Stability of Neoclassical Rotation in Edge Plasmas

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Abstract. Using a revised neoclassical theory, the stability of rotation velocities of a collisional tokamak plasma near the separatrix is investigated. First, assuming equilibrium, full coupled equations of poloidal and toroidal velocities are solved by numerical means and depending on the underlying temperature and density profiles behavior of solutions, such as shocks etc. is studied. Secondly, using a generic quasilinear differential equation for the toroidal velocity, we investigate its evolution from an initial profile depending on the coefficients of the equation and give a criterion for the development of a *breaking*, or a shock front represented by it.

1. Introduction

Within the framework of the revisited neoclassical theory, [1-4] the analysis of poloidal or toroidal rotation in a collision dominated toroidal plasma with steep gradients is based on the fluid equations with mass and momentum sources. The revised theory has introduced important modifications into the parallel momentum equation, when the parameter $\Lambda_1 \equiv (\nu_i / \Omega_i)(q^2 R^2 / r L_\psi)$ exceeds 1/3 [1,3]. It has also been speculated that the poloidal velocity as derived by the revisited neoclassical theory would not be unique and under certain conditions allow for bifurcated equilibria, since it is determined there by means of a cubic equation [3]. Poloidal plasma rotation in toroidal systems is related to various instability mechanisms. For example, plasma rotation is always accompanied by a radial electric field, whose origin appears to be complicated due to various competing effects. It is usually believed that the neoclassical transport should be ambipolar and independent of the radial electric field. However, this requirement is strictly valid only in an equilibrium plasma, in which there is no other source or damping of toroidal momentum. Stringer [6] was the first to notice that the resistive diffusion rate in a toroidal plasma can not only be non-ambipolar, but can also be negative for some values of the poloidal rotation velocity. Rosenbluth and Taylor [7], considered the stability of toroidal diffusion using a fluid model and proved that if the resistivity were the only dissipative mechanism, then even if all plasma deformations were excluded, there could be no stable poloidal rotation velocity. The unstable poloidal or toroidal rotations, can also be observed as spontaneous spin-up phenomena in tokamaks. An interaction between a spontaneous poloidal or toroidal spin-up and the turbulence driven anomalous transport is also believed to be a likely reason for the L-H mode transition in tokamaks. A further consequence of an unstable rotation is that, a poloidally asymmetric particle transport may also render the radial electric field unstable [8].

2. General Formulation of the Problem

Equations for a two-component plasma, describing the continuity of species J with sources $S_J^{\,N}(\vec{x},t)$, the momentum balance with friction $\vec{R}_J^{\,M}(\vec{x},t)$ and momentum input $\vec{S}_J^{\,M}(\vec{x},t)$, and the energy balance with similar terms and energy input $S_J^{\,E}(\vec{x},t)$ are

$$\frac{\partial N}{\partial t}+\nabla\cdot N\vec{U}_{J\perp}+\vec{B}\cdot\nabla(N\vec{B}\cdot\vec{U}_J/B^2)=S_J{}^N \quad (1)$$

$$\frac{\partial(m_J N\vec{U}_J)}{\partial t}+\nabla\cdot(m_J N\vec{U}_J\vec{U}_J+P_J\vec{I}+\vec{\Pi}_J)=e_J N(\vec{E}+\vec{U}_j\times\vec{B})+\vec{R}_J{}^M+\vec{S}_J{}^M \quad (2)$$

$$\frac{3}{2}\left(\frac{\partial P_J}{\partial t}+\nabla\cdot P_J\vec{U}_J\right)+(P_J\vec{I}+\vec{\Pi}_J)\cdot\nabla\cdot\vec{U}_J=-\nabla\cdot\vec{q}_J+(R_J{}^E+S_J{}^E)-\vec{U}_J\cdot(\vec{R}_J{}^M+\vec{S}_J{}^M)+\frac{m_J}{2}U_J{}^2S_J{}^N \quad (3)$$

