915	4116	2802	6917	3993	8764
9790	2228	9702	1690	7170	7511	1937	0723	4505	7155
3250	8860	3294	2684	6572	3415	5750	8726	2647	6596
5450	3922	0950	0890	6434	2306	2781	1066	3681	2404
5765	0765	7311	5270	5910	7009	0240	7435	4568	6484
8408	1939	0599	5347	2160	7376	4696	6969	0787	3838
8460	7658	6906	9177	1492	4680	3719	3456	8681	6736
4198	7244	3849	4819	1008	6781	3388	5253	7041	6712
9872	4441	6712	9614	2736	5533	9062	2534	0855	7946
6485	0487	0004	5563	1481	1546	8245	6116	6920	0990
2064	0512	9509	0341	8131	7778	8609	9417	1216	4189
9927	8987	5321	3125	9992	9449	5951	5872	2057	5731
4918	9690	6121	8770	6053	6931	7252	5409	1869	4229
8099	5821	3899	2685	6781	3178	0096	2986	8878	8991
1901	4974	1262	6810	4673	8772	6616	2632	7891	9970
8273	6675	4925	3924	2274	3860	1662	7480	8674	4503
2878	8213	3170	5126	0434	9481	7029	8688	4027	3340
6088	1182	3242	0835	1765	8819	3462	9820	5759	4189
5773	6600	5306	0354	8295	0148	6608	9064	3421	8570

Random Numbers *(cont.)*

2742	6731	3741	4890	8818	3208	3171	5755	2301	5517
9112	8964	7544	8932	1281	2355	5563	1638	7331	2387
0462	1288	6055	6983	0294	5271	4846	1094	4234	5361
4616	3678	1168	1044	6235	6534	5263	9716	3890	6999
4976	3383	9718	1256	9764	0393	7132	6904	2607	4733
9852	1453	9473	2914	4262	7582	9644	7969	3711	8334
7535	0446	3178	0206	5295	2991	5923	3215	8824	9362
7469	6789	4044	7141	1840	6288	7101	0746	2695	7019
1886	1976	1856	9117	7133	7428	9700	0223	3753	0230
6268	0733	1136	7479	7881	0953	5499	1158	4729	1442
2097	9561	2830	0500	3695	4707	4083	8432	0611	7583
3735	1763	8090	1457	8894	0426	0433	9165	4050	1432
9296	5868	7340	8139	7202	0465	9028	7987	3800	4555
8527	5288	8545	9063	3106	4875	5945	3058	7682	5313
7866	9022	0844	5307	0612	8102	8529	7529	6106	5733
9954	5248	6636	4988	3463	2461	5451	4552	3026	1554
2613	8184	5800	6939	5684	7008	0950	0891	6856	0228
2668	4852	8006	8893	6596	9808	3623	9518	3102	6927
3002	2998	5357	6634	7099	2299	0198	4755	0795	3344
8422	4435	0134	4587	4886	2413	2774	6136	8993	2721
3690	2492	7171	7720	6509	7549	2330	5733	4730	1759
0813	6790	6858	1489	2669	3743	1901	4971	8280	1127
6477	5289	4092	4223	6454	7632	7577	2816	9202	0811
0772	2160	7236	0812	4195	5589	0830	8261	9232	5944
5692	9870	3583	8997	1533	6466	8830	7271	3809	3291
2080	3828	7880	0586	8482	7811	6807	3309	2729	8636
1039	3382	7600	1077	4455	8806	1822	1669	7501	5622
7227	0104	4141	1521	9104	5563	1392	8238	4882	0861
8506	6348	4612	8252	1062	1757	0964	2983	2244	0253
5086	0303	7423	3298	3979	2831	2257	1508	7642	3891
0092	1629	0377	3590	2209	4839	6332	1490	3092	2590
0935	5565	2315	8030	7651	5189	0075	9353	1921	9572
2605	3973	8204	4143	2677	0034	8601	3340	8383	2818
7277	9889	0390	5579	4620	5650	0310	2082	4664	0494
5484	3900	3485	0741	9069	5920	4326	7704	6525	2503
6905	7127	5933	1137	7583	6450	5658	7678	3444	5163
8387	5323	3753	1859	6043	0294	5110	6340	9137	6137
4094	4957	0163	9717	4118	4276	9465	8820	4127	2954
4951	3781	5101	1815	7068	6379	7252	1086	8919	5677
9047	0199	5068	7447	1664	9278	1708	3625	2864	1351
7274	9512	0074	6677	8676	0222	3335	1976	1645	3914
9192	4011	0255	5458	6942	8043	6201	1587	0972	3391
0554	1690	6333	1931	9433	2661	8690	2313	6999	3436
9231	5627	1815	7171	8036	1832	2031	6298	6073	7121
3995	9677	7765	3194	3222	4191	2734	4469	8617	6694
2402	6250	9362	7373	4757	1716	1942	0417	5921	7456
5295	7385	5474	2123	7035	9983	5192	1640	6176	3818
5177	1191	2106	3351	5057	0967	4538	1246	3374	3286
7315	3365	7203	1231	0546	6612	1038	1425	2709	5780
5775	7517	8974	3961	2183	5295	3096	8536	9442	3139

