A New Coulomb Logarithm and Its Effects on the Fokker-Planck Equation, Relaxation Times and Cross-field Transport in Fusion Plasmas

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Abstract. It is shown that a new cutoff at small scattering angle should be introduced for constant relative velocity meanwhile a new cutoff on the velocity increment should be introduced for varied relative velocity based on the effective collision conditions. It is found that the Coulomb logarithm should be reduced to half of the well-known result. Consequently, the Fokker-Planck coefficients are modified and applicable to both weakly and moderately coupled plasmas. The relaxation times increase and the cross-field electrical resistivity, electron diffusion and ion thermal conductivity are reduced to half level for Maxwellian scatters. The non-Maxwellian effect can furthermore modify the Fokker-Planck coefficients, increase the relaxation times and reduce the cross-field transport coefficients.

1. Introduction

Recently, tokamak experiments have shown that the measured cross-field diffusion and thermal conductivity can be well below the neoclassical levels in the core of reversed magnetic shear (RMS) plasmas [1]. It is important to restudy Coulomb collisions since they are fundamental for plasma transport. Meanwhile, the new derivation indicates that the Coulomb logarithm $\ln\Lambda$ has a precise significance for charged-particle stopping power in inertial confinement fusion plasmas. The conventional Fokker-Planck (F-P) equation should be modified to include the non-dominant term, which is applicable to moderately coupled plasmas [2]. The conventional theory uses the scattering angle θ instead of momentum transfer $m\Delta\mathbf{v}$ to describe the Coulomb collisions in a plasma. Recently, the difference between small-θ and small-$m\Delta\mathbf{v}$ collisions of inverse-square force was clarified [3].

The Coulomb collision between a test particle (with m, $\mathbf{v}$) and a field particle (with m_F, $\mathbf{v}_F$) is equivalent to the interaction of a particle of reduced mass $\mu = mm_F/(m+m_F)$ with a fixed scattering center as in Fig. 1. The momentum transfer $m\Delta\mathrm{v}$, the impact parameter b and the Rutherford cross-section σ_R depend on the relative velocity g, and θ as follows:

$$m\Delta\mathrm{v} = 2\mu g\sin(\theta/2)\,, \tag{1}$$

$$b = (ZZ_F e^2/4\pi\varepsilon_0\mu g^2)\cot(\theta/2)\,, \tag{2}$$

and $$\sigma_R = \left(ZZ_F e^2/8\pi\varepsilon_0\mu g^2\right)^2\csc^4(\theta/2)\,. \tag{3}$$

In this paper, it is indicated that the cutoff at $\theta_{\min}$ introduced in the customary approach is questionable because the interaction distance r between the test and field particles was replaced by b. A new cutoff at $\theta_{\min}$ should be introduced for constant g. It is shown that $m\Delta\mathrm{v}$ can properly describe the Coulomb collisions in plasma rather than θ because $m\Delta\mathrm{v}$ is

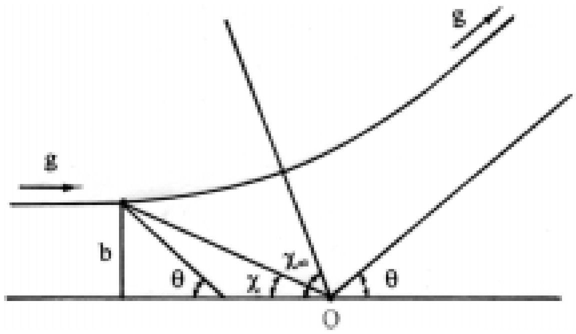

Fig. 1. The geometrical variables for the scattering process in the center-of-mass coordinates.

determined by both θ and g that can vary from zero to infinity in a plasma. A new cutoff on $\Delta \mathrm{v}$ can be introduced based on the Debye shielding theory and the effective collision conditions, which are valid for varied g. These results should be applicable to the fundamental theory of plasma kinetics. For example, F-P coefficients, the relaxation times and the cross-field transport coefficients in a plasma should be modified for Maxwellian and non-Maxwellian scatters.

