Mechanism of stabilization of ballooning modes by toroidal rotation shear in tokamaks

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Abstract. A ballooning perturbation in a toroidally rotating tokamak is expanded by square-integrable eigenfunctions of an eigenvalue problem associated with ballooning modes in a static plasma. Especially a weight function is chosen such that the eigenvalue problem has only the discrete spectrum. The eigenvalues evolve in time owing to toroidal rotation shear, resulting in countably infinite number of crossings among them. The crossings cause energy transfer from an unstable mode to the infinite number of stable modes; such transfer works as the stabilization mechanism of the ballooning mode.

1. Introduction

The theory of ideal magnetohydrodynamic (MHD) ballooning modes in a static plasma has been well established by applying the Wentzel-Kramars-Brillouin (WKB) method [1, 2]. For a static plasma, the time dependence of the mode is assumed as $e^{i\omega t}$, and we obtain an eigenvalue problem with second-order ordinary differential operator along a magnetic field line in the lowest-order WKB approximation. The WKB method was also applied for the ideal MHD ballooning modes in a toroidally rotating tokamak. For the rotating plasma, we cannot generally assume the time dependence of the mode as $e^{i\omega t}$, and we obtain coupled wave equations along a magnetic field line [3]. Then we have to solve them as an initial value problem. The behavior of the solution has not been fully understood, although several numerical solutions have been obtained [4–8]. In Ref. [7], it was found that toroidal rotation shear damps the perturbation energy of ballooning modes; the damping phase alternates with an exponentially growing phase. When the damping compensates the growth in the sense of time average, the plasma is marginally stable. However, it has not been clarified how the toroidal rotation shear damps the perturbation energy.

In the present paper, we explore how toroidal rotation shear stabilizes ballooning modes via a spectral analysis of a linear differential operator for ballooning modes in a static plasma. If we crudely try to expand the ballooning perturbation in a rotating plasma by a set of eigenfunctions for ballooning modes in a static plasma, such an attempt will fail because of difficulty in treating the continuous spectrum; the generalized eigenfunctions belonging to the continuous spectrum are singular and non-square-integrable. Thus we cannot treat them numerically. Although analytical studies are possible, however, tractable cases are limited by geometry, profiles, and so on. An earlier work tried to resolve the difficulty by replacing it with the closely spaced discrete spectrum [9], which is accomplished by replacing the covering space with a large but finite interval. However, when the ballooning mode in a rotating tokamak propagates in a wide region, we have to take the interval large enough. Then the spacings among the eigenvalues become closer and the eigenfunctions become nearly singular, which indicates the difficulty in the analysis. We have formulated an associated eigenvalue problem yielding only the discrete spectrum in the original covering space [10]; it is composed of the linear differential operator for ballooning modes in a static plasma and a weight function which is chosen to generate only the

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discrete spectrum. Thus, we obtain a complete set of square-integrable eigenfunctions which is defined in the same domain as the ballooning mode in a toroidally rotating tokamak. Then we can expand the ballooning mode by the set of eigenfunctions. Actually, since the associated eigenvalue problem include time as a parameter, we can define the set of eigenfunctions as well as the eigenvalues at each instance. By looking at the time evolution of the eigenvalues and the amplitude of each eigenmode, we have clarified that energy transfer occurs from an unstable mode to countably infinite number of stable modes owing to the toroidal rotation shear. This works as a stabilization mechanism of the ballooning mode.

In Sec. 2., we introduce a model equation for ballooning modes in toroidally rotating tokamaks. The reason why the model equation is used is that the original ballooning equations including plasma compressibility are too complicated to analyze. As will be mentioned in Sec. 2., the model equation possesses important features of the original ballooning equations. In addition, it reduces to the conventional ballooning equation when the rotation shear vanishes. Therefore, it smoothly connects ballooning modes in a rotating plasma to those in a static plasma. In Sec. 3., we modify the ballooning equation in a static plasma, and obtain an eigenvalue problem which yields only the discrete spectrum. Several examples of the numerical solutions to the eigenvalue problem are shown. In Sec. 4., a numerical solution of the model equation is expanded by the complete set of square-integrable functions. This clarifies that the ballooning mode is stabilizes by energy transfer from an unstable mode to countably infinite number of stable modes. Conclusions are given in Sec. 5.