Annex 5.1 (c)

Random Numbers (*cont.*)

5500	2276	6307	2346	1285	7000	5306	0414	3383	2137
3251	8902	8843	2112	8567	8131	8116	5270	5994	7445
4675	1435	2192	0874	2897	0262	5092	5541	4014	2086
3543	6130	4247	4859	2660	7852	9096	0576	0097	4746
3521	8772	6612	0721	3899	2999	1263	7017	8057	4983
5573	9396	3464	1702	9204	3389	5678	2589	0288	4633
7478	7569	7551	3380	2152	5411	2647	7242	2800	6183
3339	2854	9691	9562	3252	9848	6030	8472	2266	1270
5505	8474	3167	8552	5409	1556	4247	4652	2953	5394
6381	2086	5457	7703	2758	2963	8167	6712	9820	5654
6975	5239	0762	5846	2431	0543	4956	8787	9651	2605
7185	4019	7332	2820	4853	8636	9505	6575	0365	6648
4510	1658	5615	2194	1901	4975	1895	4383	0415	3771
7752	0105	4769	2994	7445	0781	4960	4253	9451	6518
4834	4043	6591	3646	8918	4603	1970	9145	7615	3905
8866	6036	9755	4508	9061	2080	3406	9856	1298	6281
6622	4612	2030	7299	8414	8822	5176	9443	6054	6462
9094	8973	3335	2183	5192	1630	0959	8143	9182	8012
5618	6445	2983	0375	2540	2735	4901	5515	4787	7058
2705	2693	1944	8074	2015	3261	5529	7193	5401	9531
1797	4334	3293	2632	3770	1675	9363	7795	3331	8995
9448	5174	5869	0448	8613	4400	6938	5161	8691	2838
3461	1304	9682	8577	4449	1896	8328	1698	7138	1141
7092	5007	5596	8522	2580	4495	4728	8946	4434	2438
5533	4294	0939	4050	1225	6414	5895	0148	7053	5935
7852	8988	5951	4919	7404	2426	4450	2358	3082	4561
8313	8456	9892	0981	6736	8021	6226	5573	1664	9489
1158	2241	9861	7588	2669	5480	9160	4267	1690	7278
9238	7226	0025	8844	8181	5565	2418	9394	0837	3106
7711	1336	3251	8902	8425	5766	3262	5848	3545	7073
2656	1863	3884	6516	6922	1808	1896	8853	0964	3089
7980	9370	2850	3818	7281	8352	9637	0618	2430	6525
1409	7865	5908	4296	1888	2792	4014	1667	1295	0814
7657	6630	5000	1493	5459	5869	0315	8134	9587	2184
2863	5450	1329	8787	8795	4604	2615	0075	1433	7707
3988	2042	2906	8995	0818	9288	1650	0803	8319	2533
4551	2815	8941	4893	8612	4844	0042	3890	7069	8512
5772	4732	2829	3931	9540	6256	5420	2179	9448	5489
9150	1435	3817	8975	4276	9569	0175	6663	0045	5549
5764	7914	8280	1337	3779	8197	9105	5985	1054	2866
5895	0044	5021	3846	7599	0398	5212	9509	0134	4656
6857	1174	8085	6503	5355	3027	1708	3626	7059	0167
2538	2669	3746	3270	1214	9983	8434	1344	1160	3292
9983	1387	1410	8891	2523	8705	9190	2986	7654	5142
5061	9529	2922	2199	8310	6954	8090	5371	0672	6281
9999	4226	2815	8817	5606	5190	0495	7867	9968	5951
9078	5936	2393	7875	6871	3163	9203	2863	5693	9973
4823	2291	8925	6306	1717	0320	2549	3107	5488	0303
1232	1384	5698	9313	3501	3238	7227	0220	6118	7655
7694	6484	0279	8528	7214	1750	0577	8418	0698	5403

Random Numbers (*cont.*)