2. A new cutoff at small scattering angle

In the customary approaches, $\Delta \mathbf{v}$ was replaced by θ to describe the Coulomb collisions in a plasma. Such replacement might be valid only if g is a constant. For the case of constant g, it is easy to obtain from Eq. (2) the relation between b and θ as follows:

$$\sin(\theta / 2) = \mathrm{b}_0 / \sqrt{\mathrm{b}^2 + \mathrm{b}_0^2} \qquad (4)$$

where $b_0 = ZZ_F e^2 / 4\pi\varepsilon_0 \mu g^2$, and $b = b_0 \cot(\theta / 2)$. In the previous theory, $\theta_{\min}$ is usually determined by setting $b_{\max} = \lambda_D$ and $g = \mathrm{v}_{th}$ so that $\sin\theta_{\min} / 2 = b_0 / (\lambda_D^2 + b_0^2)^{1/2} \approx \lambda_L / \lambda_D$ for $\lambda_D >> b_0$, where λ_D is the Debye length, λ_L Landau length and v_{th} thermal velocity of the field particle. Then, it was shown that the cutoff is made at $\theta_{\min} / 2 = \lambda_L / \lambda_D$ and the Coulomb logarithm is $-\ln\sin(\theta_{\min} / 2) = \ln\Lambda$ [4-7], where $\Lambda = \lambda_{\mathrm{D}} / \lambda_L$ is the Coulomb constant. Such a method is questionable because $b \neq r$. Substituting $b_{\max} = (\lambda_D \sin\theta)/2$ into Eq. (4), I would have $\sin(\theta_{\min} / 2) = b_0 / [(\lambda_D \sin\theta_{\min})^2 / 4 + b_0^2]^{1/2}$. After simple algebraic calculation, I obtain $\sin\theta_{\min} / 2 \approx \theta_{\min} / 2 = (b_0 / \lambda_D)^{1/2}$. Hence, the Coulomb logarithm should be modified as $-\ln\sin(\theta_{\min} / 2) = \ln\Lambda^{1/2} = (1/2)\ln\Lambda$.

3. A new cutoff on small velocity increment

In plasma physics, $\mathbf{v}_F$ can vary from zero to infinity so that g also varies from zero to infinity even if the test particle velocity $\mathbf{v}$ is a constant. Obviously, $\Delta\mathbf{v}$, b, and σ_R cannot be determined by θ alone if g varies. From Eqs. (1) - (3), it is easy to observe that the cutoff at $\theta = \theta_{\min}$ is unsuitable. It unfairly excludes the strong collisions with quite large g because small θ cannot ensure that $\Delta\mathbf{v}$ is small. Moreover, when g approaches zero, it cannot exclude the weak collisions because b and σ_R are still divergent. Hence, the cutoff on small θ alone cannot fulfill Debye shielding theory.

Actually, it is also improper to use b instead of $\Delta\mathbf{v}$ to measure the collision intensity. For example, according to the previous theories, small b implies close collisions. However, when

$b=0$, two ions cannot get closer than λ_{D} if their g is very small. It is easy to observe this point if combineing Eqs. (1) and (2) by eliminating *g* to obtain

$$b/\sin\theta = (ZZ_F e^2/4\pi\varepsilon_0)(2\mu/m^2\Delta \mathrm{v}^2)\,. \tag{5}$$

Obviously, it is $b/\sin\theta$ rather than *b* that can replace $\Delta\mathrm{v}$ to describe the Coulomb collisions in a plasma since $b/\sin\theta$ can be completely determined by $\Delta\mathrm{v}$.

First, I consider the effective collision condition from the viewpoint of energy. Assuming that the collisions are effective only if the kinetic energy associated with the component of ***g*** along the interaction direction is greater than the Coulomb potential energy, namely, $(1/2)\mu g^2\cos^2\chi \geq ZZ_F e^2/4\pi\varepsilon_0\lambda_D$ due to the Debye shielding, where χ is the angle between ***g*** and the interaction direction. For simplicity, I may use χ_∞ to replace χ, where χ_∞ is the special value of χ at the moment of closest distance r_m, and multiply by an undetermined factor α to adjust it. By using the relation $\chi_\infty = (\pi-\theta)/2$ and Eq. (1), it is easy to obtain $\Delta\mathrm{v}^2 \geq (ZZ_F e^2/4\pi\varepsilon_0)(8\mu/\alpha^2 m^2\lambda_D)$. Thus, the cutoff on $\Delta\mathrm{v}$ should be taken as

$$\Delta\mathrm{v}_{\min} = (8\mu ZZ_F e^2/4\pi\varepsilon_0\alpha^2 m^2\lambda_D)^{1/2}\,. \tag{6}$$

Secondly, I consider the effective collision condition from the viewpoint of interaction distance. I assume that the Coulomb collisions are effective only if $r = b/\sin\chi \leq \lambda_D$. For simplicity, I may use $b/\sin\theta$ to replace $b/\sin\chi$, which is reasonable according to Eq. (5), and have to multiply by an undetermined factor β to adjust it. By using Eq. (4), I would have $\Delta\mathrm{v}^2 \geq (ZZ_F e^2/4\pi\varepsilon_0)(2\beta\mu/m^2\lambda_D)$. Therefore, the cutoff on $\Delta\mathrm{v}$ should be taken as

$$\Delta\mathrm{v}_{\min} = (2\beta\mu ZZ_F e^2/4\pi\varepsilon_0 m^2\lambda_D)^{1/2} \tag{7}$$

that is consistent with the result of Eq. (6).

