Probabilistic transport models for fusion

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Abstract. A generalization of diffusive (Fickian) transport is considered, in which particle motion is described by probability distributions. We design a simple model that includes a critical mechanism to switch between two transport channels, and show that it exhibits various interesting characteristics, suggesting that the ideas of probabilistic transport might provide a framework for the description of a range of unusual transport phenomena observed in fusion plasmas. The model produces power degradation and profile consistency, as well as a scaling of the confinement time with system size reminiscent of the gyro-Bohm/Bohm scalings observed in fusion plasmas, and rapid propagation of disturbances. In the present work we show how this model may also produce on-axis peaking of the profiles with off-axis fuelling. It is important to note that the fluid limit of a simple model like this, characterized by two transport channels, does not correspond to the usual (Fickian) transport models commonly used for modelling transport in fusion plasmas, and behaves in a fundamentally different way.

1. Introduction

Transport in fusion plasmas is commonly described using Fickian diffusive transport models, in which the (particle and heat) flux is a function of the local parameters (gradients). A large body of literature testifies to the existence of phenomena that are difficult to fit into this framework (e.g. [1, 2]). Such phenomena include: Bohm scaling of confinement (approximately linearly proportional to the system size L, whereas standard diffusion would predict $\propto L^2$); power degradation (i.e. the confinement time decreases with increasing heating, P); profile consistency or stiffness of profiles, making the plasma profiles relatively insensitive to the power deposition, even leading to profile peaking during off-axis heating (e.g. [3]); and rapid phenomena in which the core plasma seems to respond almost instantaneously to e.g. edge cooling (much faster than the diffusive timescale permits) (e.g. [4]). The common approach to the problem of modelling these phenomena is to include additional convective terms in the diffusive model. However, the required convective terms are usually much larger than the available physical mechanisms (e.g. the Ware pinch), and vary from case to case, so that this approach does not permit making accurate predictions. The current paper re-examines the basic hypotheses underlying the usual modelling procedure and suggests an alternative.

2. Diffusion revisited

In this paper, we reconsider the foundations of the standard diffusive model in the framework of the Continuous Time Random Walk (CTRW) formalism [5,6]. As is well known, diffusion is based on Brownian motion, in which individual particles remain a certain (random) time at their current position and then take a (random) step. These waiting times and steps are drawn from probability distributions, which we assume statistically independent for simplicity. Let

$\psi(t\text{-}t')$ be the waiting time distribution, giving the probability that a particle, having arrived at position x' at time t', takes a step at time $t \geq t'$, and $p(x\text{-}x',x',t)$ the step distribution, giving the probability at time t that a particle takes a step from x' to x. Even though the step distribution p may depend on (x', t), e.g. via a dependence on the local plasma parameters, it can be shown [7] that a collection of such particles will obey the Generalized Master Equation:

$$\frac{\partial n}{\partial t} = \int_0^1 dt' \phi(t-t') \left[\int_{-\infty}^{\infty} dx' \, p(x-x',x',t) n(x',t) \right] - \int_0^1 dt' \phi(t-t') n(x,t') \tag{1}$$

where $n(x,t)$ is the particle density. The function ϕ is related to the waiting time ψ above [7]. Naturally, if the waiting time is chosen exponential, $\psi(t\text{-}t') = exp[-(t\text{-}t')/\tau_D]/\tau_D$, and the step distribution Gaussian, $p(x\text{-}x',x',t) = exp[-(x\text{-}x')^2/4\sigma^2]/2\sigma\sqrt{\pi}$, normal Fickian diffusion is recovered and Eq. (1) is reduced to:

$$\frac{\partial n}{\partial t} \cong \frac{\sigma^2}{\tau_D} \frac{\partial^2 n}{\partial x^2} \tag{2}$$

in the limit of small σ. However, the CTRW formalism allows other distributions ψ and p. This immediately raises the question of whether the standard choice of exponential waiting time distributions and Gaussian step size distributions is always justified. The Gaussian distribution is the limit distribution of the sum of independent identically distributed (i.i.d.) random variables whose individual distribution has finite width. Removing the latter restriction, the sums of general i.i.d. variables are distributed according to the Lévy distribution. Thus, the Lévy distribution is a natural generalization to the Gaussian distribution. It is characterized, among other things, by a parameter α. The Lévy distribution at $\alpha = 2$ is just the Gaussian distribution, with an exponential tail for large x, $p(x) \propto \exp[-x^2]$, but for $0 < \alpha < 2$ the tail is algebraic, $p(x) \propto x^{-\alpha-1}$. This tail has profound consequences, since it leads to long-range correlations and general unusual ("non-local") behaviour. In particular, it means that the boundary conditions cannot be ignored (up to a constant) when constructing steady state solutions.

