Non-diffusive Transport in 3-D Pressure Driven Plasma Turbulence

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Abstract. Numerical evidence of non-diffusive transport in 3-dimensional, resistive, pressure-gradient-driven plasma turbulence is presented. It is shown that the probability density function (pdf) of tracers is strongly non-Gaussian and exhibits algebraic decaying tails. To describe these results, a transport model using fractional derivative operators in proposed. The model incorporates in a unified way non-locality (i.e., non-Fickian transport), memory effects (i.e., non-Markovian transport), and non-diffusive scaling features known to be present in fusion plasmas. There is quantitative agreement between the model and the turbulent transport numerical calculations. In particular, the model reproduces the shape and space-time scaling of the pdf, and the super-diffusive scaling of the moments.

1. Introduction

Experimental and theoretical evidence suggests that transport in fusion plasmas deviates from the standard diffusion paradigm. Some examples include the confinement time scaling in L-mode plasmas [1], rapid propagation phenomena and non-local behavior observed in perturbative transport experiments [2,3,4], and inward transport observed in off-axis fueling experiments [5,6]. The limitations of the diffusion paradigm can be traced back to the restrictive assumptions in which it is based. In particular, Fick's law, one of the cornerstones of diffusive transport, assumes that the fluxes, which contain the dynamical information of the transport process, only depend on local quantities, i.e. the spatial gradient of the field(s). Another key issue is the Markovian assumption that neglects memory effects in the transport process. Also, at a microscopic level, standard diffusion assumes the existence of an underlying Gaussian, uncorrelated stochastic process (i.e. a Brownian random walk) with well defined characteristic spatio-temporal scales.

Motivated by the need to develop models of non-diffusive transport, we discuss here a class of transport models that incorporate in a unified way non-Fickian transport, non-Markovian processes or "memory" effects, and non-diffusive scaling. At a microscopic level, the proposed models assume an underlying stochastic process without characteristic spatio-temporal scales that generalizes the Brownian random walk. As discussed below, these stochastic processes are intimately linked to Levy stable distributions.

As a concrete case study to motivate and test the model, we consider tracers transport in three-dimensional, pressure-gradient-driven turbulence. In this system changes in the pressure gradient trigger instabilities at rational surfaces that locally flatten the pressure profile and increase the gradient in nearby surfaces. This is turn leads to successive instabilities and intermittent, avalanche-like transport [7], which together with the trapping effects of the turbulent eddies has been observed to cause anomalous diffusion [8]. By anomalous diffusion we mean that the moments of the radial displacement of tracers grow as $\left\langle x^n \right\rangle \sim t^{n\nu}$, where contrary to the standard diffusion case, $\nu \neq 1/2$. Our goal is to construct a macroscopic model for the tracer particles probability density function (pdf) using fractional derivative operators in space and time.

The rest of this paper is organized as follows. In the next section, we describe the turbulence model, and present the numerical results of tracers transport. The proposed fractional transport model is discussed in Sec. 3. In Sect. 4, we compare the turbulent transport numerical results with the fractional model, and discuss preliminary results on transport studies in no-steady turbulence. The conclusions are presented in Sect. 5.

2. Pressure-gradient-driven turbulence model

The turbulence model used in the transport calculations is based on an electrostatic approximation of the reduced magnetohydrodynamic equations [9]. The model describes the evolution of the $\vec{E}\times\vec{B}$ velocity stream-function $\tilde{\Phi}$, and the pressure fluctuation $\tilde{p}$:

$$\left(\frac{\partial}{\partial\tau}+\tilde{\vec{V}}\cdot\nabla\right)\nabla_\perp^2\tilde{\Phi}=-\frac{1}{\eta m_i n_0 R_0}\nabla_\parallel^2\tilde{\Phi}+\frac{B_0}{m_i n_0}\frac{1}{r_c}\frac{1}{r}\frac{\partial\tilde{p}}{\partial\theta}+\mu\nabla_\perp^4\tilde{\Phi} \tag{1}$$

$$\left(\frac{\partial}{\partial\tau}+\tilde{\vec{V}}\cdot\nabla\right)\tilde{p}=\frac{\partial\langle p\rangle}{\partial r}\frac{1}{r}\frac{\partial\tilde{\Phi}}{\partial\theta}+\chi_\perp\nabla_\perp^2\tilde{p}+\chi_\parallel\nabla_\parallel^2\tilde{p} \tag{2}$$

