Long-Wavelength Instability of Periodic Flows and Whistler Waves in Electron Magnetohydrodynamics

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Abstract. Stability analysis of periodic flows and whistlers with respect to long-wavelength perturbations within the framework of dissipative electron magnetohydrodynamics (EMHD) based on two-scale asymptotic expansion technique is presented. Several types of flows are considered: two-dimensional Kolmogorov-like flow, helical flow, and anisotropic helical flow. It is shown that the destabilizing effect on the long-wavelength perturbations is due to either the negative resistivity effect related to flow anisotropy or α-like effect related to its microhelicity. The criteria of the corresponding instabilities are obtained. Numerical simulations of EMHD equations with the initial conditions corresponding to two types of periodic flows are presented.

1. It is known that at some critical amplitude of the velocity both two-dimensional (Kolmogorov) [1]–[4] and three-dimensional (Beltrami) [3, 5] periodic flows in viscous fluid become unstable with respect to long-wavelength perturbations with the wavevector perpendicular to the wavevector of basic flow. A similar instability takes place for the Rossby waves in the ocean and atmosphere and for drift waves in magnetized plasmas [6]. Being unstable the above mentioned periodic flows and waves are capable of generating large-scale coherent structures which are of interest for researchers due to their assumed role in a variety of physical phenomena such as an enhanced (compared to molecular) transport in fluids, anomalous transport of particles and energy in magnetized plasmas, self-organization. In the present paper in the framework of electron magnetohydrodynamics (EMHD) stability analysis of periodic flows and whistlers with respect to long-wavelength perturbations is presented.

2. EMHD equations describe small-scale ($l < c/\omega_{pi}$) phenomena on the time scales $1/\omega_{Be} < t < 1/\omega_{Bi}$, where ω_{pi} is the Langmuir frequency of ions and $\omega_{Be,i}$ are the gyrofrequencies of electrons and ions, correspondingly. Such phenomena are important in systems in which there is substantial electron drift motion across the magnetic field, e.g., z-pinches, plasma erosion switches, laser-produced coronas [7], and possibly in tokamaks with an intensive electron cyclotron resonant heating [8]. In the ranges of length and time scales mentioned above, the electrons can be described by fluid equations; the ions form an immobile, charge neutralizing background. In the simplest case when the background plasma density is homogeneous ($n_0 = \text{const}$), the perturbations are considered to be quasineutral (the density remains unperturbed) and the electron finite-Larmor-radius effects are negligible, EMHD equations reduce to (see, e.g., [7, 9]):

$$\frac{\partial}{\partial t}\left(\mathbf{B} - d_e^2\nabla^2\mathbf{B}\right) = \text{curl}\ \left[\left(\mathbf{B} - d_e^2\nabla^2\mathbf{B}\right) \times \text{curl}\ \mathbf{B}\right] + \nu\nabla^2\mathbf{B} + \mathbf{f}. \qquad (1)$$

Here the spatial variables are normalized to $l = \lambda/2\pi$ (λ is the wavelength of the periodic flow (or whistler), time – to whistler time $t_0 = \omega_{Be}^{-1}(l\omega_{pe}/c)^2$, the magnetic field strength is expressed in terms of B_0, which corresponds to the amplitude of magnetic field of the initial periodic flow (or in the case of whistlers – to the background magnetic field), $d_e = c/\omega_{pe}l$ is the normalized electron collisionless skin depth, and $\nu = 1/\omega_{Be}\tau_e$ is related to plasma resistivity (τ_e is the ion-electron collision time). The r.h.s. of the equation is augmented with the source term $\mathbf{f}$, the role of which is to compensate the decay of the initial periodic flow (or whistler) due to finite plasma conductivity.

It is assumed that as a result of some kind of plasma instability (primary instability) or by means of the external source the periodic flow (or whistler) $\mathbf{B}^0$, which satisfies Eq.(1), is created in plasma. To maintain its amplitude constant the source term is taken in the form $\mathbf{f} = -\nu\nabla^2\mathbf{B}^0$. The instabilities of such a periodic flow with respect to the small-amplitude, long-wavelength perturbations are studied (secondary instabilities). Due to an assumption of the distinct separation of the characteristic space and time scales of the initial flow and its perturbations the two-scale asymptotic expansion method is applied in the analytical analysis.