The question whether Lévy distributions are relevant for transport in fusion plasmas can only be partially answered at this moment. Generally speaking, if fusion plasmas are governed to some degree by Self-Organized Criticality (SOC) and transport is avalanche-like, as suggested by various authors [8,9], Lévy distributions can model the supercritical transport channel. Furthermore, Lévy distributions have been observed in numerical simulations of resistive pressure-gradient driven turbulence using particle tracking methods [10, 11].

3. Toy model for transport

To test what effect these assumptions might have on transport, we have constructed a toy model with an exponential waiting time distribution (for simplicity and to retain Markovian temporal behaviour) and a Gauss/Lévy step distribution. To produce the required profile consistency, we also included a critical gradient mechanism. Thus, we design our 1-D, single-field toy model such that:

$|\nabla n| < [\nabla n]_{crit}$: "normal" transport, p = Gaussian
$|\nabla n| \geq [\nabla n]_{crit}$: "anomalous" transport, p = Lévy

A Generalized Master Equation can be derived for this model [7]:

$$\frac{\partial n}{\partial t} = \frac{1}{\tau_D} \left[\int_0^L dx' \, p[x-x', f(n(x',t))] n(x',t) \right] - \frac{n(x,t)}{\tau_D} + S(x) \tag{3}$$

where the step distribution p depends on local parameters via a function f. We also include an external drive $S(x)$ to compensate for continuous edge losses.

Recently, we have shown [12] that the fluid limit of this nonlinear CTRW model leads to a fractional differential equation. Here and for simplicity, we illustrate this limit for the particular situation when the supercritical transport is described by a Gaussian. In this case, the fluid limit of Eq. (3) is:

$$\frac{\partial n(x,t)}{\partial t} = S_0 + \frac{\partial^2}{\partial x^2}\left\{\left[H\left(1-\frac{|\partial n/\partial x|}{Z_c}\right)\sigma_1^2 + \left[1-H\left(1-\frac{|\partial n/\partial x|}{Z_c}\right)\right]\sigma_2^2\right]n(x,t)\right\} \qquad (4)$$

Here, H is the Heaviside step function and σ_1 and σ_2 are the widths of the two Gaussians. The time average of the square bracket gives the effective diffusivity. This diffusivity, resulting from the combined effects of the two mechanisms, has a strong radial dependence, as shown in Fig. 1. Note that Eq. (4) differs from the usual diffusion equation, Eq. (2). Eq. (4) produces both an effective diffusivity and a pinch, its velocity being equal to the radial derivative of the effective diffusion. This type of equation is typical for avalanche-like transport processes [13]. If the supercritical mechanism involves a Lévy step distribution, Eq. (4) is more complicated because it then leads to a fractional derivative in space [12].

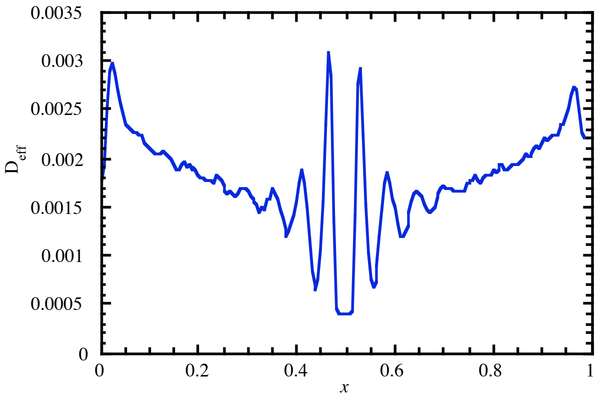

Fig. 1 – Effective diffusivity for a model incorporating criticality and 2 Gaussian step distributions (σ_1 = 0.08, σ_2 = 0.02)

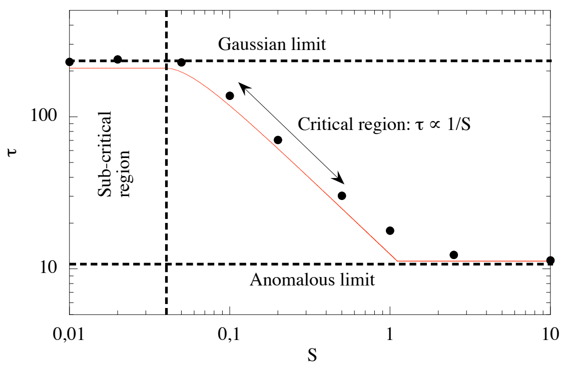

Fig. 2 – Power degradation.