9207	6903	9703	2028	3460	0778	3795	0698	3974	8522
1886	2080	3719	3602	3896	1214	9862	1969	6782	9327
6963	4197	6405	8683	7573	0842	9306	2596	7404	9999
1797	2315	5434	0787	3809	9129	4511	0708	2181	9119
6534	5578	4158	6256	3721	7515	3905	1905	7153	3552
2325	4238	8861	6098	8837	7690	0497	6848	6601	1553
6598	4628	1023	9747	4660	3437	7414	7609	9938	8335
4592	5016	4434	7133	7216	4602	1690	7914	8819	3600
1765	8822	5278	2324	3715	0431	7780	4955	9683	8998
6139	3275	7731	3351	5306	0323	5387	3901	4151	2922
3911	8334	5465	6647	8773	7456	9954	5141	3573	5570
6840	0366	6962	3462	1724	6661	7221	6074	9262	3461
5572	8838	8132	9398	0737	7125	7388	7686	9814	1760
2357	5303	3720	3917	7238	9925	7940	7818	1676	9780
3138	6014	4909	1143	7551	3380	2713	7649	2784	0175
1921	9046	6300	7460	8271	5547	6701	0098	8559	0490
6735	5991	6857	0756	1943	7657	6399	1052	1816	7694
6089	1958	8748	8310	6953	7775	9274	1579	2548	7028
1767	9764	0628	5037	1549	9048	0644	3850	5555	6104
1748	3753	1385	0381	9998	4075	7729	2301	5831	3510
7529	4195	5691	2192	0770	0900	2608	5048	5751	9322
9717	9635	7972	1198	6177	0745	2199	8205	4559	9159
9063	3318	6412	4632	6904	2713	7258	3089	7970	4058
0915	5751	9112	1040	0561	9128	4285	5149	3002	2868
6456	8457	3348	4322	6493	8656	7532	1384	6198	1236
8700	0874	3291	7324	4471	5375	6769	2113	0950	0682
8050	3921	0741	8814	4262	7477	7042	7027	4275	2143
6663	2373	0845	8119	6862	7477	7148	9840	7154	4495
3648	5449	3166	7921	3125	9884	5739	1775	4927	5394
4238	9071	4578	2589	0076	6636	7915	8739	4907	6195
6015	5642	4642	2904	7841	4004	7299	8310	6953	7986
1088	7727	1060	9589	3315	9763	6349	5029	6314	9714
2027	6143	5814	9032	9856	1146	5864	6943	8234	0896
1929	8697	3478	5184	6468	9513	6333	1699	7348	6139
7651	4561	1036	9359	6992	8957	0203	8199	9733	7220
1629	0688	9118	7121	7870	5877	2438	1767	9065	4264
5650	0473	0543	5273	5716	1335	0448	8490	6438	4090
0604	0231	1836	4004	5457	8963	7017	8226	9427	8318
0602	9868	2575	8785	8517	8975	4276	9677	2064	0599
5913	4536	4377	5975	3607	6362	6467	5009	1142	7238
9111	0238	9718	4707	3767	0311	2967	1839	5788	9242
7787	7707	4838	5910	3042	4640	2299	4400	4446	0423
5731	3738	3526	2380	8557	8789	9577	1122	8310	7057
4827	9203	2758	2946	5915	0771	1426	2919	8014	4156
1653	8923	5386	3684	7891	5200	9884	5545	2742	7205
3561	7178	2486	1353	6928	9463	7229	1545	7616	4224
8301	8225	7398	4084	8697	9670	7404	9893	1467	3681
5658	7221	6074	8944	6446	7141	2731	6380	4621	2703
4819	1007	6258	4628	0705	0757	4546	8906	7679	4180
9239	0523	3369	6234	6115	1524	8575	3502	3971	9376

Annex 5.1 (d)

Random Numbers (*cont.*)