In addition to analytical analysis Eq.(1) with the initial conditions corresponding to periodic flows is solved numerically. The numerical simulation allows to check the validity of the analytical approximations applied and to generalize the results to the case of the perturbations with the wavelengths comparable to that of the initial flow. Below the analytical and in some cases numerical results for three different types of periodic EMHD flows and whistlers are presented.

3. *The Kolmogorov-like flow.* The simplest exact solution of Eq.(1) is a two-dimensional flow characterized by

$$\mathbf{B}^0 = \cos x\,\mathbf{e}_z, \quad \mathbf{v}_e \propto \text{curl}\;\mathbf{B}^0 = \sin x\,\mathbf{e}_y. \tag{2}$$

The flow velocity is directed along the y-axis, and its amplitude depends on x, and such a flow is similar to the Kolmogorov flow in hydrodynamics (see, e.g., [3]). In the case of large magnetic Reynolds number ($\nu \ll 1$), the multi-scale expansion analysis results in the following asymptotic dispersion relation of perturbations with the wavevector perpendicular to the wavevector of initial flow

$$\gamma = \frac{1}{2\nu}(1 + d_e^2)(q_z^2 - q_y^2), \tag{3}$$

where γ is the growth rate of the perturbation and $\mathbf{q}$ is its wavevector, $q \simeq \nu^2$. This asymptotic dispersion relation is similar to that of in the case of the Kolmogorov flow at large Reynolds numbers [1, 2]. As it follows from Eq.(3) the most unstable perturbations have the wavevector $\mathbf{q} = (0, 0, q_z)$ which is perpendicular to the direction of the basic flow $\mathbf{v}_e$. The source of instability is related to the initial small-scale periodic flow which results in the negative resistivity effect on b_x-component of the long-wavelength magnetic field. The y- and z-components of magnetic field of the perturbation damp due to plasma resistivity. The growth rate of instability is proportional to magnetic Reynolds number ν^{-1}. A similar effect has been found in the recent paper [10] where the effect of anisotropic small-scale EMHD turbulence on the larger-scale perturbations has been considered. The results of numerical simulation of the Kolmogorov-like flow are presented on Fig. 1. These results are in a good agreement with the analytical ones when the latter are applicable. The numerical analysis also shows that the perturbations are stable when their wavelength is comparable to that of the initial flow.

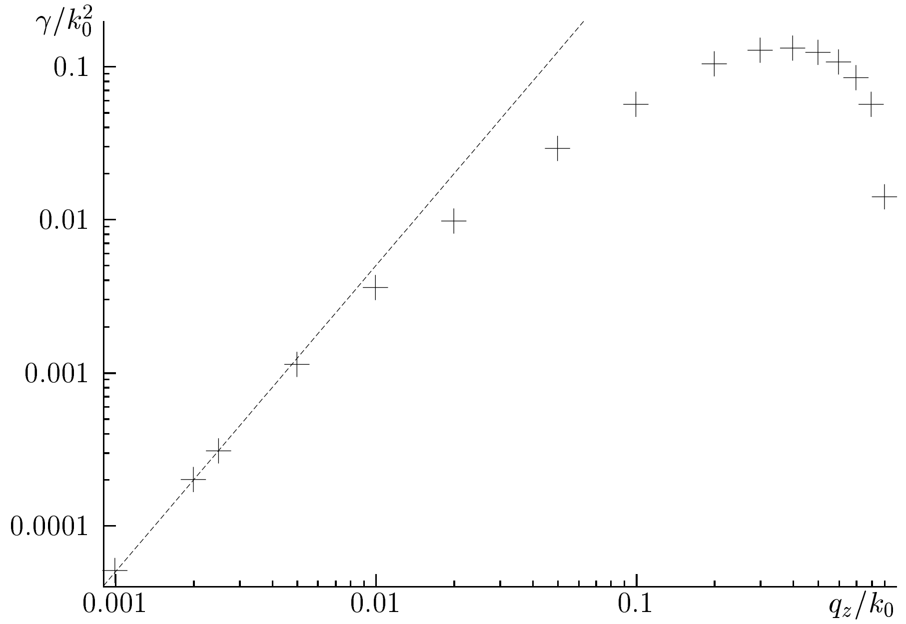

Figure 1: *The dependence of the growth rate on* q_z *for the Kolmogorov-like flow,* $d_e = \pi$, $\nu = 0.1$, $q_y/q_z = 0.01$. *Here: the crosses correspond to the results of numerical simulation and the stroke line – to the asymptotic formula (3). In the region* $q_z \leq \nu^2 k_0$ *the analytical and numerical results coincide, when* $q_z \approx 0.4k_0$ *there is a maximum of the growth rate, and when* $q_z \approx k_0$ *the mode becomes stable*