It is interesting to rewrite Eq. (3) as $\sigma_R = (ZZ_F e^2/4\pi\varepsilon_0)^2(4\mu^2/m^4\Delta\mathrm{v}^4)$, which can be completely determined by $\Delta\mathrm{v}$. This confirms that $\Delta\mathrm{v}$ is really a sole variable to measure the collision intensity. σ_R is very large for small $\Delta\mathrm{v}$ and approaches to ∞ when $\Delta\mathrm{v}\to 0$. Thus, I have to introduce a cutoff at $\Delta\mathrm{v} = \Delta\mathrm{v}_{\min}$. Using Eqs. (6)-(7), and letting $\alpha=\sqrt{2}$ and $\beta = 2$, I would obtain the cross section $\sigma_d = \lambda_D{}^2/4$ for the most distant collision. By using θ and $\Delta\mathrm{v}$ as independent variables instead of θ and g, I obtain the total cross section $\sigma^t = 2\pi\int_0^\pi \sigma_R \sin\theta d\theta = 4\pi(ZZ_F e^2/4\pi\varepsilon_0)^2(4\mu^2/m^4\Delta\mathrm{v}^4)$. Substituting Eqs. (6) and (7) into this expression, I would have $\sigma_d^t = \pi\lambda_D{}^2$ for the most distant collision, which is consistent with the Debye shielding theory, and $\Delta\mathrm{v}_{\min} = (4\mu ZZ_F e^2/4\pi\varepsilon_0 m^2\lambda_D)^{1/2}$ so that $u_{\delta\min} = \Delta\mathrm{v}_{\min}/a\mathrm{v}_{\mathrm{th}} = \Lambda^{-1/2}$, where $a = 2\mu/m$. With the help of this result, the divergent part of the Fokker-Planck integral [3] becomes

$$\int_{\Lambda^{-1}}^{\infty}\Gamma\left(\mathrm{n}+1,\mathrm{u}_\delta^2\right)\mathrm{u}_\delta^{-2}\mathrm{du}_\delta^2 \approx \mathrm{n}!2\ln\Lambda^{1/2}\,. \tag{8}$$

Obviously, the Coulomb logarithm is $\ln\Lambda^{1/2}$ which is the half of the usual $\ln\Lambda$. This conclusion is consistent with the result obtained in Sec. 2.

4. Modification for Maxwellian and non-Maxwellian scatters

All F-P coefficients can be obtained for Maxwellian scatters. For example, the first two-order F-P coefficients are as follows:

$$<\Delta \mathrm{v}_{\|}>=-4\omega\, a\mathrm{v}_{th}\ln\Lambda^{1/2}G(u)\,, \tag{9}$$

$$<\Delta \mathrm{v}_{\|}^2>=2\omega\,(a\mathrm{v}_{th})^2[(\ln\Lambda^{1/2}/u+u)G(u)-(3/4)u^2\gamma^*(5/2,u^2)]\,, \tag{10}$$

$$<\Delta \mathrm{v}_{\perp 1}^2>=<\Delta \mathrm{v}_{\perp 2}^2>=-[<\Delta \mathrm{v}_{\|}^2>-2\omega(av_{th})^2\ln\Lambda^{1/2}\Phi(u)/u]/2\,, \tag{11}$$

and $<\Delta \mathrm{v}^2>=<\Delta \mathrm{v}_{\|}^2>+<\Delta \mathrm{v}_{\perp 1}^2>+<\Delta \mathrm{v}_{\perp 2}^2>=2\omega(av_{th})^2\ln\Lambda^{1/2}\Phi(u)/u$ (12)

where $u=\mathrm{v}/\mathrm{v}_{\mathrm{th}}$, $\omega=n_{\mathrm{F}}\mathrm{v}_{\mathrm{th}}\pi\lambda_{\mathrm{L}}^2$, $\Phi(u)$ is the error function, $G(u)$ Chandrasekhar function, and $\gamma^*(t,x)$ the analytic incomplete gamma function. The dynamic friction coefficient and the dominant part of the diffusion tensor reduce to half. The condition for non-dominant part exceeding dominant part becomes $\mathrm{v}^2>\mathrm{v}_{th}^2\ln\Lambda^{1/2}$ instead of $\mathrm{v}^2>\mathrm{v}_{th}^2\ln\Lambda$. Hence, the definition for weakly, moderately and strongly coupled plasmas should be modified.