0356	6390	2107	3875	4488	2911	1727	8108	3484	6370
3686	8812	8754	2758	3079	2994	3642	1580	1475	0366
4195	4602	1481	7324	8570	6913	6228	1934	6165	0554
3180	5460	0134	4469	8619	7723	8084	3293	1895	4886
1498	7883	5280	0692	7202	1273	3334	1554	3303	8569
9428	8633	9606	7679	4182	4035	6849	5593	6712	9822
9630	5879	9342	9618	8513	4399	9734	7744	4600	0224
1086	8918	7713	5909	2620	6612	0616	1298	2476	2386
2478	3551	1247	8004	0301	6672	6176	0682	2493	6381
2508	1133	5853	8737	9804	2404	7400	5904	8803	0377
6934	2047	4963	4531	6391	9064	3526	2482	9328	5556
1156	1191	1182	3032	8640	4681	3932	6975	4926	4870
5677	7494	0987	8870	4837	5267	4119	4163	1953	3553
3719	3586	5775	7309	5111	0919	7721	7032	1164	2105
6556	8472	1848	1056	3670	7509	0854	7210	9336	8127
1246	3476	4027	3654	2444	9040	5331	2363	4738	9822
6591	3387	4109	7956	5837	6914	6435	2624	8610	4005
3197	9026	4868	6372	2695	7143	2783	1925	3383	9060
5035	4569	7158	8531	8891	0975	6329	1329	8746	0989
1563	9650	2139	7696	7511	1725	7292	0664	8440	8593
6034	4512	1505	3857	0290	3270	8389	9612	1892	8707
2435	0238	6478	5727	0862	1621	5228	5038	2000	0433
9418	4486	5992	7172	8353	6516	6605	6387	8126	1603
3116	1295	0563	6475	4382	9902	6621	9209	8060	1787
5426	5517	5603	3722	4965	5892	8135	5214	9877	6429
2494	6696	5881	1198	2055	4624	4592	4788	7477	7149
1362	2650	8867	6503	5250	7622	5989	5909	2623	7875
5622	8415	9553	7882	1402	4723	7101	1917	8305	0440
6687	5386	9837	9111	8123	3859	1134	6321	4756	1325
0045	5546	2340	7068	6692	3802	8740	0563	8253	1589
3441	4562	1126	6427	7674	6564	1996	9167	4995	6200
9354	3914	6037	7309	5111	3080	3616	2152	2426	4450
8655	6422	1264	7859	3622	8979	7253	4257	5523	4808
0143	0292	0220	2205	4773	4964	5055	5460	0240	7505
5860	4714	6437	3670	5781	1131	7609	9690	3736	7266
8400	6939	5684	7116	3472	4006	1069	5272	5209	8271
3262	4214	5901	1064	7064	4286	1038	1178	3658	4628
2220	1426	2920	8956	8142	4642	3008	9816	5548	7753
9734	7954	9700	1489	3213	8400	7043	7552	4019	9938
3178	1061	8942	8397	4898	3793	6603	2864	6014	5225
4189	6015	5328	8242	0427	1270	1992	4789	8075	7632
4774	5282	1202	5496	8949	8940	9032	6872	3581	6631
9541	6606	6881	4916	5257	3207	9530	4546	9880	0479
4560	8877	8779	1690	6959	1916	2049	7214	0761	5111
2719	2098	7631	2574	5660	8600	2922	1570	6442	8082
7081	8366	4236	6582	9193	4328	8842	1588	1391	7714
2300	5410	2186	6846	4440	6180	6021	5258	3080	3723
4090	3091	2193	1295	0563	6579	6249	9151	1959	8949
2656	1861	2833	0067	2726	3697	5862	6058	8434	1240
9465	8924	6068	1461	0656	2718	1468	5401	9638	0931

Random Numbers (*cont.*)