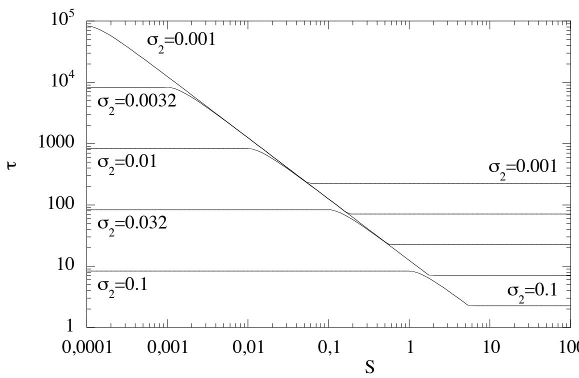

Fig. 3 – Scaling with system size.

4. Numerical results

With an appropriate choice of parameters, this system will self-regulate its gradient around the critical value for a range of values of the drive, leading to stiff profiles. Fig. 2 shows the confinement time obtained from a power scan. It is seen that the critical gradient mechanism indeed leads to power degradation. In addition, at low power the confinement time scales diffusively ($\tau \propto L^2$), whereas as high power the system scales anomalously ($\tau \propto L^\alpha$), which is reminiscent of Bohm/gyro-Bohm scaling behaviour. This is clarified in Fig. 3, which schematically shows the scaling behaviour of Fig. 2 for various different choices of the system size L ($\propto 1/\sigma_2$). In the critical region between the two power extremes, the confinement time is seen to depend exclusively on the fuelling rate S and the critical gradient, and not on either the step size or the waiting time. Since any (effective) diffusion coefficient must always depend on the step size and the waiting time, this means that the system behaviour cannot be described by an (effective) diffusion model in the critical region.

The stiffness of the model also leads to the fast propagation of disturbances, e.g. a “cold pulse” (produced by a sudden reduction of n at the edge) was found to travel almost three orders of magnitude faster than what would be expected from a diffusive estimate [7].

A sensitive test of non-standard behaviour is provided by an off-axis fuelling experiment [14]. Figs. 4 and 5 show the response of the profile of n for two off-axis fuelling amplitudes. At low S, the system is not very critical and responds almost diffusively; but at high values of S, the system is critical and the profile peaks, even though the source S is zero in the centre. This behaviour is due to the self-regulation of the profile around its critical value, in combination with the finite particle excursions permitted by the step distribution. Due to the latter circumstance, particles can move in the direction of the gradient, which would be impossible in the usual diffusive limit (in which the single-step particle excursions are assumed to be negligibly small).

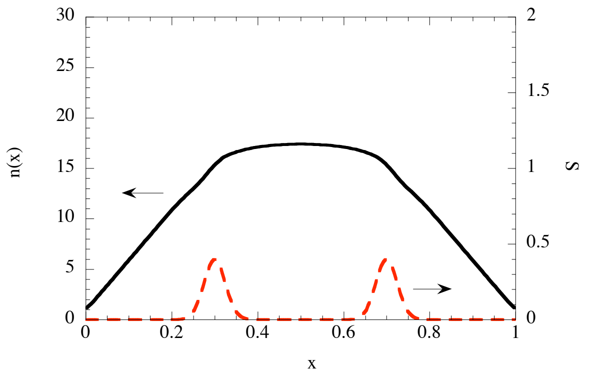

Fig. 4 – Off-axis fuelling, low power.

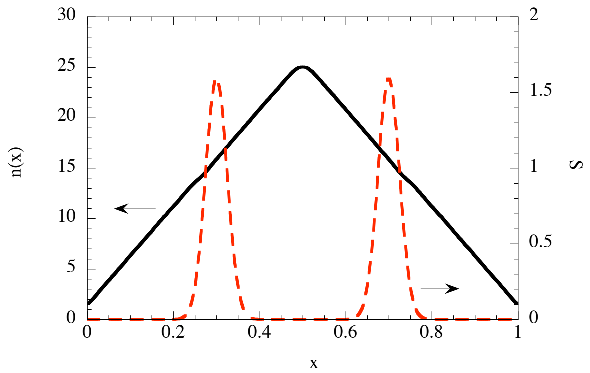

Fig. 5 – Off-axis fuelling, high power

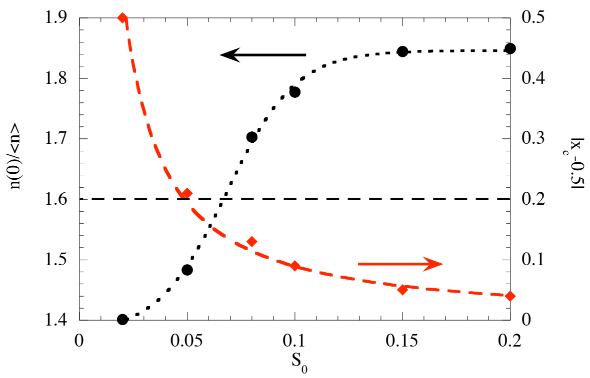

Fig. 6 – Profile peaking factor and width of central non-critical region as a function of heating.

