A New Matching Scheme for Resistive Wall Mode Analysis

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Abstract. A new matching scheme is proposed as a numerically tractable implement for linear magnetohydrodynamic (MHD) stability analysis. The scheme divides the plasma region into inner layers and outer regions as the conventional asymptotic matching method. The essential difference is that each inner layer in the new scheme has finite width including the rational surface and thus the outer regions, which are governed by the Newcomb equation, have no singularity at their terminal points. The matching condition in this scheme is that the normal component of the plasma displacement vector is connected smoothly among outer regions and inner layers. It is demonstrated that the new scheme can be applied to the initial value problem of the Frieman-Rotenberg equation to analyze rotation effects on MHD modes. The present status of RWMs analysis code in JAEA is also reported.

1. Introduction

Stabilization of the Resistive Wall Modes (RWMs) by plasma rotation is one of the most important physical issues for future reactors operated in an advanced tokamak regime[1]. To study the stability of rotating plasmas, some numerical codes have been developed such as MARS[2]. However, for rotating plasmas, the problem of linear ideal MHD stability becomes non-self-adjoint because it is governed by the Frieman-Rotenberg equation[3], and thus the conventional normal mode decomposition is not complete. Therefore, a numerical code that solves the initial value problem is desirable.

For linear MHD stability, there exists a powerful method, the asymptotic matching scheme[4]. The scheme divides the plasma region into outer regions and inner layers. The outer regions are far from rational surfaces, and the inertial and non-ideal effects can be neglected there. Consequently, the regions are governed by the Newcomb equation[5, 6, 7, 8]. Each inner layer is regarded as an infinitely thin region around a rational surface; all effects are retained in the inner layer. Stretching time and space coordinates, the MHD equations in the inner layer reduce to simpler equations called the inner layer equations[9]. The solutions in outer regions and inner layers are asymptotically matched at the rational surfaces to determine plasma motion globally.

The underlying idea in the asymptotic matching scheme should be valid for RWMs because the characteristic time of them is much longer than the Alfvén transit time, and the inertial effects in the plasma can be neglected except thin layers around rational surfaces. However, this scheme assumes that the stability is to be analyzed as an eigenvalue problem[9]. It seems to be out of question to apply this to the initial value problem for such as the Frieman-Rotenberg equation.

To overcome the above limitation, we propose a new matching scheme for initial value problems in the MHD stability analysis. The new scheme, whose basic idea has been reported in Ref.[10], uses the inner layers and outer regions as in the conventional asymptotic matching scheme; however the new scheme gives finite width to each layer. Therefore, the outer regions have no singularity. The matching condition is that the normal component of the plasma displacement vector is connected smoothly among outer regions and inner layers. Since there are singularities neither in outer regions nor in inner layers, the new scheme provides a tractable implement from a numerical point of view. Another feature of the new scheme is that we can flexibly change the physical model governing inner layers and get insight into the effects of the new physics on MHD modes.

This paper comprises two parts. We propose in Sec.2 the new matching scheme for the initial value problem of the Frieman-Rotenberg equation to study rotation effects on MHD modes. In Sec.3, we report the present status of developing the RWMs analysis code in a tokamak configuration in JAEA, and summary is given in Sec.4.

2. A new matching scheme for linear MHD stability of rotating plasmas

2.1. Full implicit form of Frieman-Rotenberg equation

We assume that the plasma obeys the ideal MHD model. Introducing the Lagrange displacement

$$\tilde{\mathbf{v}} = \partial_t \xi + \mathbf{v} \cdot \nabla \xi - \xi \nabla \mathbf{v},$$

where $\mathbf{v}$(*res.* $\tilde{\mathbf{v}}$) is the equilibrium (*res.* perturbation) of flow, and assuming appropriate initial conditions, the linearized MHD equation can be cast into the Frieman-Rotenberg equation[3]

$$\rho \partial_t^2 \xi + 2\rho \mathbf{v} \cdot \nabla \partial_t \xi = \mathbf{F}[\xi], \tag{1}$$

where $\mathbf{F}[\xi]$ is the force operator generalized to include the rotation effect

$$\mathbf{F}[\xi] = \mathbf{F}_s[\xi] + \nabla \cdot [\rho \xi (\mathbf{v} \cdot \nabla) \mathbf{v} - \rho \mathbf{v} (\mathbf{v} \cdot \nabla) \xi], \tag{2}$$

and $\mathbf{F}_s$ is the static part of the force operator

$$\mathbf{F}_s[\xi] = \nabla[\xi \cdot \nabla p + \Gamma p \nabla \cdot \xi] + \mathrm{rot}\mathbf{Q} \times \mathbf{B} + \mathbf{j} \times \mathbf{Q}. \tag{3}$$