0430	2613	7878	5595	8076	3068	4113	7962	4989	3987
6095	8618	7024	5831	8995	0902	3972	4894	2153	5764
6720	4690	2123	7101	1166	0087	3042	4745	4690	3173
8948	4609	9126	2814	8383	2951	1958	8212	4131	5313
3268	0189	9175	0473	4295	2993	7027	4153	3864	7542
5473	1540	5681	3933	7826	3823	9933	3101	6742	2364
2951	2169	3096	8663	8147	5255	1756	0650	8493	7024
1031	4981	9118	7969	2688	8229	2863	5320	6124	9920
6010	3219	2640	2737	6055	7298	7628	8430	8159	8506
7534	1369	9935	4358	1490	3193	2906	8893	6850	0891
1529	4760	9897	8352	9742	3222	4402	5286	7490	8293
8640	4369	2154	0167	8358	4086	2082	4454	8300	6078
4802	5490	5581	5666	2838	6789	4990	4780	8211	9244
5575	7527	5181	5048	6590	2984	3675	3713	9279	2444
8567	8238	4885	2201	9949	6557	9036	0762	5741	0075
3274	7170	8828	3802	8740	0349	3301	0854	6672	7755
7241	5444	3332	9411	0974	9305	1968	6590	2879	8846
9775	0637	0620	3371	1320	2326	5367	2505	7106	8327
5967	8343	1255	6275	8485	2979	4725	8253	1482	7908
7751	6866	9343	2573	9731	6516	8446	2564	4320	5098
5351	8360	4701	0334	6325	6623	5137	9967	3771	2139
9489	7125	7285	6006	9644	8078	4301	8991	7505	7908
4123	5242	6491	7400	5045	7933	0171	4898	3955	2712
6300	7627	8988	6159	9272	7495	1516	3011	4975	1997
8692	3259	4187	0708	1821	5341	9440	0223	3858	0821
3971	0814	5119	6038	1296	2652	0170	3593	1459	0425
1458	9604	6418	7922	3438	3212	8105	6187	9651	2711
1532	5944	6461	2186	6531	9642	7027	4512	1871	1547
4358	4732	2836	7479	7775	9372	5429	3719	3704	0959
4771	3705	1207	6469	3964	3735	6951	6726	1822	6737
8320	3056	6632	6155	3391	6854	2468	7444	0255	3133
0760	1789	7331	2176	2761	8899	6595	9499	3543	5813
4070	5232	3454	6773	8100	2066	1208	3083	6906	8123
5346	1848	1476	0788	5314	8818	2993	6710	8478	8989
9998	3687	9759	6679	2548	1029	3827	7357	1508	7640
3645	8649	3296	0892	7380	0197	4126	0455	6421	0426
2557	6761	7808	5045	9141	5541	4223	6136	9505	4995
1087	9755	4751	5061	2832	4231	1508	7534	0844	5725
6355	5538	2461	5869	0447	6704	1861	2725	8901	8212
8426	6186	9022	1265	8066	7173	8981	0181	0794	0397
3513	0498	9207	7115	3367	8342	0415	4787	7058	6874
8415	9346	4334	2976	3382	5732	4626	8733	5606	0797
5604	4248	5282	1402	4514	2419	9922	0440	3763	4844
3293	2554	1972	9984	1488	1936	0002	4826	8885	4399
0482	6386	7228	6668	5710	7267	1942	7027	4138	9070
4496	8432	0507	5735	8649	3503	4284	4731	2598	8563
9909	4716	7697	8245	5921	3583	7679	4077	1490	4665
8029	7193	5378	7464	5161	8795	4537	2816	8367	6919
8435	7008	0739	8069	8406	0862	1831	1584	5723	8733
4520	9775	0803	8260	7956	4495	6670	6761	6847	4544

Random Numbers (cont.)

1306	1189	5731	3968	5606	5084	8947	3897	1636	7810
0422	2431	0649	8085	5053	4722	6598	5044	9040	5121
6597	2022	6168	5060	8656	6733	6364	7649	1871	4328
7965	6541	5645	6243	7658	6903	9911	5740	7824	8520
7695	6937	0406	8894	0441	8135	9797	7285	5905	9539
5160	7851	8464	6789	3938	4197	6511	0407	9329	2232
2961	0551	0539	8288	7478	7565	5581	5771	5442	8761
1428	4183	4312	5445	4854	9157	9158	5218	1464	3634
3666	5642	4539	1561	7849	7520	2547	0756	1206	2033
6543	6799	7454	9052	6689	1946	2574	9386	0304	7945
9975	6080	7423	3175	9377	6951	6519	8287	8994	5532
4866	0956	7545	7723	8085	4948	2228	9583	4415	7065
8239	7068	6694	5168	3117	1586	0237	6160	9585	1133
8722	9191	3386	3443	0434	4586	4150	1224	6204	0937
1330	9120	8785	8382	2929	7089	3109	6742	2468	7025
2296	2952	4764	9070	6356	9192	4012	0618	2219	1109
3582	7052	3132	4519	9250	2486	0830	8472	2160	7046
5872	9207	7222	6494	8973	3545	6967	8490	5264	9821
1134	6324	6201	3792	5651	0538	4676	2064	0584	7996
1403	4497	7390	8503	8239	4236	8022	2914	4368	4529
3393	7025	3381	3553	2128	1021	8353	6413	5161	8583
1137	7896	3602	0060	7850	7626	0854	6565	4260	6220
7437	5198	8772	6927	8527	6851	2709	5992	7383	1071
8414	8820	3917	7238	9821	6073	6658	1280	9643	7761
8398	5224	2749	7311	5740	9771	7826	9533	3800	4553
0995	8935	2939	3092	2496	0359	0318	4697	7181	4035
6657	0755	9685	4017	6581	7292	5643	5064	1142	1297
8875	8369	7868	0190	9278	1709	4253	9346	4335	3769
8399	6702	0586	6428	7985	2979	4513	1970	1989	3105
6703	1024	2064	0393	6815	8502	1375	4171	6970	1201
4730	1653	9032	9855	0957	7366	0325	5178	7959	5371
8400	6834	3187	8688	1079	1480	6776	9888	7585	9998
3647	8002	6726	0877	4552	3238	7542	7804	3933	9475
6789	5197	8037	2354	9262	5497	0005	3986	1767	7981
2630	2721	2810	2185	6323	5679	4931	8336	6662	3566
1374	8625	1644	3342	1587	0762	6057	8011	2666	3759
1572	7625	9110	4409	0239	7059	3415	5537	2250	7292
9678	2877	7579	4935	0449	8119	6969	5383	1717	6719
0882	6781	3538	4090	3092	2365	6001	3446	9985	6007
0006	4205	2389	4365	1981	8158	7784	6256	3842	5603
4611	9861	7916	9305	2074	9462	0254	4827	9198	3974
1093	3784	4190	6332	1175	8599	9735	8584	6581	7194
3374	3545	6865	8819	3342	1676	2264	6014	5012	2458
3650	9676	1436	4374	4716	5548	8276	6235	6742	2154
7292	5749	7977	7602	9205	3599	3880	9537	4423	2330
2353	8319	2850	4026	3027	1708	3518	7034	7132	6903
1094	2009	8919	5676	7283	4982	9642	7235	8167	3366
0568	4002	0587	7165	1094	2006	7471	0940	4366	9554
5606	4070	5233	4339	6543	6695	5799	5621	3953	9458
8285	7537	1181	2300	5294	6892	1627	3372	1952	3028