The magnetic field corresponds to a stellarator-like equilibrium with a toroidal component B_0, and a q –profile, $q=1/(0.53+0.5r^2)$. The tildes indicate fluctuating quantities (in space and time), and the angular brackets, $\langle\ \rangle$, denote flux surface averaging over the cylinder at a fixed radius. The equilibrium density is n_0, the ion mass is m_i, the averaged radius of curvature of the magnetic field lines is r_c, and the resistivity is η. The subindices $\perp$ and $\parallel$ denote the direction perpendicular and parallel to the magnetic field respectively. The dissipative terms in Eqs. (1) and (2) have characteristic coefficients μ (the collisional viscosity) and $\chi_\perp$ (the collisional cross-field transport) respectively. A parallel dissipation term, proportional to $\chi_\parallel$, is also included in the pressure equation. The instability drive is the flux surface averaged pressure gradient, $\partial_r\langle p\rangle$, where

$$\frac{\partial\langle p\rangle}{\partial\tau}+\frac{1}{r}\frac{\partial}{\partial r}r\left\langle\tilde{V}_r\,\tilde{p}\right\rangle=S_0+D\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\langle p\rangle\right) \tag{3}$$

The term S_0, which is a function of r, represents a source of particles and heat due, for instance, to neutral beam heating and fueling. Here we assume $S_0=\bar{S}_0\left[1-(r/a)^2\right]$.

To study transport we follow particle tracers advected by the $\vec{E}\times\vec{B}$ turbulent velocity field,

$$\frac{d\vec{r}}{d\tau}=\tilde{\vec{V}}=\frac{1}{B_0^2}\nabla\tilde{\Phi}\times\vec{B}_0 \tag{4}$$

Since the magnetic field is static, the tracers provide a Lagrangian description of the electrostatic turbulence in the model. The numerical calculations were performed using 25×10^3 tracer particles with initial conditions randomly distributed in θ and z on the surface of a cylinder of radius *r=a/2*. In their evolution, the tracer particles either get trapped in the eddies shown in Fig.1 for long times, or jump over several eddies in a single "flight". The combination of these trapping and flight events leads to anomalous diffusion.

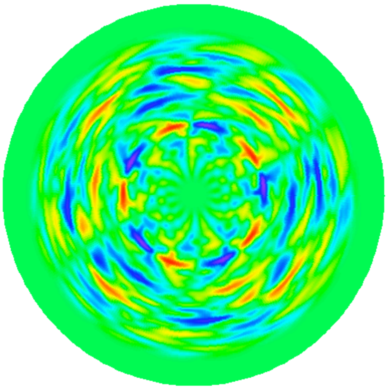

FIG. 1. Fluctuating electrostatic potential $\tilde{\Phi}$ at a fixed time obtained from the numerical integration of the resistive, pressure-gradient-driven turbulence model in Eqs.(1)-(3). The observed eddies are responsible for the trapping of particle tracers that causes anomalous diffusion.

Our main object of study is the probability density function (pdf) of radial displacements of tracers, *P(x,t)*, where $x = \delta r / a$ and $t = \tau / \tau_R$, where $\tau_R = a^2 \mu_0 / \eta$ is the resistive time. By definition, at *t=0*, $P = \delta(x)$. As *t* advances, the pdf broadens and develops tails. Figure 2 shows the pdf *P(x,t)* as function of *x* at *t=0.64* obtained from the histogram of particle displacements. The log-normal scale of the plot makes evident the strong non-Gaussianity of the density function (in this scale a Gaussian is a parabola).