The terms S_J on the r.h.s. of (3) denote heating due to energy sources and losses, whereas R_J denote terms relating to collisional energy transfer and frictional heating. The correct forms of these terms require a kinetic and atomic approach. However, here we shall assume them, as given functions. Plasma layer just inside the separatrix is collisional enough to be treated by these fluid equations including the parallel, perpendicular and gyro stress tensor expressions given by Braginskii (recently extended and completed by Mikhailovsky and Tsypin [5]). The radial electric field satisfies Ohm's law, $E_r+(\vec{U}_i\times\vec{B})_r=(1/eN)\partial P_i/dr$. In the revisited neoclassical theory [1], a plausible ordering inside the separatrix is introduced by means of a small parameter μ (~0.1) as

$$\left(\frac{qR\nu_J}{c_J}\right)^{-1}\sim\frac{L_\psi}{r}\sim\frac{r}{qR}\sim\frac{B_\vartheta}{B_\phi}\sim\mu \qquad \text{and} \qquad \frac{e_J\upsilon}{T_J}\sim\left(\frac{m_e}{m_i}\right)^{\frac{1}{2}}\sim\frac{a_i}{L_\psi}\sim\mu^2 \quad (4)$$

where L_ψ is the radial gradient scale, c_J and a_J are the thermal speed and the gyro radius of the species J, respectively; υ is the loop voltage and ν_J is the collision frequency between like particles J. Using the magnetic field aligned orthonormal unit vectors $(\hat{p},\hat{b},\hat{n})$ in radial, binormal and parallel directions, and the small parameter μ, the velocity of species J can be assumed as,

$\hat{p}\cdot\vec{U}_J\equiv\mu^6U^{(6)}{}_{\psi,J}(\psi,\chi)+\cdots;\quad \hat{b}\cdot\vec{U}_J\equiv\mu^2U^{(2)}{}_{\beta,J}(\psi)+\mu^3U^{(3)}{}_{\beta,J}(\psi,\chi)+\cdots;$ and

$$\hat{n}\cdot\vec{U}_J\equiv\mu U^{(1)}{}_{||,J}(\psi)+\mu^2U^{(2)}{}_{||,J}(\psi,\chi)+\cdots. \quad (5)$$

Assuming that the magnetic field, density, temperature, potential, etc., are independent of the poloidal angle in dominant order, these are also expanded in perturbation series. For example, the density and the magnetic field are written as, $N(\psi,\chi)\approx N^{(0)}(\psi)[1+\mu n^{(1)}(\psi,\chi)+\ldots]$ and $B(\psi,\chi)\approx B^{(0)}(\psi)[1+\mu b^{(1)}(\psi,\chi)+\ldots]$, respectively. For a tokamak plasma with circular cross section, also the use is made of the toroidal unit vectors $(\vec{e}_r,\vec{e}_\vartheta,\vec{e}_\varphi)$. Taking toroidal and parallel projections of the momentum equation and averaging them over the magnetic flux surfaces, and imposing the ambipolarity condition, one obtains a pair of coupled nonlinear equations for the toroidal and poloidal ion velocities in terms of other plasma variables, such as temperature, density, and electric field [1]. Main results describing the radial transport of toroidal momentum in a collisional subsonic plasma with steep gradients, are [2,3],

$$m_iN_i\frac{\partial U^{(1)}{}_{\varphi,i}}{\partial t}=\frac{\partial}{\partial r}\left[\eta_{2,i}\left(\frac{\partial U^{(1)}{}_{\varphi,i}}{\partial r}-\frac{0.107q^2}{1+Q^2/S^2}\frac{\partial\ln T_i}{\partial r}\frac{B_\varphi}{B_\theta}U^{(2)}{}_{\theta,i}\right)\right]$$
$$+J_rB_\theta-m_i\oint\frac{d\vartheta}{2\pi}h^2S_i{}^NU_{i\varphi}+\oint\frac{d\vartheta}{2\pi}h^2\vec{S}_i{}^M\cdot\vec{e}_\varphi \quad (6)$$

where, $h=1+(r/R_0)\cos\vartheta$, $Q=[4B_\varphi U^{(2)}{}_{\theta,i}-2.5(T_i/e_i)\partial\ln N_i^2T_i/\partial r]B^{-1}$, J_r is radial polarization current, $S=(2r\chi_{\|,i}N_i^{-1})/q^2R^2$; and the parallel heat diffusion coeff. is $\chi_{||,i}=3.9\ P_i/m_i\nu_i$. The poloidal rotation driven by the temperature gradient seen inside the paranthesis on the right hand side of (6) results from the gyro-stress tensor and acts like another source term, i.e., as a toroidal momentum source or sink, depending on the sign of its radial gradient. External momentum sources can be direct, such as fast ions provided by the neutral beam injection, collisions by alpha particles, or indirect and due to particle sources such as charge exchange with cold recycling neutrals. Important modifications of the toroidal momentum equation are manifested by its nonlinear coupling to the equation for poloidal rotation.