2. A model equation

Ballooning equations in toroidally rotating tokamaks were derived as coupled wave equations for two components of a displacement vector [3]. As same as for ballooning modes in a static plasma, the plasma displacement is expressed in an eikonal form as $\boldsymbol{\xi} = \hat{\boldsymbol{\xi}} e^{\mathrm{i} nS}$ where $\boldsymbol{\xi}$ is the plasma displacement vector in the covering space, n is the toroidal mode number and S is the eikonal. The key point of the formulation is that the eikonal S is chosen so that it satisfies not only $\mathbf{B} \cdot \nabla S = 0$ but also $\mathrm{d}S/\mathrm{d}t = 0$. The resultant wave vector, which is a gradient of the eikonal, depends on time as

$$\mathbf{k} \equiv \nabla\zeta - q\nabla\theta - (\vartheta - \theta_k + \dot{\Omega}t)\nabla q, \tag{1}$$

where θ and ζ are a poloidal angle and a toroidal angle, respectively, ϑ is an extended poloidal angle in the covering space, θ_k is a ballooning angle, q is a safety factor, $\dot{\Omega}$ is a ratio of the rotation shear to the magnetic shear, defined by $\mathrm{d}\Omega/\mathrm{d}q$, and Ω is the toroidal rotation frequency. Features of the coupled wave equations are as follows; (i) convection terms exist and (ii) the coefficients of the equations depends on time through the wave vector and have the dynamical lattice symmetry [3]. We proposed a model equation which has the above two features in Ref. [11],

$$\bar{\rho}\left(\frac{\partial^2\xi_\perp}{\partial t^2} - U\frac{\partial\xi_\perp}{\partial t}\right) = \mathcal{L}\xi_\perp \tag{2}$$

where the ballooning operator $\mathcal{L}$ is defined as

$$\mathcal{L}\xi_\perp \equiv \frac{\partial}{\partial\vartheta}\left(f\frac{\partial\xi_\perp}{\partial\vartheta}\right) - g\xi_\perp \tag{3}$$

and the coefficients are

$$\bar{\rho} \equiv \frac{\mu_0 \rho |\mathbf{k}|^2 \sqrt{g}}{B^2}, \tag{4}$$

$$U \equiv \frac{2\mathbf{k} \cdot \nabla\Omega}{|\mathbf{k}|^2}, \tag{5}$$

$$f \equiv \frac{|\mathbf{k}|^2}{B^2 \sqrt{g}}, \tag{6}$$

$$g \equiv -\frac{2\mu_0}{B^4}(\mathbf{B} \times \mathbf{k} \cdot \boldsymbol{\kappa})(\mathbf{B} \times \mathbf{k} \cdot \nabla p). \tag{7}$$

It should be noted that Eq. (2) reduces to the conventional ballooning equation in a static plasma when the toroidal rotation shear Ω' is set to zero. Here, $\xi_\perp$ is a perpendicular (both to the magnetic field and to the wave vector) component of the displacement vector, μ_0, ρ and p are vacuum permeability, mass density and pressure, respectively, $\mathbf{B}$ is a magnetic field, $\boldsymbol{\kappa}$ is a magnetic curvature and $\sqrt{g}$ is the Jacobian. Fourier transformation in time is not effective for this problem since the coefficients of Eq. (2) include t^2 and therefore the Fourier transformation yields a second-order derivative with respect to the transformation parameter. Instead, we have solved Eq. (2) as an initial value problem. We have confirmed numerically that damping phases appear in the time evolution of $\int_{-\infty}^{\infty} \mathrm{d}\vartheta |\xi_\perp|^2$ [11] as same as the numerical solutions to the original equations [7]. The damping phase alternates with an exponentially growing phase, which is essential for the stabilization of the ballooning modes [7] (see also Fig. 4(a)).