Random Numbers (cont.)

2444	9039	4803	8568	1590	2420	2547	2470	8179	4617
5748	7767	2800	6289	2814	3381	1549	9519	3341	1192
7761	8583	0852	5619	6864	5506	9643	7763	9611	1289
6838	9280	2654	0812	3985	2146	5095	0150	8043	9079
6440	2631	3033	9167	4995	7036	0133	7428	9702	1376
6829	0094	2687	3802	5497	0318	5168	6377	9216	2802
9845	4796	2951	4449	1999	2691	5328	7674	7004	6212
5072	9000	3887	5739	7920	6074	4715	3681	2721	2701
9035	0553	1272	2600	3525	5197	8852	9092	8027	6144
5562	1180	2222	0336	1411	0303	7424	3713	9278	1918
2757	2650	8727	3953	9570	2442	8041	9869	2887	3933
6397	1848	1476	0787	4090	4666	1208	2769	3922	1158
9208	7641	3575	4279	1282	1640	5999	1806	7809	5885
2418	9289	6120	8141	3909	5577	3590	2317	8975	4593
7300	9066	5659	8258	3652	0332	5369	3640	0563	7939
6870	2535	8916	3245	2256	4350	6064	2438	2002	1272
2914	7309	4045	7513	3195	4166	0878	5184	6680	7655
0668	8657	8118	6340	9451	7460	3291	5778	1167	0312
7994	6579	6461	2292	9554	5309	5036	0974	9517	8293
8587	0764	6687	9150	1642	2050	4934	0027	1376	5040
8016	8345	2257	5084	8004	7949	3205	3972	7640	3478
5581	5775	7517	9076	4699	5313	8401	7147	9416	7184
2015	3364	6688	2631	2152	2220	1637	8333	4838	5699
7327	8987	5741	0102	1173	7350	7080	7420	1847	0741
3589	1991	1764	8355	9684	9423	7101	1063	4151	4875
2188	6454	7319	1215	0473	6589	2355	9579	7004	6209
2924	0472	9878	7966	2491	5662	5635	2789	2564	1249
1961	1669	2219	1113	9175	0260	4046	8142	4432	2664
2393	9637	0410	7536	0972	5153	0708	1935	1143	1704
7585	4424	2648	6728	2233	3518	7267	1732	1926	3833
0197	4021	9207	7327	9213	7017	9060	6216	1942	6817
9719	5336	5532	8537	2980	5252	4971	0110	6209	1556
8666	4785	6007	8006	9643	4109	5570	9249	9905	2152
5744	3957	8786	9023	1472	7275	1014	1104	0832	7680
7149	5721	1389	6581	7196	7072	6360	3084	7009	0239
7710	8479	9345	7773	9056	1202	8845	3163	7937	6163
5246	5651	0432	8644	6341	9661	2361	8377	5673	6098
3576	0013	7381	0124	8559	9513	9080	6984	0926	2169
3026	1464	2671	4691	0353	5289	8754	2442	7793	8983
6591	4365	8717	2365	5666	5377	8675	9798	7745	6360
0402	3257	0480	5038	1998	2935	1306	1190	2406	2596
7135	7654	4745	4482	8471	1424	2031	7803	4367	6816
7181	4146	1046	0885	1264	7755	1653	8924	5822	4401
3655	2287	2178	8134	3291	7262	8229	2866	7065	4806
5121	6717	3117	1901	5184	6467	8954	3884	0279	8635
3618	2098	9205	7429	1575	1917	7927	2696	3704	0833
0166	3638	4947	1414	4794	9189	2459	5056	5982	6154
6187	9653	3658	4730	1651	8096	8289	9368	5531	7788
1234	1448	0276	7290	1667	2823	3755	5642	4854	8844
8949	8731	4875	5724	2962	1182	2930	7539	4526	7252