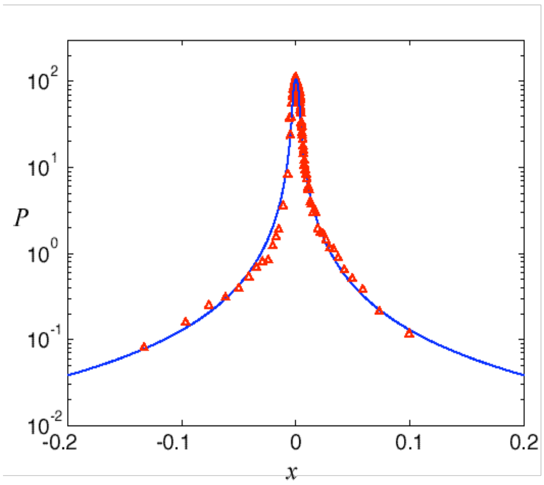

FIG. 2. Probability density function of tracers as function of x, at fixed time, in pressure gradient driven plasma turbulence. The triangles are the results from the turbulent transport calculation, and the solid line denotes the pdf according to the fractional transport model with $\alpha = 3/4$, *and* $\beta = 1/2$. *In agreement with the model, the tails of the pdf decay algebraically as* $P \sim x^{-(1+\alpha)}$.

The evolution in time of the pdf at a fixed radial position is shown in Fig. 3. In addition to the spatio-temporal dynamics of the pdf, we have computed the moments of the radial displacements, and consistent with [8], have observed superdiffusive scaling $\langle x^n \rangle \sim t^{n\nu}$ with $\nu \approx 0.66 \pm 0.002$.

3. Fractional diffusion transport model

The generic form of the proposed model for tracer transport in pressure-gradient-driven plasma turbulence in the one-dimensional domain $x \in (a,b)$ is [10]

$$ {}_0^c D_t^\beta \, P = \chi \left(w^- \, {}_a D_x^\alpha + w^+ \, {}_x D_b^\alpha \right) P \tag{5}$$

where the left hand side is the fractional derivative in time of order β, and the two terms on the right hand side are the left and right Riemann-Liouville fractional derivatives of order α respectively, and χ is a constant. Fractional derivatives are integro-differential operators that naturally generalize the concept of differentiation to fractional orders. As expected, for $\beta = 1$, ${}_0^cD_t^\beta = \partial_t$, and for $\alpha = 2$, ${}_aD_x^\alpha = \partial_x^2$. Most results from regular calculus directly translate to the fractional calculus formalism that in recent years have found an increasing number of applications in science and engineering [11].

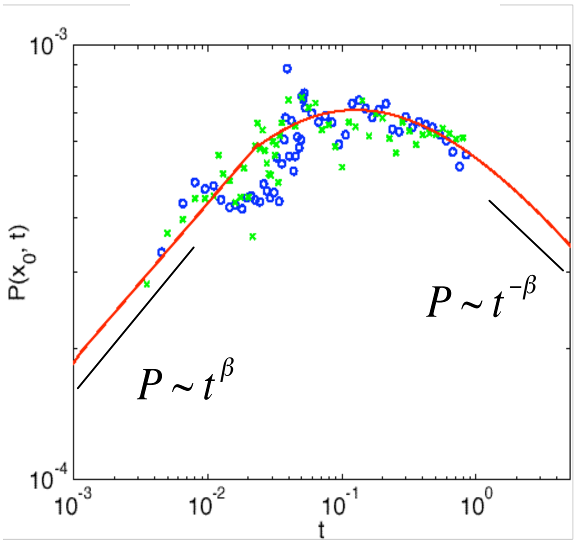

FIG. 3. Probability density function of tracers as function of t, at a fixed radial postion, in pressure-gradient-driven plasma turbulence. The circles and crosses are the results from the turbulent transport calculation, and the solid line denotes the pdf according to the fractional transport model with $\alpha = 3/4$, and $\beta = 1/2$. In agreement with the model, the rise and decay of the tails follow the algebraic scaling $P \sim t^{\pm\beta}$.