3. Eigenvalue problem without continuum

In the growing phases of the time evolution, the instantaneous growth rates are obviously related to the unstable eigenvalues of the eigenvalue problem

$$\mathcal{L}\xi = -\bar{\rho}\lambda\xi, \tag{8}$$

which is the ballooning equation in a static plasma. In fact, especially when the rotation shear Ω' is small enough, the the coefficients of Eq. (2) does not change significantly in time. Also, the second term of the left hand side almost vanish. Then the time evolution of the ballooning mode can be approximated by the spectral resolution of Eq. (8). Therefore, it seems natural to expand $\xi_\perp$ in Eq. (2) by the eigenfunctions of Eq. (8) for further analysis of the stabilization mechanism. However, as mentioned in the Introduction, this choice of the eigenfunction leads to intractable difficulty because of the continuous spectrum; the generalized eigenfunction corresponding to the continuous spectrum is singular and non-square-integrable which cannot be treated numerically. Therefore, we consider another eigenvalue problem

$$\mathcal{L}\xi = -\bar{\rho}h\lambda\xi, \tag{9}$$

which is composed of the ballooning operator $\mathcal{L}$ and a weight function $\bar{\rho}h$. If we appropriately choose the function h, the eigenvalue problem yields only the discrete spectrum. Similar technique has been already applied for solving the Newcomb equation [12] in a tokamak [13].

The spectrum, or whether the eigenfunction is square-integrable or not, depends on the behavior of ξ at large $|\vartheta|$. Thus changing the coordinate as $x \equiv 1/\vartheta$ and we obtain

$$x^2 \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 f \frac{\mathrm{d}\xi}{\mathrm{d}x}\right) - g\xi = -\lambda\bar{\rho}h\xi. \tag{10}$$

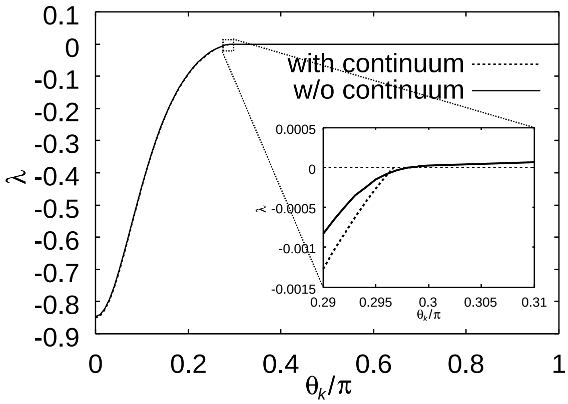


FIG. 1. The smallest eigenvalues of Eqs. (8) and (9) as functions of θ_k. For $|\theta_k/\pi| < 0.297$, both Eqs. (8) and (9) yield the discrete eigenvalues. The eigenvalues nearly coincide. For $|\theta_k/\pi| > 0.297$, only Eq. (9) yields the discrete eigenvalue.

By changing the variable as $\varphi \equiv \sqrt{x^2 f}\xi$, this equation becomes

$$-\frac{x^4 f}{\bar{\rho}h}\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} + \frac{x^2}{\bar{\rho}h}\left[\sqrt{x^2 f}\frac{\mathrm{d}^2\sqrt{x^2 f}}{\mathrm{d}x^2} + \frac{g}{x^2}\right]\varphi = \lambda\varphi. \tag{11}$$

Taking $h = f x^4/\bar{\rho}$ yields an equation of the form of the Schrödinger equation as

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} - \frac{1}{x^2 f}\left[\sqrt{x^2 f}\frac{\mathrm{d}^2\sqrt{x^2 f}}{\mathrm{d}x^2} + \frac{g}{x^2}\right]\varphi = -\lambda\varphi. \tag{12}$$