Annex 5.2

A_{n-i+1} Coefficients

i \ n	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	0.7071	0.6872	0.6646	0.6431	0.6233	0.6052	0.5888	0.5739	0.5601	0.5475	0.5359	0.5251	0.5150	0.5056	0.4968	0.4886
2		0.1677	0.2413	0.2806	0.3031	0.3164	0.3244	0.3291	0.3315	0.3325	0.3325	0.3318	0.3306	0.3290	0.3273	0.3253
3				0.0875	0.1401	0.1743	0.1976	0.2141	0.2260	0.2347	0.2412	0.2460	0.2495	0.2521	0.2540	0.2553
4						0.0561	0.0947	0.1224	0.1429	0.1586	0.1707	0.1802	0.1878	0.1939	0.1988	0.2027
5								0.0399	0.0695	0.0922	0.1099	0.1240	0.1353	0.1447	0.1524	0.1587
6										0.0303	0.0539	0.0727	0.0880	0.1005	0.1109	0.1197
7												0.0240	0.0433	0.0593	0.0725	0.0837
8														0.0196	0.0359	0.0496
9																0.0163

i \ n	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34
1	0.4808	0.4734	0.4643	0.4590	0.4542	0.4493	0.4450	0.4407	0.4366	0.4328	0.4291	0.4254	0.4220	0.4188	0.4156	0.4127
2	0.3232	0.3211	0.3185	0.3156	0.3126	0.3098	0.3069	0.3043	0.3018	0.2992	0.2968	0.2944	0.2921	0.2898	0.2876	0.2854
3	0.2561	0.2565	0.2578	0.2571	0.2563	0.2554	0.2543	0.2533	0.2522	0.2510	0.2499	0.2487	0.2475	0.2463	0.2451	0.2439
4	0.2059	0.2085	0.2119	0.2131	0.2139	0.2145	0.2148	0.2151	0.2152	0.2151	0.2150	0.2148	0.2145	0.2141	0.2137	0.2132
5	0.1641	0.1686	0.1736	0.1764	0.1787	0.1807	0.1822	0.1836	0.1848	0.1857	0.1864	0.1870	0.1874	0.1878	0.1880	0.1882
6	0.1271	0.1334	0.1399	0.1443	0.1480	0.1512	0.1539	0.1563	0.1584	0.1601	0.1616	0.1630	0.1641	0.1651	0.1660	0.1667
7	0.0932	0.1013	0.1092	0.1150	0.1201	0.1245	0.1283	0.1316	0.1346	0.1372	0.1395	0.1415	0.1433	0.1449	0.1463	0.1475
8	0.0612	0.0711	0.0804	0.0878	0.0941	0.0997	0.1046	0.1089	0.1128	0.1162	0.1192	0.1219	0.1243	0.1265	0.1284	0.1301
9	0.0303	0.0422	0.0530	0.0618	0.0696	0.0764	0.0823	0.0876	0.0923	0.0965	0.1002	0.1036	0.1066	0.1093	0.1118	0.1140
10		0.0140	0.0263	0.0368	0.0459	0.0539	0.0610	0.0672	0.0728	0.0778	0.0822	0.0862	0.0899	0.0931	0.0961	0.0988
11				0.0122	0.0228	0.0321	0.0403	0.0476	0.0540	0.0598	0.0650	0.0697	0.0739	0.0777	0.0812	0.0844
12						0.0107	0.0200	0.0284	0.0358	0.0424	0.0483	0.0537	0.0585	0.0629	0.0669	0.0706
13								0.0094	0.0178	0.0253	0.0320	0.0381	0.0435	0.0485	0.0530	0.0572
14										0.0084	0.0159	0.0227	0.0289	0.0344	0.0395	0.0441
15												0.0076	0.0144	0.0206	0.0262	0.0314
16														0.0068	0.0131	0.0187
17																0.0062