For $0 < \alpha < 1$ and $0 < \beta < 1$, the range of parameter values of interest here, the model can be rewritten in the more familiar form

$$\frac{\partial P}{\partial t} = -\chi \frac{\partial}{\partial x}\left[w^- \, \Gamma_\ell + w^+ \, \Gamma_r \right] \tag{6}$$

where the fluxes are defined as

$$\Gamma_\ell = -\frac{\partial}{\partial t}\int_0^t d\tau \int_a^x dy \; K(x-y;t-\tau)\, P(y,\tau) \qquad \Gamma_r = \frac{\partial}{\partial t}\int_0^t d\tau \int_x^b dy \; K(y-x;t-\tau)\, P(y,\tau) \tag{7}$$

with K an algebraic decaying function of the form

$$K(x-y;t-\tau) = \frac{1}{\Gamma(1-\alpha)\Gamma(\beta)} \; \frac{1}{(t-\tau)^{1-\beta}\,(x-y)^{\alpha}} \tag{8}$$

Thus, in the model, non-Fickian effects due to avalanche-like events that induce Levy flights in the tracers, are described using non-local, integro-differential operators in space. The flux at a point x consists of a "left-sided" contribution Γ_ℓ from the (a,x) interval, and a "right-sided" contribution Γ_r from the (x,b) interval. The time integrals in the fluxes, account for non-Markovian effects due the trapping of tracers in eddies. The relative weight of Γ_ℓ and Γ_r is determined by w^+ and w^- that are functions of α, and θ, a parameter that determines the asymmetry of the transport process. For a symmetric process, the case of interest here, $\theta = 0$, and $w^+ = w^- = -\sec(\pi\alpha/2)/2$.

Because the individual tracers follow the turbulent velocity field $\tilde{\vec{V}}$ according to Eq.(4), the pdf of the tracers satisfies $\left(\partial_t + \tilde{\vec{V}} \cdot \nabla\right)P = 0$. Comparing this equation with Eq.(6), we can intuitively envision the fractional model as a way to encapsulate or renormalize the non-linear effects of the turbulent velocity field $\tilde{\vec{V}}$ into an effective transport operator involving non-local fluxes according to the prescription $\tilde{\vec{V}} \cdot \nabla \Rightarrow \chi \partial_x \left[w^- \Gamma_\ell + w^+ \Gamma_r\right]$. In this regard, the results presented here represent a first step in a phenomenological renormalization of plasma turbulence using fractional operators.

The physics behind the fractional model can be further understood from the connection between transport models and the theory of random walks [12]. To explain this, recall that the diffusion model is a macroscopic description of the Brownian random walk which assumes that particles experience uncorrelated, random displacements, or jumps, $\zeta = \zeta_1, \zeta_2, \cdots \zeta_i \cdots$, where ζ is drawn from a pdf $\lambda(\zeta)$ with finite second moment. In the problem of interest here, this simple Brownian walk picture does not apply because the turbulent eddies tend to trap the tracers, and "avalanche-like" transport events induce large displacements. An elegant and powerful model that incorporates these phenomena is the continuous time random walk (CTRW) [13,14]. This model introduces in addition to the jump pdf $\lambda(\zeta)$, a waiting-time pdf $\psi(\tau)$. That is, contrary to the Brownian model in which particles jump at regular time intervals, the CTRW assumes that the waiting time between jumps, $\tau_i = t_i - t_{i-1}$, is a random variable with pdf $\psi(\tau)$. Most importantly, the CTRW model place no restrictions on the trapping and jumps pdfs, allowing the possibility of describing a large variety of non-Gaussian transport processes including, for example, non-Markovian effects [15]. In the CTRW the probability of finding a particle at point *x* at time *t* is determined by the master equation

$$P(x,t) = \delta(x) \int_t^\infty \psi(t')\,dt' + \int_0^t \psi(t-t') \left[\int_{-\infty}^\infty \lambda(x-x') P(x',t')\,dx'\right] dt' \tag{9}$$

The first term on the right hand side is the contribution to *P* of particles that have not moved during the time interval *(0,t)*, and the second term denotes the contribution to *P* of particles located at x' and jumping to *x* during this time interval. Given ψ and λ, in the continuum, fluid, limit Eq. (9) leads to an evolution equation for P*(x,t)*. In particular, for an exponential decaying ψ, and a Gaussian distributed λ, Eq.(9) reduces to the diffusion equation. However, for algebraic decaying trapping times and jump distributions of the form $\psi \sim \tau^{-(1+\beta)}$, $\lambda \sim \zeta^{-(1+\alpha)}$ Eq.(9) reduces in the fluid limit to the fractional Eq.(5). Note that in this case, because of the algebraic decay, $\langle\tau\rangle$ and $\langle\zeta^2\rangle$diverge, that is, there are no characteristic spatio-temporal scales. Thus, based on the CTRW model, the proposed fractional transport model can be conceived as a macroscopic description of an underlying "microscopic", non-Gaussian stochastic process exhibiting long trapping events and Levy flights with no characteristic scale.

