TWO-POINT BOUNDARY VALUE AND CAUCHY FORMULATIONS IN AN AXISYMMETRICAL MHD EQUILIBRIUM PROBLEM [1]

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Abstract

In this paper we present two equilibrium solvers for axisymmetrical toroidal configurations, both based on the expansion in poloidal angle method. The first one has been conceived as a two-point boundary value solver in a system of coordinates with straight field lines, while the second one uses a well-conditioned Cauchy formulation of the problem in a general curvilinear coordinate system. In order to check the capability of our moment methods to describe equilibrium accurately, a comparison of the moment solutions with analytical solutions obtained for a Solov'ev equilibrium has been performed.

1. INTRODUCTION

Since most MHD stability codes us a flux coordinate system, a mapping procedure is required to map various quantities from the (r, z) space to the (ψ, θ) space in the flux coordinate system, with θ a poloidal angle. This is the inverse equilibrium solution. Several types of inverse equilibrium solvers (iterative metric methods, direct inverse solution methods and methods of expansion in poloidal angle) were developed [1, 2].

Solving the elliptic equations as an initial value problem failed due to the ill-posed nature of the Cauchy problem for elliptic equations. In Ref. [3] a very efficient "sweeping" technique to solve the general problem of the evaluation of a "small" solution in presence of a "large" solution is given. In our Cauchy formulation, the solving of the free boundary equilibrium problem is well-conditioned.

Two equilibrium solvers for axisymmetrical toroidal configurations have been developed, both based on the expansion in poloidal angle method. The first solver has been conceived for a two-point boundary value problem, while the second one for a free boundary problem.

2. TWO-POINT BOUNDARY VALUE EQUILIBRIUM SOLVER

In a system of coordinates (a, θ, ζ), with straight field lines on the a =const surface (the current lines remain "unrectified") [4], the contravariant and covariant components of the magnetic field and current density reads as

$$B^i = \left\{0,\ \ \chi^{'} / \left(2\pi\sqrt{g^r}\right),\ \ \Phi^{'} / \left(2\pi\sqrt{g^r}\right)\right\}, \tag{1}$$

$$j^i = \left\{0,\ \left(I^{'} - \frac{\partial \nu}{\partial \zeta}\right) \Big/ \left(2\pi\sqrt{g^r}\right),\ \ \left(J^{'} + \frac{\partial \nu}{\partial \theta}\right) \Big/ \left(2\pi\sqrt{g^r}\right)\right\}, \tag{2}$$

$$B_i = \left\{\frac{1}{2\pi}\left(\frac{\partial \varphi}{\partial a} - \nu\right),\ \frac{1}{2\pi}\left(\frac{\partial \varphi}{\partial \theta} + J\right),\ \frac{1}{2\pi}\left(\frac{\partial \varphi}{\partial \zeta} - I\right)\right\}, \tag{3}$$

[1] Part of this work has been performed during a stay of one of the authors (C.V.Atanasiu) at the Max-Planck-Institut für Plasmaphysik, Garching, Germany

where χ and Φ are the poloidal and the toroidal magnetic field fluxes, I and J are the poloidal and toroidal currents, $\sqrt{g^r}$ is the Jacobian in the "rectified" coordinate system, ν and φ are periodic functions of θ and ζ characterizing the charge separation current and the scalar potential of the irrotational part of the magnetic field, respectively [5, 6].

The equations for the determination of the θ and ζ coordinates are given by relating the covariant components of the magnetic field to its contravariant components:

$$-\nu+\frac{\partial\varphi}{\partial a}=\frac{g^r_{12}}{\sqrt{g^r}}\chi^{\prime}+\frac{g^r_{13}}{\sqrt{g^r}}\Phi^{\prime},\quad J+\frac{\partial\varphi}{\partial\theta}=\frac{g^r_{22}}{\sqrt{g^r}}\chi^{\prime}+\frac{g^r_{23}}{\sqrt{g^r}}\Phi^{\prime},\quad -I+\frac{\partial\varphi}{\partial\zeta}=\frac{g^r_{23}}{\sqrt{g^r}}\chi^{\prime}+\frac{g^r_{33}}{\sqrt{g^r}}\Phi^{\prime}. \quad (4)$$