Using the ambipolarity condition and the extended forms of the stress tensors in the parallel momentum equation, one can cancel the time derivative and the source terms. The result in the lowest order is a nonlinear equation between the radial derivatives of the poloidal and toroidal plasma velocities:

$$U^{(2)}{}_{\theta,i} + 1.833(e_i B_\varphi)^{-1}\frac{\partial T_i}{\partial r} = 0.36\frac{\eta_{2,i}/\eta_{0,i}}{1+Q^2/S^2} q^2 R^2 \frac{e_i B_\varphi}{T_i}\frac{\partial \ln T_i}{\partial r}\left[\frac{T_i}{e_i B_\theta}\frac{\partial U^{(1)}{}_{\varphi,i}}{\partial r} + \frac{1}{2}U^{(1)2}_{\varphi,i}\right.$$
$$\left. - U^{(1)}{}_{\varphi,i}\frac{B_\varphi}{B_\theta}\left(U^{(2)}{}_{\theta,i} - \frac{T_i}{e_i B_\varphi}\frac{\partial \ln N_i^2 T_i}{\partial r}\right) + 1.90\frac{B_\varphi^2}{B_\theta^2}\left(U^{(2)}{}_{\theta,i} - 0.8\frac{T_i}{e_i B_\varphi}\frac{\partial \ln N_i^{1.6} T_i}{\partial r}\right)^2\right]$$
$$-\frac{2R^2}{3\eta_{0,i}} J_r B_\varphi \qquad (7)$$

The particle source and momentum input terms can also be introduced into (6). Using the definitions, $<A>$ ≡ $\overline{A} \equiv \oint (d\vartheta/2\pi) hA$, and $\tilde{A} \equiv A - \overline{A}$, for the surface-averaged and the poloidal-angle-dependent parts, these terms can be written as,

$$< S_i{}^N U_{i\varphi} h > = < (\overline{S}_i{}^N + \tilde{S}_i{}^N)[U_{i\varphi}{}^{(1)}(r) + U_{i\varphi}{}^{(2)}(r,\vartheta)](1+\varepsilon\cos\vartheta) > \quad \text{where } (\varepsilon\sim q\mu) \qquad (8)$$
$$< S_{i\varphi}{}^M h > = < (\overline{S}_{i\varphi}{}^M + \tilde{S}_{i\varphi}{}^M)(1+\varepsilon\cos\vartheta) >,$$

To calculate (8), we need $U_{i\varphi}{}^{(2)}$. According to the revised neoclassical theory [1], we find that $U_{i\varphi}{}^{(2)}(r,\vartheta) \approx -\mu U_{i\varphi}{}^{(1)}(r) A(r,\vartheta) + \mu C(r,\vartheta)$, where $A \equiv n^{(1)} - b^{(1)}$, and

$$C \equiv -\frac{cB_0}{eN^{(0)}} RB_\varphi \frac{P^{(0)}}{B_0{}^2}\left(\frac{e}{T^{(0)}}\frac{\partial V}{\partial r} - \frac{\partial \ln N^{(0)}}{\partial r}\right) n^{(1)} - 2\left[\frac{e}{T^{(0)}}\frac{\partial V^{(0)}}{\partial r} + \left(1 + \frac{N^{(0)}}{T^{(0)}}\frac{\partial T^{(0)}/\partial r}{\partial N^{(0)}/\partial r}\right)\frac{\partial \ln N^{(0)}}{\partial r}\right] b^{(1)} \qquad (9)$$

The perturbed density $N^{(1)}(r,\theta) \equiv n^{(1)}(r,\theta) N^{(0)}(r)$ in Eq.(8) can be found from the expanded and averaged forms of Eqs.(1-3), reflecting the symmetry behaviour of magnetic field $B^{(1)}(r,\theta)$ and the sources. For example, from the energy equations, omitting the sources [3]

$$2\chi_{\|i}\frac{B_\chi}{rB}\frac{\partial^2 n^{(1)}}{\partial\vartheta^2} = N_i\left[RB_\varphi\frac{T_i}{e_i B}\left(4e_i\frac{\partial V}{T_i \partial r} + \frac{\partial \ln T_i^{3/2} N_i^{-1}}{\partial r}\right) + 4U_{\|i}\right]\frac{\partial n^{(1)}}{\partial\vartheta} + 5RB_\varphi\frac{P_i}{e_i B}\frac{\partial \ln T_i}{\partial r}\frac{\partial b^{(1)}}{\partial\vartheta}$$