4. *Helical Beltrami-like flow.* Another stationary solution of EMHD equations is

$$\mathbf{B}^0 = \mathbf{e}_y \sin x + \mathbf{e}_z \cos x. \tag{4}$$

It is similar to three-dimensional Beltrami flow in hydrodynamics (see, e.g.,[3]). This flow has the structure of magnetic field similar to whistlers in the limiting case of zero frequency and the wavenumber $k_z = 0$ (see below). It belongs to the so-called force-free equilibria with curl $\mathbf{B}^0 = \mathbf{B}^0$ and possesses non-zero microhelicity $\mathbf{B}^0 \cdot \text{curl}\ \mathbf{B}^0 = 1$. Assuming that $\nu \ll 1$, it is found that the initial helical flow results in the α-like effect on the long-wavelength perturbations which is proportional to its microhelicity, and the asymptotic dispersion relation takes the form

$$(\gamma + \nu q^2)^2 = \frac{1}{4\nu^2}(1 - d_e^4)q^6, \qquad q = \sqrt{q_y^2 + q_z^2} \simeq \nu^2. \tag{5}$$

If $d_e < 1$ the perturbations with $q > 2\nu^2/\sqrt{1 - d_e^4}$ are unstable. At the same time, when $d_e > 1$ the right hand side of Eq.(5) is negative, and in this case the basic periodic flow results in the mode with $\text{Im}\,\gamma \neq 0$, i.e. $\text{Re}\,\omega \neq 0$, which damps due to plasma resistivity. Therefore in this case the helical flow is stable with respect to long-wavelength perturbations. The numerical simulation (see Fig. 2) shows the existence of the instability thresholds with respect to q both from below (in accordance with the analytical predictions) and from above (when the wavelength of the perturbation is of the order of that of the initial flow). Also the numerical calculations clearly show the existence of stability boundary with respect to d_e – the perturbations become stable when $d_e \geq 1$.

5. *Anisotropic helical flow.* Along with the isotropic helical flow Eq.(1) admits the solution which can be called **anisotropic helical** flow

$$\mathbf{B}^0 = \mathbf{e}_y \sin x \cos \alpha + \mathbf{e}_z \cos x \sin \alpha. \tag{6}$$

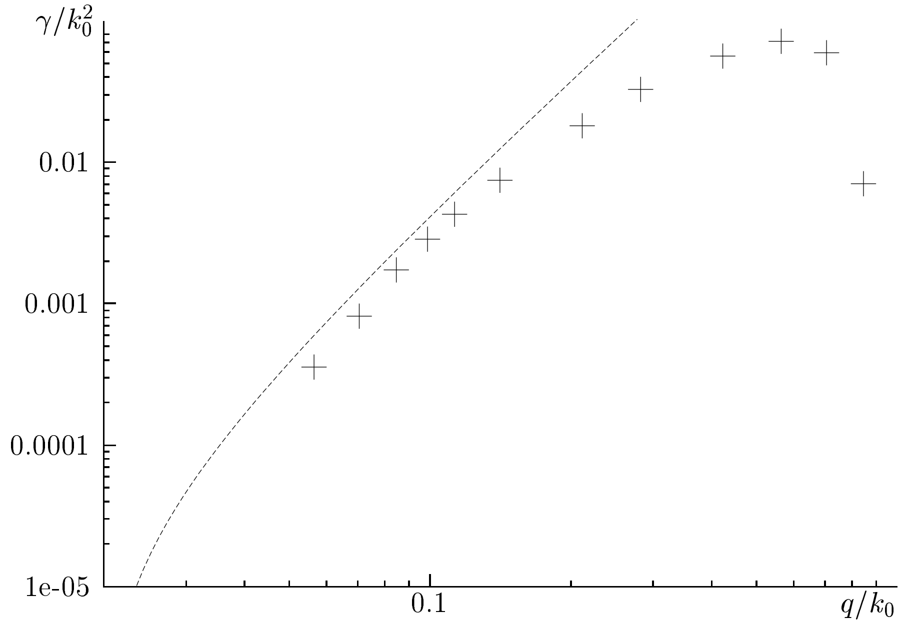

Figure 2: *The dependence of the growth rate on q for the helical flow,* $d_e = 0.1$, $\nu = 0.1$. *Here: the crosses correspond to the results of numerical simulation and the stroke line – to the asymptotic formula (5). Transfer to stability regions is observed both at small* ($q/k_0 < 0.01$) *and at large* ($q/k_0 > 0.8$) *values of the wavenumber of perturbations*