Let us consider a general curvilinear coordinate system ("unrectified") (a,ω,ζ) with the connection between the poloidal angles given by $\theta=\omega+\Lambda(a,\omega)$, with Λ a periodic function of ω to be determined. In a local coordinate system (x,y,ζ), associated with the magnetic axis of the equilibrium configuration, we consider the x axis directed along the normal and the y axis directed along the binormal to the magnetic axis. A coordinate transformation through Fourier series in ω is given by [7] :

$$x=\rho(a,\omega)\cos\omega,\quad y=\rho(a,\omega)\sin\omega,\quad \rho^2(a,\omega)=a^2+\mathrm{Re}\left[\sum_{m=-\infty,m\neq0}^{\infty}\delta_m\mathrm{e}^{\mathrm{i}m\omega}\right], \quad (5)$$

where $\delta_m(a)$ are the complex moments ($\delta_m=\delta^*_{-m}$), and by definition $a^2=\Phi(a)/\pi B_0$, with B_0 the toroidal magnetic field at the magnetic axis ($a=0$). The metric coefficients g_{ik} have been separated in the form

$$g_{ik}=g^{(0)}_{ik}+g^{(1)}_{ik}+g^{(2)}_{ik},\quad \tilde{g}^{(0)}_{ik}=0,\quad \overline{g}^{(1)}_{ik}=0,\quad \overline{g}^{(2)}_{ik}\neq0, \quad (6)$$

where$()^{(0)}$, $()^{(1)}$ and $()^{(2)}$ represent the averaged part, the linear periodic part, and the nonlinear periodic part, respectively.

Eliminating φ from Eqs. (4), separating the different components: those with the superscripts (1) in the l.h.s. and those with the superscripts (2) in the r.h.s. of the equations and identifying each moment m in the l.h.s. of the equations, we have obtained the following system of complex differential equations

$$Y^{''}_m+\left(3+2\frac{\mu^{'}a}{\mu}\right)\frac{Y^{'}_m}{a}-(m^2-1)\frac{Y_m}{a^2}=-2\frac{W_m}{\mu a^2}, \quad (7)$$

where $Y_m=\delta_m/a$, $\mu=1/q$ is the rotational transform, $W_m=W_m(a,\mu(a),g_{ik}(a,\omega),p^{'}(a),B_0,$ $R,\delta_{l,l\neq m})$ is a nonlinear functional and prime indicates differentiation with respect to a. By solving this system as a Cauchy problem for $m=1$: $\delta_1(0)=\delta^{'}_1(0)=0$, and as a two-point boundary value problem for $m>1$: $\delta_m(0)=0$ and given $\delta_{m,m\neq1}(1)$, one obtains the $\delta_m(a)$ dependence over the full plasma region and thus the full equilibrium description of the considered plasma.

3. FREE-BOUNDARY EQUILIBRIUM SOLVER

For nested axisymmetrical flux surfaces, the Grad-Shafranov equation has the well-known form

$$\frac{1}{\sqrt{g}}\left[-\frac{\partial}{\partial a}\frac{g_{22}}{\sqrt{g}}\Psi^{*'}+\frac{\partial}{\partial\omega}\frac{g_{12}}{\sqrt{g}}\Psi^{*'}\right]=4\pi^2\mu_0\frac{p^{*'}}{\Psi^{*'}}+\frac{\mu_0^2}{r^2}\frac{F^*F^{*'}}{\Psi^{*'}}, \quad (8)$$

This equation can be put in the form

$$\left(\frac{D'}{D}-\frac{\Psi''}{\Psi'}+\frac{H'}{H}-\frac{g'_{22}}{g_{22}}\right)\Psi'Q+\frac{\partial}{\partial\omega}G\Psi'=S. \quad (9)$$

where

$$\Psi = \Psi^*/T, \quad F = \mu_0 F^*/T, \quad T = 2\pi R_0 B_0, \quad p = \mu_0 p^*/B_0^2, \quad H = r/R_0,$$
$$j(a) = FF^{'}/\Psi^{'} + p^{'}/\Psi^{'}, \quad D = r^{'} y_\omega - r_\omega y^{'} H = 1 + Kx, \quad K = 1/R_0, \quad x = r - R_0,$$
$$S = p'/\Psi' \left(HD - D/H\right) + jH/D, \quad G = g_{12}/(HD), \quad Q = g_{22}/(HD). \tag{10}$$

with R_0 the radius of the magnetic axis and B_0 the magnetic induction at the magnetic axis. For sake of simplicity, flux surfaces possessing up-down symmetry will be presented only. For this case it is possible to represent the coordinate transformations as a Fourier series in ω

$$x = \frac{a^2}{b}\cos\omega + \sum_{m=0,\quad m\neq 1}^{\infty} x_m \cos m\omega, \quad y = b\sin\omega, \quad b = \lambda^{1/2} a, \tag{11}$$

where x_0 is the shift, λ is the ellipticity, x_2 is the triungularity, x_3 the quadrangularity of the flux surfaces, etc. By substituting these Fourier series for x into the Grad-Shafranov equation (9), and retaining only a finite number M of amplitude functions, one obtains an approximate solution for the flux surface geometry. To identify each moment, we follow the "classical" way of averaging Eq. (9) multiplied by $\cos m\omega$ with respect to ω

The Grad-Shafranov equation has to be solved now in its Cauchy formulation: with given ellipticity and initial conditions $x_m(0) = 0$ at the magnetic axis. One can prove that near the magnetic axis all the $x_m, \quad m \geq 2$, moments have a $x_m \sim C_m a^m$ dependence, with C_m a free parameter, while the magnetic shift x_0 has an a^2 dependence.