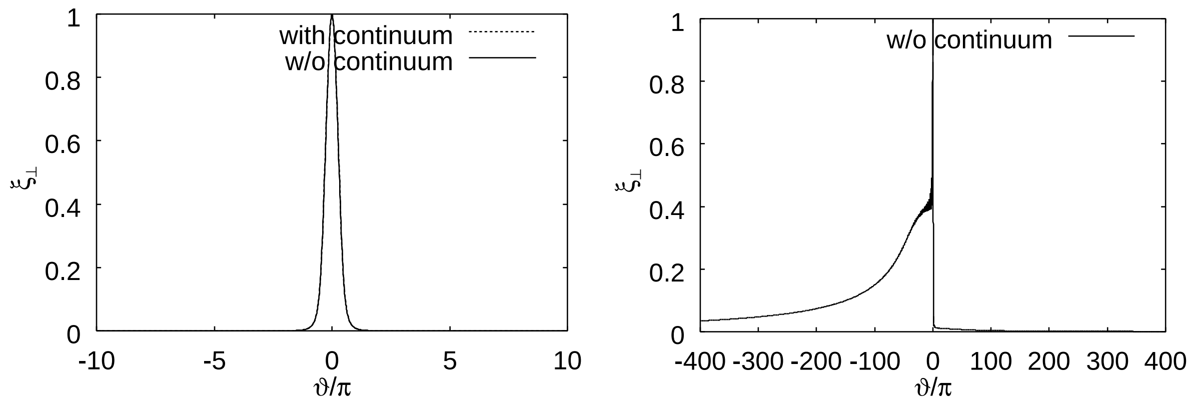
It is noted that the secularity of f and $\bar{\rho}$ are both ϑ^2, then h should be proportional to ϑ^{-4} as $x \to 0$ (or $|\vartheta| \to \infty$). Then the potential term diverges (although it oscillates); this indicates the existence of infinitely high potential wall at $|\vartheta| \to \infty$. It is noted that h in Eq. (9) is taken as $h \propto \vartheta^{-4}$ at large $|\vartheta|$ and $h = 1$ at moderate $|\vartheta|$, since (i) it is sufficient to choose h proportional to ϑ^{-4} at large $|\vartheta|$ for obtaining only the point spectrum and (ii) the eigenfunctions within the region with $h = 1$ may not be much different from those of Eq. (8) (we will see this below). Actually $h = 1$ and $h \propto \vartheta^{-4}$ are smoothly connected at $\vartheta/\pi = \pm 40$ in the following results. It is possible to take the connection point much farther from the origin, however, we have to take much wider computational domain to achieve good convergence of the numerical solutions. It is also possible to move the connection point in time, however, it is fixed in the following.

Here, we should point out three features of Eq. (9). Firstly, Eq. (9) has the Sturmean properties. Therefore, the eigenfunctions form a complete set and the orthogonality condition is given by

$$\int_{-\infty}^{\infty} \mathrm{d}\vartheta\bar{\rho}h\xi_j\xi_k^* = \delta_{jk} \tag{13}$$

where δ_{jk} is the Kronecker's delta function. Secondly, the marginally stable state is the same as that of Eq. (8); the quadratic forms of them are given by

$$\int_{-\infty}^{\infty} \mathrm{d}\vartheta\left(f\left|\frac{\mathrm{d}\xi}{\mathrm{d}\vartheta}\right|^2 + g|\xi|^2\right) = \begin{cases} \lambda\int_{-\infty}^{\infty}\mathrm{d}\vartheta\bar{\rho}|\xi|^2 & \text{for Eq. (8),} \\ \lambda\int_{-\infty}^{\infty}\mathrm{d}\vartheta\bar{\rho}h|\xi|^2 & \text{for Eq. (9).} \end{cases} \tag{14}$$



(a) $\theta_k/\pi = 0$ (b) $\theta_k/\pi = 0.5$

FIG. 2. Eigenfunctions corresponding to the smallest eigenvalues of Eqs. (8) and (9) as functions of ϑ. For $\theta_k/\pi = 0$, the eigenfunctions are well localized in ϑ-space and both curves are indistinguishable. For $\theta_k/\pi = 0.5$, only Eq. (9) yields the square-integrable eigenfunction.