i \ n	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
1	0.4096	0.4068	0.4040	0.4015	0.3989	0.3964	0.3940	0.3917	0.3894	0.3872	0.3850	0.3830	0.3808	0.3789	0.3770	0.3751
2	0.2834	0.2813	0.2794	0.2774	0.2755	0.2737	0.2719	0.2701	0.2684	0.2667	0.2651	0.2635	0.2620	0.2604	0.2589	0.2574
3	0.2427	0.2415	0.2403	0.2391	0.2380	0.2368	0.2357	0.2345	0.2334	0.2323	0.2313	0.2302	0.2291	0.2281	0.2271	0.2260
4	0.2127	0.2121	0.2116	0.2110	0.2104	0.2098	0.2091	0.2085	0.2078	0.2072	0.2065	0.2058	0.2052	0.2045	0.2038	0.2032
5	0.1883	0.1883	0.1883	0.1881	0.1880	0.1878	0.1876	0.1874	0.1871	0.1868	0.1865	0.1862	0.1859	0.1855	0.1851	0.1847
6	0.1673	0.1678	0.1683	0.1686	0.1689	0.1691	0.1693	0.1694	0.1695	0.1695	0.1695	0.1695	0.1695	0.1693	0.1692	0.1691
7	0.1487	0.1496	0.1505	0.1513	0.1520	0.1526	0.1531	0.1535	0.1539	0.1542	0.1545	0.1548	0.1550	0.1551	0.1553	0.1554
8	0.1317	0.1331	0.1344	0.1356	0.1366	0.1376	0.1384	0.1392	0.1398	0.1405	0.1410	0.1415	0.1420	0.1423	0.1427	0.1430
9	0.1160	0.1179	0.1196	0.1211	0.1225	0.1237	0.1249	0.1259	0.1269	0.1278	0.1286	0.1293	0.1300	0.1306	0.1312	0.1317
10	0.1013	0.1036	0.1056	0.1075	0.1092	0.1108	0.1123	0.1136	0.1149	0.1160	0.1170	0.1180	0.1189	0.1197	0.1205	0.1212
11	0.0873	0.0900	0.0924	0.0947	0.0967	0.0986	0.1004	0.1020	0.1035	0.1049	0.1062	0.1073	0.1085	0.1095	0.1105	0.1113
12	0.0739	0.0770	0.0798	0.0824	0.0848	0.0870	0.0891	0.0909	0.0927	0.0943	0.0959	0.0972	0.0986	0.0998	0.1010	0.1020
13	0.0610	0.0645	0.0677	0.0706	0.0733	0.0759	0.0782	0.0804	0.0824	0.0842	0.0860	0.0876	0.0892	0.0906	0.0919	0.0932
14	0.0484	0.0523	0.0559	0.0592	0.0622	0.0651	0.0677	0.0701	0.0724	0.0745	0.0765	0.0783	0.0801	0.0817	0.0832	0.0846
15	0.0361	0.0404	0.0444	0.0481	0.0515	0.0546	0.0575	0.0602	0.0628	0.0651	0.0673	0.0694	0.0713	0.0731	0.0748	0.0764
16	0.0239	0.0287	0.0331	0.0372	0.0409	0.0444	0.0476	0.0506	0.0534	0.0560	0.0584	0.0607	0.0628	0.0648	0.0667	0.0685
17	0.0119	0.0172	0.0220	0.0264	0.0305	0.0343	0.0379	0.0411	0.0442	0.0471	0.0497	0.0522	0.0546	0.0568	0.0588	0.0608
18		0.0057	0.0110	0.0158	0.0203	0.0244	0.0283	0.0318	0.0352	0.0383	0.0412	0.0439	0.0465	0.0489	0.0511	0.0532
19				0.0053	0.0101	0.0146	0.0188	0.0227	0.0263	0.0296	0.0328	0.0357	0.0385	0.0411	0.0436	0.0459
20						0.0049	0.0094	0.0136	0.0175	0.0211	0.0245	0.0277	0.0307	0.0335	0.0361	0.0386
21								0.0045	0.0087	0.0126	0.0163	0.0197	0.0229	0.0259	0.0288	0.0314
22										0.0042	0.0081	0.0118	0.0153	0.0185	0.0215	0.0244
23												0.0039	0.0076	0.0111	0.0143	0.0174
24														0.0037	0.0071	0.0104
25																0.0035