In another numerical experiment, we found that the amount of profile peaking depends on the inverse of the size of the central non-critical region, as shown in Fig. 6. In other words, the peaking is reduced when the diffusive transport channel is enhanced. This behaviour is possibly in agreement with actual fusion plasma experiments in which the collisionality was varied [15].

5. Discussion

In this paper, we have presented a straight-forward generalization of the common diffusive transport model. The generalization consists of allowing the particle step distributions to include Lévy distributions. Based on this idea, we have designed a numerical 1-D, 1-field toy model, which incorporates a critical gradient mechanism to switch between normal (Gaussian) transport and anomalous (Lévy) transport. The model was shown to produce interesting behaviour: (1) stiff profiles, (2) power degradation, (3) system size scaling reminiscent of Bohm/gyro-Bohm behaviour, (4) rapid transport events, and (5) on-axis profile peaking during off-axis heating. This behaviour is in qualitative agreement with observations in fusion plasmas.

The fluid limit of this model leads to a fractional differential equation. Even in the case in which both transport mechanisms are Gaussian, the partial differential equation is not given by the simple diffusion equation, and it combines the effects of diffusion and a pinch. Such a combination is typical for avalanche-like transport processes.

The model needs to be extended to at least two fields (n and T) before any modelling of actual experiments can be attempted; this work is in progress.

An important message from this work is the following. Even under the assumption that the current model is essentially correct, it is still possible to model any individual experiment using the conventional description (involving effective diffusion and convection coefficients) [7]. However, the long-range correlation of the Lévy distributions and the non-linearity due to the critical mechanism introduce a fundamental dependence of these effective parameters on the system size L. Therefore, the effective diffusion and convection coefficients obtained in a system of size L cannot be used to predict the outcome of an experiment in a system of size $L' \neq L$! The only correct way to generalize the results of the transport analysis to systems of any size is therefore to determine the probability distributions directly, and use a description in terms of a Master Equation. Also, a model based on the conventional Fickian description is not capable of both matching the steady state situation and reproducing the transient effects the toy model exhibits. In this sense, transient transport experiments may provide a definitive test of these ideas.

References

[1] CALLEN, J.D., and KISSICK, M.W., Plasma Phys. Control. Fusion **39** (1997) B173
[2] STROTH, U., Plasma Phys. Control. Fusion **40** (1998) 9
[3] PETTY, C.C., and LUCE, T.C., Nucl. Fusion **34** (1994) 121
[4] GENTLE, K.W., BRAVENECK, R.V., CIMA, G., *et al.,* Phys. Plasmas **2** (1995) 2292
[5] MONTROLL, E.W., and WEISS, G.H., J. Math. Phys. **6** (1965) 167.
[6] BALESCU, R., Phys. Rev. E **51** (1995) 4807.
[7] VAN MILLIGEN, B.Ph., SÁNCHEZ, R., and CARRERAS, B.A., Phys. Plasmas **11** (2004) 2272.
[8] DIAMOND, P.H., and HAHM, T.S., Phys. Plasmas **2** (1995) 3640.
[9] NEWMAN, D., CARRERAS, B.A., DIAMOND, P.H., and HAHM, T.S., Phys. Plasmas **3** (1996) 1858.
[10] CARRERAS, B.A., LYNCH, V.E., and ZASLAVSKY, G.M., Phys. Plasmas **8** (2001) 5096.
[11] DEL CASTILLO NEGRETE, D., CARRERAS, B. A., and LYNCH, V., Phys. Plasmas **11** (2004) 3854, and this conference.
[12] SÁNCHEZ, R, CARRERAS, B.A., and VAN MILLIGEN, B.Ph., Phys. Rev. E (to be published).
[13] DÍAZ-GUILERA, A., Phys. Rev A **45** (1992) 8551.
[14] VAN MILLIGEN, B.Ph., CARRERAS, B.A., and SÁNCHEZ, R., Phys. Plasmas **11** (2004) 3787.
[15] ANGIONI, C., PEETERS, A.G., PREVERZEV, G.V., et al., Phys. Rev. Lett. **90** (2003) 205003.