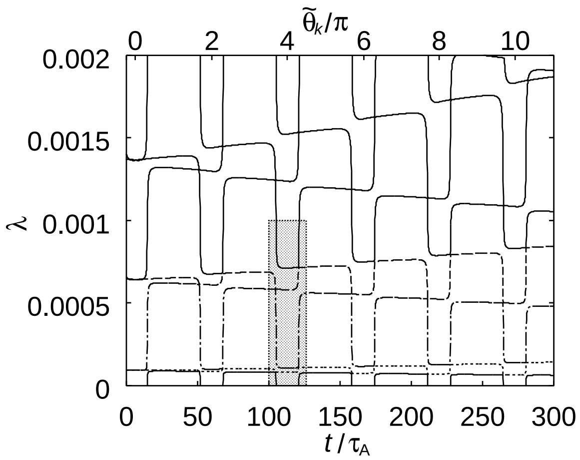
Here f and g are not affected by the choice of h and the integral parts of the right-hand sides are both positive. Then, if we find an unstable eigenvalue for Eq. (8), or we find any ξ which makes the left-hand side of Eq. (14) negative, the same ξ also yields an unstable eigenvalue for Eq. (9). Lastly, the unstable eigenfunctions for Eq. (9) almost coincide with those of Eq. (8) since the eigenfunction is localized around $\vartheta = 0$ where h is chosen to be 1. We have also found that the eigenfunctions of Eq. (9) effectively approximate the dominant behavior of ballooning modes in toroidally rotating tokamaks even in the stable regime [10]. We do not explain it in detail in the present paper, although we will show some examples of eigenvalues and eigenfunctions below.

Figure 1 shows the smallest eigenvalues of both eigenvalue problems (8) and (9) by changing the ballooning angle θ_k for $t = 0$. The square root of the eigenvalue has dimension of frequency, thus it is normalized by the Alfvén frequency defined by (Alfvén velociry)/(connection length). The pressure gradient is large enough and the unstable eigenvalue is obtained for small $|\theta_k/\pi| < 0.297$. Equation (9) yields the discrete eigenvalue for every θ_k/π, although Eq. (8) does only when the eigenvalue is negative. The stable eigenvalue of Eq. (9) looks as zero, however, it is actually small but finite.

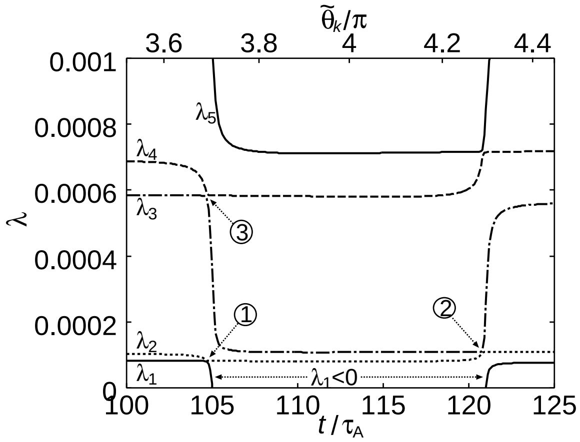
Figure 2 shows the eigenfunctions corresponding to the smallest eigenvalues for $\theta_k/\pi = 0$ and 0.5. For $\theta_k/\pi = 0$, both Eqs. (8) and (9) yield the square-integrable eigenfunctions; they are localized in ϑ-space and almost coincide since $h = 1$ except for large $|\vartheta|$. The corresponding eigenvalues also nearly coincide (see Fig. 1). For $\theta_k/\pi = 0.5$, on the other hand, only Eq. (9) yields the discrete eigenvalue and the square-integrable eigenfunction. It looks singular around $\vartheta = 0$, however, it is a smooth and oscillating function.

4. Energy transfer due to crossing of eigenvalues

In the previous Section, we have obtained a complete set of square-integrable functions which are suitable for expanding the ballooning perturbation. The toroidal rotation shear changes the wave vector in time [3], and the coefficients of the ballooning equation depends on time only through the wave vector. Thus Eq. (9) also have a parameter t. Then we can solve Eq. (9) for



(a)



(b)

FIG. 3. Eigenvalues λ_j's as functions of t or $\tilde{\theta}_k$. Around the time when the smallest eigenvalue λ_1 changes its sign, countably infinite number of crossings of eigenvalues occur. The shaded region in (a) is enlarged in (b).

every t, and obtain a set of eigenfunctions at each instance. We can hence expand $\xi_\perp$ in Eq. (2) as

$$\xi_\perp(t,\vartheta) = \sum_j a_j(t)\xi_j(t,\vartheta), \tag{15}$$

where a_j's are amplitudes and ξ_j's are the eigenfunctions of Eq. (9) with the ballooning angle $\tilde{\theta}_k \equiv \theta_k - \dot{\Omega} t$ taken to be a constant. It is noted that the discrete eigenvalues of Eq. (9) do not mean the spectral resolution of the Alfvén wave. Also, they are not a subset of the spectrum of Eq. (8). However, a few of the eigenfunctions can approximate dominant behavior of the ballooning mode, and also the eigenfunctions form a complete set which can expand any square-integrable function [10].