We note that in-out or up-down symmetry behaviour of A and C imposed by the magnetic field, may be further modified by the influence of the sources. Hence, returning to the averages, we find

$$\begin{aligned} &< S_i{}^N U_{i\varphi} h > \\ &\approx U_{i\varphi}{}^{(1)}[< S_i{}^N > (1 - q\mu^2 < \cos\vartheta\tilde{A} >) + q\mu^2 < \cos\vartheta\tilde{S}_i{}^N > - \mu^2 < \tilde{A}\tilde{S}_i{}^N > - q\mu^3 < \cos\vartheta\tilde{A}\tilde{S}_i{}^N >] \\ &\quad + \mu < C >< S_i{}^N > + q\mu^2 < \cos\vartheta\tilde{C} > + q\mu^2 < \cos\vartheta\tilde{S} > + \mu^2 < \tilde{S}\tilde{C} > + q\mu^3 < \cos\vartheta\tilde{S}\tilde{C} > \end{aligned} \qquad (10)$$

and

$$< S_\varphi{}^M h > \approx < \overline{S}_\varphi{}^M > + 2q\mu < \cos\vartheta\tilde{S}_\varphi{}^M > \qquad (11)$$

For a numerical solution of (6) and (7) in the equilibrium case, we transform r to a stretched boundary layer coordinate,.$\xi \equiv (r-a)/L_\psi$, where a is the radial position of the separatrix, and assume that temperature and density decrease outwards exponentially. Velocities U_φ and U_θ are normalized by μc_i and μc_i^2, respectively. Such solutions, U_θ, for example, can start from the neoclassical B.C. values far inside the separatrix and for some B.C., we find that do display jump discontinuties close to the separatrix. This tendency is related to the algebraic nonlinearity of these equations. In Fig.1, we present one of the smooth set of solutions.

In order to study the evolution and stability of rotation velocity profiles, we rewrite P.D.Eq.(6) as an initial value problem. Substituting in Eq.(6) the first radial derivative of U_φ from Eq.(7), we obtain for U_φ a quasilinear differential equation of the first order. Namely, U_φ satisfies in $t > 0$, $-\infty < \xi < 0$,

$$\frac{\partial U_\varphi}{\partial t} + [\kappa(t,\xi)U_\varphi + \lambda(t,\xi)]\frac{\partial U_\varphi}{\partial \xi} = \alpha(t,\xi)U_\varphi^{\ 2} + \beta(t,\xi)U_\varphi + \gamma(t,\xi) \qquad (12)$$

and an initial profile $U_\varphi(0,\zeta) = F(\zeta)$. Some of the coefficients $\kappa,\lambda,\alpha,\beta$, and γ in Eq.(12) depend on ξ, (and maybe on t), not only through temperature and density profiles but also through U_ϑ and the sources. For simplicity, however, we shall ignore the latter dependencies and treat Eq.(12) as if uncoupled from the poloidal rotation equation, and assume its coefficients here as slowly varying, or averaged quantities, i.e., in the first approximation, we treat them as constants. Defining a modified toroidal velocity $W \equiv \kappa U_\varphi + \lambda$, Eq.(12) becomes

$$\frac{\partial W}{\partial t} + W\frac{\partial W}{\partial \xi} = A + 2BW + CW^2 \qquad (13)$$

where $A \equiv \kappa\gamma - \beta\lambda + \alpha\lambda^2/\kappa$, $B \equiv \beta/2 - \alpha\lambda/\kappa$, $C \equiv \alpha/\kappa$.. Let the initial profile for W be given as, $W(0,\zeta) = f(\zeta)$. For t>0, W can then be found using the method of characteristics, depending on the value of B^2-AC. Namely, if $B^2 - AC > 0$, then

$$W(t,\varsigma) = -\frac{B}{C} - \frac{\sqrt{B^2 - AC}}{C}\,\frac{g(\varsigma)e^{2\sqrt{B^2-AC}\,t}+1}{g(\varsigma)e^{2\sqrt{B^2-AC}\,t}-1}, \text{ where } g(\varsigma) = \frac{Cf(\varsigma)+B-\sqrt{B^2-AC}}{Cf(\varsigma)+B+\sqrt{B^2-AC}} \qquad (14)$$