4. Comparison of fractional model with turbulence model

The solution of the fractional diffusion model in Eq.(5) for a general initial condition *P(x,0)* can be written as

$$P(x,t) = \int G_{\alpha\beta\theta}(x - x',t)\, P(x',0)\, dx' \qquad (10)$$

where $G_{\alpha\beta\theta}$ is the Green's function or propagator giving the transition probability that a tracer located in x' at $t=0$ exhibits a displacement $x - x'$ at time t. As expected, in the standard diffusion case, $\alpha = 2$, $\beta = 1$ and $\theta = 0$, $G_{\alpha\beta\theta}$ reduces to a Gaussian distribution. For different values of α, β and θ, Eq.(10) reproduces a wide variety of non-Gaussian distribution functions. In particular, for $\alpha \neq 2$ and $\beta = 1$, $G_{\alpha\beta\theta}$ reduces to an important class of distributions known as Levy, α-stable distributions that are ubiquitous in non-Gaussian stochastic processes. For transport in pressure-gradient-driven plasma turbulence, it turns out that $\alpha = 3/4$, $\beta = 1/2$ and $\theta = 0$ [10].

Consistent with the radially localized initial condition used in the tracer particles numerical calculations, we consider an initial condition of the form $P = P_0/\varepsilon$ for $|x| \le \varepsilon/2$ and zero otherwise, where ε is a small parameter. Figure 2 shows a very good agreement between the solution of the fractional diffusion model and the turbulent transport results at a fixed time. Consistent with the asymptotic properties of the solution of the fractional model [10], and in agreement with the turbulent transport results, the pdf exhibits the algebraic decay $P(x,t_c) \sim x^{-(1+\alpha)}$. That is, α, the order of the fractional derivative operator in space determines the decay exponent of the pdf in space at a fixed time.

To understand the role of the parameter β we show in Fig. 3 the pdf as function of time at a fixed spatial location. Again, consistent with the asymptotic properties of the solution the pdf exhibits an algebraic growth and eventual decay of the form $P(x_c,t) \sim t^{\pm\beta}$. That is, β, the order of the fractional derivative operator in time, determines the algebraic growth and decay of the pdf in time at a fixed position. The scaling of the moments $\langle x^n \rangle \sim t^{n\nu}$ provides another test of the model. From the scaling properties of the fractional equation it follows that $\nu = \beta/\alpha = 2/3$ a value in very good agreement with the turbulence calculation result $\nu \approx 0.66 \pm 0.002$.

In all the previous calculations, we restricted attention to steady-state turbulence. That is, in the numerical calculations the tracers were followed after the transient effects have died and the turbulence have reached a steady state. Figure 4 shows some preliminary results in the non-steady turbulence regimen. In this case the focus is in the short time evolution of a large pulse perturbation in the pressure. Figure 4a shows the space-time evolution of a negative pulse (with the steady-state turbulence background substracted). Some interesting asymmetries are observed, in particular in this transient regimen, the inward propagation of the pusle is significantly faster that then outward spreading. To quantify the spreading of the turbulence we have plotted in Fig. 4b the time evolution of the normalized second moment of positive and negative pressure pulses as function of time. In both cases it is observed the moments exhibit superdiffusive scaling $\langle \delta\tilde{p}^2 \rangle \sim t^{2\nu}$ with an anomalous exponent $\nu \approx 0.63$

remarkably close to the exponent of the fractional model. Despite this encouraging preliminary result, more work in needed in the fractional diffusion description of non-steady turbulent transport. A crucial aspect of this problem is the modeling of the inward-outward transport asymmetry.