Substituting Eq. (15) into Eq. (2) and using the orthogonality condition (13), we obtain coupled evolution equations for a_j's as

$$\frac{\mathrm{d}^2 a_j}{\mathrm{d}t^2} + \sum_k C_{1jk}\frac{\mathrm{d}a_k}{\mathrm{d}t} + \sum_k C_{2jk}a_k = -\sum_k C_{3jk}\lambda_k a_k, \tag{16}$$

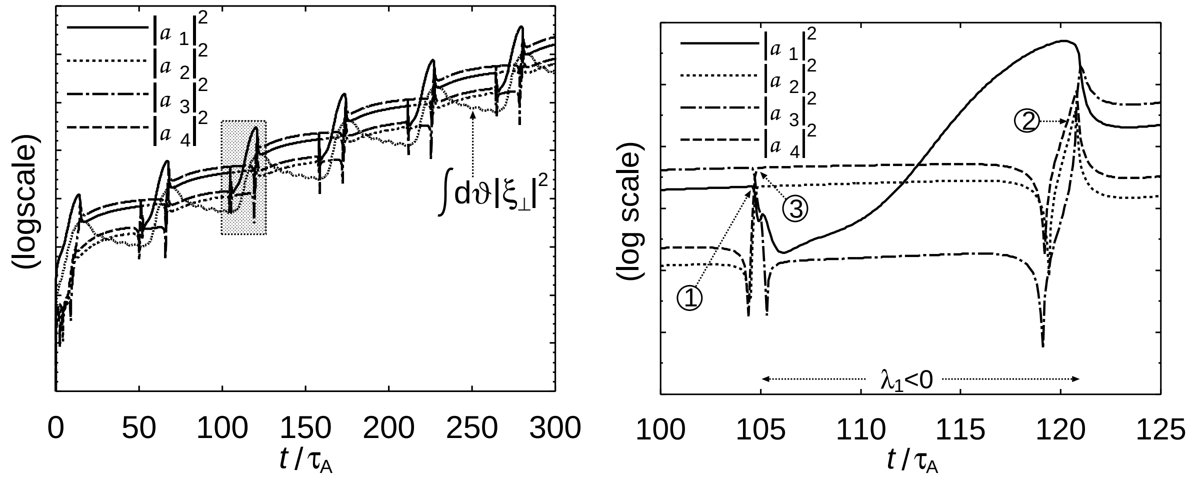
where the coupling parameters are defined as

$$C_{1jk} \equiv \int_{-\infty}^{\infty} \mathrm{d}\vartheta\bar{\rho}h\xi_k^*\left(2\frac{\partial\xi_j}{\partial t} - U\xi_j\right), \tag{17}$$

$$C_{2jk} \equiv \int_{-\infty}^{\infty} \mathrm{d}\vartheta\bar{\rho}h\xi_k^*\left(\frac{\partial^2\xi_j}{\partial t^2} - U\frac{\partial\xi_j}{\partial t}\right), \tag{18}$$

$$C_{3jk} \equiv \int_{-\infty}^{\infty} \mathrm{d}\vartheta\bar{\rho}h\xi_k^* h\xi_j. \tag{19}$$

The coefficients C_{1jk} and C_{2jk} come both from the convection term U and from the time dependence of the wave vector. When $\Omega' = 0$, C_{1jk} and C_{2jk} vanish. Numerical solutions of Eq. (16) will be presented in another paper.