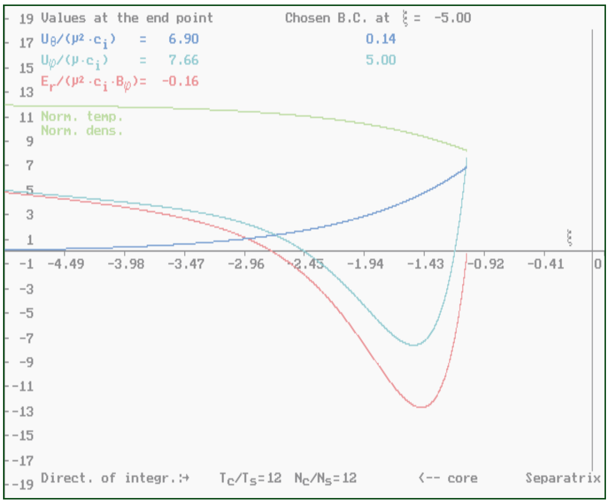

FIG.1. Equilibrium Solutions: Normalized rotation velocities and radial electric field over ξ. On the left border U_θ starts from the neoclassical value. Green curve represents the model temperature and density profiles. The red curve denotes the radial electric field. The cyan curve is U_φ

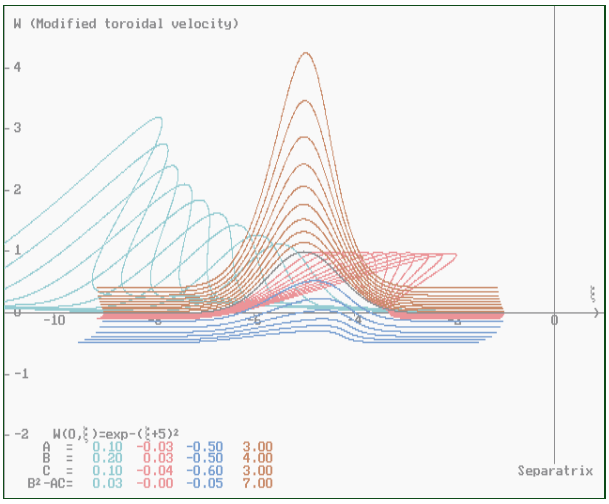

FIG.2. Evolution of an Initial Modified- Toroidal-Velocity Profile: Starting from the same bell-shaped initial profile, at the consecutive time steps, different sets of coefficients A,B, and C lead to various paths of evolution, such as stable, unstable,damping or breaking profiles.

and the equation for characteristics is

$$\xi(t,\varsigma)=\varsigma-\frac{B+\sqrt{B^2-AC}}{C}t-\frac{1}{C}\ln\left|\frac{g(\xi)\,e^{2\sqrt{B^2-AC}\,t}-1}{g(\xi)-1}\right| \tag{15}$$

Depending on the numerical values of the coefficients, A, B, and C, these solutions indicate either a temporal damping or an enhancement of an initial velocity profile, with or without an evolution towards a many-valued profile. Evolution of a many valued wave profile is called *breaking*. However, in our case, it must be interpreted as an indication of a jump discontinuity, or a shock front. For a given initial velocity profile, f(ς), characteristics forming an envelope is an indication for the forming of a *breaking* , or a shock front [9]. We find the condition for such a discontinuity as,

$$f'(\varsigma)<C\left[f(\varsigma)+\frac{B-\sqrt{B^2-AC}}{C}\right] \tag{16}$$

If $B^2-AC<0$, then the solution of (13) is found as

$$W(t,\varsigma)=-\frac{B}{C}+\frac{\sqrt{AC-B^2}}{C}\tan\left[\sqrt{AC-B^2}\,t+\arctan K\right] \tag{17}$$

where K = arctan[(Cf(ζ)+B)/(AC-B^2)^{½}] and

$$\xi(t,\varsigma)=\varsigma-\frac{B}{C}t-\frac{1}{C}\ln\left|\frac{\cos\left(\sqrt{AC-B^2}\,t+K\right)}{\cos K}\right| \tag{18}$$

Finally, if $B^2-AC=0$, then the solution and the characteristics are found as

$$W(t,\varsigma)=-\frac{B}{C}-\frac{1}{C}\frac{Cf(\varsigma)+B}{\left[Cf(\varsigma)+B\right]t-1}, \qquad \xi(t,\varsigma)=\varsigma-\frac{B}{C}t-\frac{1}{C}\log\left|\left[Cf(\varsigma)+B\right]t-1\right| \tag{19}$$

Thus, using above approach, we gain some valuable analytical information about the stability behaviour of the revised neoclassical toroidal rotation velocity, since the coefficients A,B, and C in (13) are functions of the plasma parameters. Whereas, due to its high degree of nonlinearity, a time dependent analytical analysis of the coupled system (6) and (7) with both toroidal and poloidal velocity variables seems to be unfeasible.

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