For numerical calculations it is more convenient to introduce the following new functions

$$w_m = \frac{x_m}{a^{m-1}}, \quad m \geq 2. \tag{12}$$

Therefore, the averaged Eq. (9) can be written in a matrix form

$$\mathbf{A}\mathbf{W}^{''} = \mathbf{B}, \tag{13}$$

where the vector $\mathbf{W}$ has the form

$$\mathbf{W}^T = [\Psi \;\; x_0 \;\; b \;\; w_2 \;\; ... \;\; w_M], \tag{14}$$

while the a_{ij} and b_j elements result by averaging. Thus, the Grad-Shafranov equation has been put in the form of a system of differential equations, appropriate for numerical computation

$$\mathbf{W}^{''} = \mathbf{A}^{-1}\mathbf{B} = \mathbf{F}(a, \mathbf{W}, \mathbf{W}^{'}) \tag{15}$$

To solve the system of equations (15), the initial conditions for $\mathbf{W}$ and $\mathbf{W}^{'}$ at the magnetic axis have been specified. For this Cauchy formulation, the problem is well-conditioned: for each moment w_m only the "large" solutions $\sim a^m$ play a role. The "weak" solutions $\sim a^{-m}$, which could appear due to numerical errors, are "forced" to zero in the vicinity of the magnetic axis, while, near the plasma boundary, vanish "naturally".

4. EXACT EQUILIBRIUM SOLUTIONS

To check the capability of our moment method to accurately describe equilibria, a comparison of the moment solutions with analytical solutions obtained for a quasiuniform current density distribution (Solov'ev equilibrium) [8]

$$\frac{\partial^2\Psi}{\partial z^2} + \frac{\partial^2\Psi}{\partial r^2} - \frac{1}{r}\frac{\partial\Psi}{\partial r} = Ar^2 + B, \quad A = -\frac{dp}{d\Psi}, \quad B = -\frac{1}{2}\frac{dF^2}{d\Psi}, \tag{16}$$

with A and B constants. A particular solution of Eq. (16) is

$$\Psi = \frac{A}{8} r^4 + \frac{B}{2} z^2. \tag{17}$$

The general solution can be written in the form

$$\Psi = C_0 \Psi_0 + C_1 \Psi_1 + C_2 \Psi_2 + C_3 \Psi_3 + ... \tag{18}$$

with C_0, C_1, C_2, ... some constants. By separating the variables, two groups of eigenfunctions can be distinguished. The first group has the form

$$\Psi_{2n} = \sum_{k=0}^{n-1} \alpha_k r^{2(n-k)} z^{2k}, \quad \Psi_{2n+1} = z \sum_{k=0}^{n-1} \beta_k r^{2(n-k)} z^{2k}, \tag{19}$$

with a finite number of terms in the sum for any n, while the α_k and β_k coefficients are given by recurrence formulas with $\alpha_0 = 1$ and $\beta_0 = 1$. The second group has the form

$$\Psi_{2n} = \sum_{k=0}^{n} (a_k \ln r + b_k) z^{2(n-k)} r^{2k}, \quad \Psi_{2n+1} = z \sum_{k=0}^{n} (c_k \ln r + d_k) z^{2(n-k)} r^{2k}, \tag{20}$$

with the a_k, b_k, c_k and a_k coefficients determined by recurrence relations, with $a_0 = 0, \quad b_0 = 1$, $c_0 = 0$ and $d_0 = 1$.

With these two groups of solutions, representing a generalized Solov'ev equilibrium, one can describe a very large range of MHD equilibria.

5. CONCLUSIONS

Two equilibrium solvers for axisymmetrical toroidal configurations have been presented. The first one has been conceived as a two-point boundary value problem to solve the fixed boundary equilibrium problem, while the second one uses the Cauchy formulation to solve both the free boundary problem and the fixed boundary one.

In our Cauchy formulation, the solving of the free boundary equilibrium problem is well-conditioned: for each moment, only the "large" solutions play a role, while the "weak" solutions are "forced" to zero or vanish "naturally". Thus, the integration of the system of differential equations, resulting from this formulation, takes place in one "iteration" only.

Acknowledgments

The stimulating discussions with Prof. K.Lackner and dr.L.E.Zakharov are greatly appreciated. Parts of this work were supported by the Max-Planck-Institut für Plasma Physik under the auspices of BMBF Project No. ROM-010/97 and the Russian Foundation of Fundamental Research Projects No. 97-02-17695 and 96-12-96815.

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