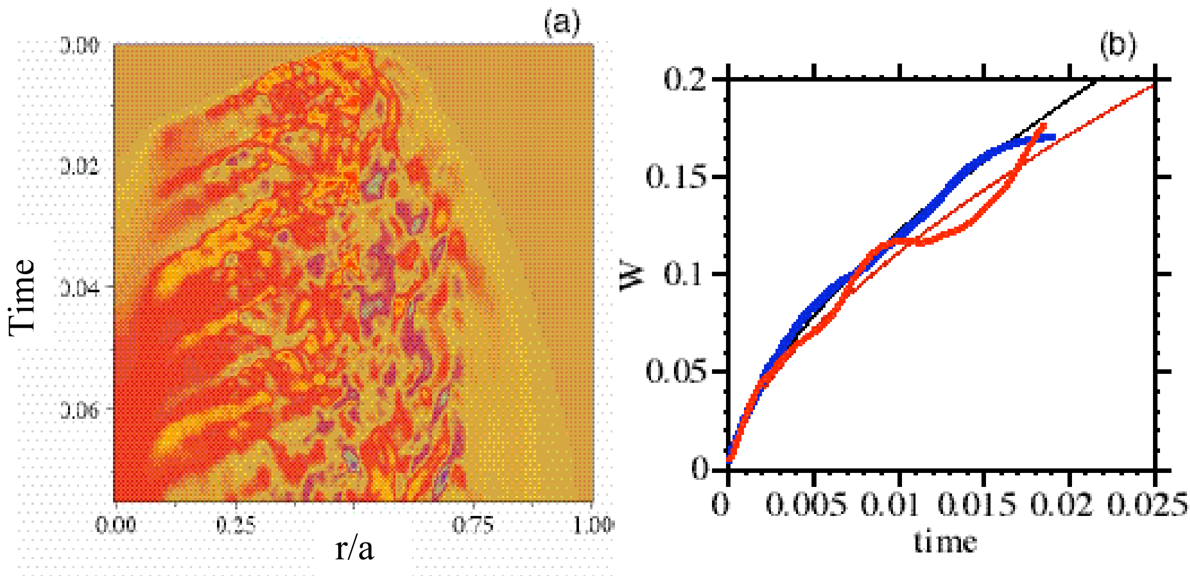

FIG 4. Anomalous diffusion of a pressure pulse in non-steady pressure-gradient-driven plasma turbulence. Panel (a) shows the space time evolution of a localized , large negative pressure pulse initial condition. The thick lines in panel (b) show the evolution in time of the normalized second moment of a negative (red) and positive (blue) pulse. The thin lines are power law fits. Consistent with the fractional diffusion scaling, the second moments exhibit superdiffusive scaling with $\nu \approx 0.63$.

5. Conclusions

In this paper we have proposed and tested a transport model for tracer particles in plasma turbulence. The model is formulated using fractional derivative operators that generalize the concept of differentiation to fractional orders. From a physical point of view, fractional derivatives are a natural tool to model non-local effects in space and time. In the case of fractional derivatives in space, the slow, algebraic decay of the fractional derivative kernel accounts for long-range effects in the flux. In a similar way, fractional derivatives in time allow the incorporation of memory, i.e. non-Markovian effects. Both effects, spatial non-locality and memory, are likely to be present in fusion plasmas. The fractional model can be conceived as a macroscopic description of an underlying microscopic, non-Gaussian stochastic process (non-Brownian random walk) with no characteristic scales. It was shown that there is quantitative agreement between the fractional model and the turbulent transport numerical calculations.

In the case of homogenous, isotropic, fully developed turbulence, Gaussian closure approximations lead to transport models based on effective diffusivities. However, the complexity of pressure-gradient-driven plasma turbulence invalidates the Gaussian assumptions. As a first step to overcome the limitations of Gaussian closures, here we have shown that $\vec{E} \times \vec{B}$ turbulent transport can be modeled with non-diffusive operators involving fractional derivatives.

The fractional transport model discussed here is linear, and an interesting important problem is to understand the interplay of nonlinearity and fractional diffusion. As a first step in this direction we have added to Eq.(5) a nonlinearity of the form $P(1-P)$, typically used in reduced models of the L-H transition. Numerical and analytical results on this model indicate that nonlinearity and fractional diffusion lead to exponential propagation of fronts [16]. This results might be relevant in the study of rapid propagation phenomena in the L-H transition.

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