On the one hand, this solution possesses non-zero microhelicity, $\mathbf{B}^0 \cdot \mathrm{curl}\, \mathbf{B}^0 = (1/2)\sin 2\alpha$, on the other hand, it is anisotropic in yz-plane in the sense that the difference of squared amplitudes of the magnetic field in y- and z-directions is proportional to $\cos 2\alpha$ and is not equal to zero. Unlike isotropic helical flow the solution Eq.(6) does not belong to the class of force-free equilibria and satisfies the condition

$$[(\mathbf{B}^0 - d_e^2\nabla^2\mathbf{B}^0) \times \mathrm{curl}\, \mathbf{B}^0] = (1 + d_e^2)\boldsymbol{\nabla}\frac{(\mathbf{B}^0)^2}{2} \tag{7}$$

In the case of strong anisotropy ($\cos 2\alpha \simeq 1$) microhelicity effect on the long-wavelength perturbations is negligible and stability conditions found in the case of the Kolmogorov-like flow are applicable to anisotropic helical flow with a substitution $1 \to \cos 2\alpha$ in Eq.(3). In the case of weak anisotropy $\cos 2\alpha \simeq \nu^2$ the effects due to anisotropy (resistivity-like term in the equation of the evolution of x-component of the long-wavelength magnetic field) and due to microhelicity (α-like effect) are comparable, and the asymptotic dispersion relation takes the form ($q \simeq \nu^2$)

$$\gamma = -\nu q^2 + \frac{\cos 2\alpha}{4\nu}(1 + d_e^2)(q_y^2 - q_z^2) \pm \frac{1}{4\nu}\sqrt{\cos^2 2\alpha(1 + d_e^2)^2(q_y^2 - q_z^2)^2 + \sin^2 2\alpha(1 - d_e^4)q^6}. \tag{8}$$

6. *Stability of whistlers.* If plasma is imbedded in the homogeneous magnetic field B_0 directed along z-axis, EMHD equations admit the waves with nonzero frequency called **whistlers**

$$\mathbf{B} = \mathbf{e}_z + \mathbf{B}^0, \quad \mathbf{B}^0 = C_0(-k_z\mathbf{e}_x \cos\psi + \mathbf{e}_y \sin\psi + k_x\mathbf{e}_z\cos\psi), \tag{9}$$

where the first term describes the background magnetic field and

$$\psi = k_x x + k_z z - \omega(\mathbf{k})t, \quad k = \sqrt{k_x^2 + k_z^2} \equiv 1, \quad \omega(\mathbf{k}) = \frac{k_z}{1 + d_e^2}. \tag{10}$$

Here the magnetic field strength is normalized to B_0. Since curl $\mathbf{B}^0 = \mathbf{B}^0$ whistler is a force-free solution and possesses non-zero microhelicity $\mathbf{B}^0 \cdot$ curl $\mathbf{B}^0 = C_0^2$. The perturbations are considered to be periodic with respect to ψ and also to depend on slow variables defined by the ordering $\mathbf{X} = \nu^2\mathbf{x}, T = \nu^4 t, \tau = \nu^5 t$. Two slow time scales correspond to the period of long-wavelength whistler (T) and to its slow evolution due to dissipation and the effect of initial whistler (τ). Then the asymptotic dispersion relation of the long-wavelength perturbations is

$$\Omega = \pm\left(q_z q + \frac{1}{4} i \frac{C_0^2}{\nu}(1 - d_e^2)q^3\right) - i\nu q^2. \tag{11}$$

Independently of the sign of $1 - d_e^2$ one of two long-wavelength whistlers is unstable if its wavenumber satisfies the condition $qC_0^2|1 - d_e^2|/4\nu^2 > 1$.

Thus, one can conclude that, generally speaking, the periodic EMHD solutions tend to be unstable with respect to long-wavelength perturbations which enables to predict the possible energy transfer to large scales in EMHD turbulence (the inverse cascade).

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