(a) (b)

FIG. 4. Time evolution of $|a_j|^2$*'s and* $\int_{-\infty}^{\infty} d\vartheta|\xi_\perp|^2$ *obtained from numerical solution of Eq. (2). When* λ*'s cross (see Fig. 3),* $|a_j|^2$ *is smoothly converted to* $|a_{j+1}|^2$*. The shaded region of (a) is enlarged in (b).*

The expansion of $\xi_\perp$ by the square-integrable eigenfunctions enables us to clarify the stabilization mechanism. When Ω' is finite, $\tilde{\theta}_k$ changes with t in Eq. (9). Accordingly the eigenvalues of Eq. (9) are obtained as functions of $\tilde{\theta}_k$ or t as shown in Fig. 3. The shaded region in Fig. 3(a) is enlarged in Fig. 3(b). We find that the toroidal rotation shear, or time dependence of the coefficients of the ballooning equation, yields crossings of eigenvalues; from the numerical results, we see countably infinite number of crossings seems to occur at the same time. The smallest eigenvalue λ_1 becomes negative around $\tilde{\theta}_k = 2\pi m$ ($m = 0, \pm 1, \cdots$). The nearly periodic behavior of the eigenvalues comes from the dynamical lattice symmetry of the wave vector, which originates in the periodicity of a torus. It is noted that we have used an equilibrium with up-down symmetry. The deviation from the complete periodicity is because the point $\vartheta = \tilde{\theta}_k$, where the ballooning mode tends to localize, goes away from the origin (for an initial condition localized around the origin) while the connection point is fixed. If the connection point is moved as same as the point $\vartheta = \tilde{\theta}_k$, we obtain completely periodic behavior of λ as a function of t.

Figure 4 shows the time evolution of $|a_j|^2$'s and $\int_{-\infty}^{\infty} d\vartheta|\xi_\perp|^2$ obtained from the numerical solution of Eq. (2). As shown in Fig. 4(a), $\int_{-\infty}^{\infty} d\vartheta|\xi_\perp|^2$ grows during λ_1 is negative. The instantaneous growth rate of $\int_{-\infty}^{\infty} d\vartheta|\xi_\perp|^2$ is almost the same as that of $|a_j|^2$. Thus the set of eigenfunctions of Eq. (9) nicely capture the behavior of the solution of Eq. (2). Figure 4(b) enlarges the shaded region in Fig. 4(a). Around the time when λ_1 changes its sign, the eigenvalues cross as indicated by ①, ② and ③. At ① in Fig. 3(b), λ_1 and λ_2 cross. At that time, $|a_1|^2$ is smoothly converted to $|a_2|^2$ as indicated by ① in Fig. 4(b). The mode couples through the convection term in Eq. (2), and the energy is transferred when the eigenvalues cross. As time proceeds, λ_2 and λ_3 cross (② in Fig. 3(b)). At that time, $|a_2|^2$ is converted to $|a_3|^2$ (② in Fig. 4(b)). Thereby, $|a_1|^2$ is converted to $|a_3|^2$ during $100 < t/\tau_A < 125$. As time proceeds further, λ_3 and λ_4 cross at $t/\tau_A \simeq 155$ and $|a_3|^2$ is converted to $|a_4|^2$ at that time (as same as ③ in Figs. 3(b) and 4(b)). Therefore, even if the unstable mode grows during λ_1 is negative, its energy is transferred to the infinite number of stable modes successively. For large Ω', the coupling among a_j's becomes strong, and the energy transfer completely stabilizes the ballooning mode. This is confirmed in Ref. [11]. Other interesting behavior will be studied elsewhere.

5. Conclusions

We have succeeded in clarifying how the toroidal rotation shear stabilizes the ballooning mode. We have found a complete set of square-integrable functions suitable for expanding the ballooning perturbation by devising the weight function of the ballooning equation in a static plasma. The associated eigenvalue problem includes time as a parameter, and therefore it defines the complete set of square-integrable eigenfunctions at each instance. Then the ballooning mode is expanded by this set, which resolves the difficulty of the continuous spectrum. The eigenvalues are also obtained at each instance. We have found that crossings among the eigenvalues occur around when the smallest eigenvalue changes its sign. When the eigenvalues cross, the energy of the unstable mode is transferred to the stable modes successively owing to the mode couplings through the convection. This mechanism can be interpreted as phase mixing which damps the unstable mode connected to the stable continuum through the toroidal rotation shear. The knowledge as well as the methodology acquired here will lead to an advanced experimental data analysis including sheared plasma rotation.

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