In other hand, the relaxation times are doubled due to $\ln\Lambda$ reducing to $\ln\Lambda^{1/2}$ if only the dominant part of the F-P coefficients is taken into account. The slowing-down time, deflection time and energy-exchange time are respectively

$$\tau_S=-u/(\partial u/\partial t)=u/[4\omega a\mathrm{v}_{th}^2\ln\Lambda^{1/2}G(u)], \tag{13}$$

$$\tau_D=-u^2/(\partial u_\perp^2/\partial t)=u^3/\{\omega(a\mathrm{v}_{\mathrm{th}})^2\ln\Lambda^{1/2}[\Phi(u)-G(u)]\}\,, \tag{14}$$

$$\tau_{ex}=-w/(\partial w/\partial t)=u^3/[8\omega(a\mathrm{v}_{\mathrm{th}})^2\ln\Lambda^{1/2}\Phi(u)]\,. \tag{15}$$

Consequently, the cross-field transport coefficients in a plasma should be reduced to half of the convention values. For example, the cross-field electrical resistivity, electron diffusion and ion thermal conductivity are reduced respectively to

$$\sigma_\perp\approx m_e/ne^2\tau_e=\sqrt{2\pi m_e}e^2\ln\Lambda^{1/2}/3(kT_e)^{3/2}\,, \tag{16}$$

$$D_\perp^e\approx kT_e/m_e\omega_{ce}^2\tau_e=4\sqrt{2\pi m_e}ne^2c^2\ln\Lambda^{1/2}/3B^2(kT_e)^{1/2}\,, \tag{17}$$

$$\kappa_\perp^i=2nkT/m_i\omega_{ci}^2\tau_i=8\sqrt{\pi m_i}n^2e^2c^2\ln\Lambda^{1/2}/3B^2(kT_i)^{1/2}\,. \tag{18}$$

For some special form of non-Maxwellian scatters, I can derive all F-P coefficients as well. For example, if the field particle has a drifting beam in the tail of a Maxwellian distribution, namely, $f(\mathrm{v})=(\pi\mathrm{v}_{\mathrm{th}}^2)^{-3/2}\{\exp(-\mathrm{v}^2/\mathrm{v}_{\mathrm{th}}^2)+b\exp[-c^2(\mathrm{v}/\mathrm{v}_{\mathrm{th}}-u_d)^2]\}$, the coefficient of dynamic friction is reduced to:

$$< \Delta \mathrm{v}_{\|} >= -2\omega\, a \mathrm{v}_{th} \ln \Lambda \{G(u) + bG[c(u-u_d)]\} . \tag{19}$$

Hence, the relaxation times are longer than in the case of Maxwellian scatters. For instant, the slowing-down time becomes

$$\tau_S = -u/(\partial u/\partial t) = u/\{4\omega a \mathrm{v}_{th}^2 \ln \Lambda^{1/2}[G(u) + bG[c(u-u_d)]\} . \tag{20}$$

The cross-field electrical resistivity, electron diffusion and ion thermal conductivity are furthermore reduced due to the non-Maxwellian effect.

5. Conclusion

It is found that a new cutoff at $\theta_{\min}$ should be introduced for constant relative velocity because the interaction distance cannot be replaced by the impact parameter. It is shown that $m\Delta\mathrm{v}$ can properly describe the Coulomb collisions in a plasma rather than θ because $m\Delta\mathrm{v}$ is determined by both θ and g that can vary from zero to infinity in a plasma. A new cutoff on $\Delta\mathrm{v}$ should be introduced for varied g because collisions are effective only if the kinetic energy is greater than the Coulomb potential energy, and/or the interaction distance is shorter than the Debye length. In both cases, it is found that the Coulomb logarithm should be reduced to half the level of conventional plasma kinetic theory. Consequently, Fokker-Planck coefficients are modified and applicable to both weakly and moderately coupled plasmas. The relaxation times increase and the cross-field electrical resistivity, electron diffusion and ion thermal conductivity are reduced to half level for Maxwellian scatters. The drifting beam of electrons in the tail of the Maxwellian distribution can furthermore modify the Fokker-Planck coefficients, increase the relaxation times and reduce the cross-field transport coefficients. The reduction of the cross-field transport might be useful for explaining why the measured cross-field diffusion and thermal conductivity well below the neoclassical level in the core of RMS plasmas The reduction of the Coulomb logarithm will modify charged-particle stopping power in inertial confinement fusion plasmas. The conventional Fokker-Planck equation is modified to include the non-dominant term, which is applicable to moderately coupled plasmas.

Acknowledgments

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