Here, ρ is the mass density, p the pressure, $\mathbf{B}$ the magnetic field and $\mathbf{j}$ the current density in an equilibrium, and $\mathbf{Q}$ is the perturbed magnetic field given by

$$\mathbf{Q} = \mathrm{rot}(\xi \times \mathbf{B}). \tag{4}$$

The operator $\mathbf{F}$ is self-adjoint, but $\rho \mathbf{v} \cdot \nabla$ is anti-self-adjoint, and the Frieman-Rotenberg equation constitutes a non-self-adjoint problem. Therefore, the initial value approach is more appropriate to capture the non-exponential modes. From the numerical point of view, we recast Eq.(1) into the Hamilton form

$$\rho \partial_t \mathbf{\Pi} = \tilde{\mathbf{F}}[\xi] - \rho \mathbf{v} \cdot \nabla \mathbf{\Pi}, \tag{5}$$

$$\rho \partial_t \xi = -\rho \mathbf{v} \cdot \nabla \xi + \rho \mathbf{\Pi}, \tag{6}$$

where $\mathbf{\Pi} = \partial_t \xi + \mathbf{v} \cdot \nabla \xi$ is the momentum vector, and $\tilde{\mathbf{F}}[\xi] = \mathbf{F}[\xi] + \rho \mathbf{v} \cdot \nabla (\mathbf{v} \cdot \xi)$. Note that $\tilde{\mathbf{F}}[\xi]$ does not contain the term $\rho \mathbf{v} \cdot \nabla (\mathbf{v} \cdot \xi)$. We utilize this aspect since this term can be numerically very large when studying the stability of rotating plasmas. To study the slowly growing modes such as RWMs, we should execute a long time calculation, for example, $t \sim 10^6 \tau_a$, where τ_a is the Alfvén transit time. Therefore, the full implicit scheme is applied to make the time step Δt sufficiently large, and Eqs.(5) and (6) are reduced to

$$\rho \mathbf{\Pi}^{n+1} + (\Delta t) \rho \mathbf{v} \cdot \nabla \mathbf{\Pi}^{n+1} - (\Delta t) \tilde{\mathbf{F}}[\xi^{n+1}] = -\rho \mathbf{\Pi}^n, \tag{7}$$

$$\rho \xi^{n+1} + (\Delta t) \rho \mathbf{v} \cdot \nabla \xi^{n+1} - (\Delta t) \rho \mathbf{\Pi}^{n+1} = \rho \xi^n. \tag{8}$$

The solution is calculated by invoking the weak form to use the finite element method.

2.2. Matching condition among an inner layer and outer regions

For simplicity, we consider a one-dimensional problem in the cylindrical coordinate system (r, θ, z) with $0 \le r \le 1$, and assume that $\xi, \mathbf{\Pi} \propto \exp(im\theta - ikz)$, where m is the poloidal mode number of the MHD mode. We first solve the initial value problem of Frieman-Rotenberg equation in the inner layer $r \in [r_L, r_R]$. Let $r\xi_{in,L}$ be the solution to the homogeneous equation of Eqs.(7) and (8),

$$\rho\mathbf{\Pi} + (\Delta t)\rho\mathbf{v}\cdot\nabla\mathbf{\Pi} - (\Delta t)\tilde{\mathbf{F}}[\xi] = 0, \tag{9}$$

$$\rho\xi + (\Delta t)\rho\mathbf{v}\cdot\nabla\xi - (\Delta t)\rho\mathbf{\Pi} = 0. \tag{10}$$

with inhomogeneous boundary condition

$$r_L\xi_r(r_L) = 1, \ \ r_R\xi_r(r_R) = 0. \tag{11}$$

Similarly, let $r\xi_{in,R}(r)$ be the solution to Eqs.(9) and (10) under the boundary condition

$$r_L\xi_r(r_L) = 0, \ \ r_R\xi_r(r_R) = 1. \tag{12}$$

We also solve the inhomogeneous equations, Eqs.(7) and (8), with homogeneous boundary condition

$$r_L\xi_r(r_L) = 0, \ \ r_R\xi_r(r_R) = 0. \tag{13}$$

This solution depends on time, and we write as $r\xi_{in}^{n+1}(r)$. Using these solutions, the general solution to Eqs.(7) and (8) can be written as

$$\xi^{n+1}(r) = c_L^{n+1}\xi_{in,L}(r) + c_R^{n+1}\xi_{in,R}(r) + \xi_{in}^{n+1}(r), \tag{14}$$

where c_p^{n+1} $(p = L, R)$ will be determined by the matching condition.

Next we solve the Newcomb equation,

$$\mathbf{F}_s[\xi] = 0, \tag{15}$$

in the outer regions, $[0, r_L]$ and $[r_R, 1]$. Note that we neglect the dynamic term in the force operator. Let $r\xi_{out,L}$ be the solution in $r \in [0, r_L]$ with the boundary condition

$$r_L\xi_r(r_L) = 1. \tag{16}$$

Similarly, let $r\xi_{out,R}$ be the solution to Eq.(15) in $r \in [r_R, 1]$ under the boundary condition

$$r_R\xi_r(r_R) = 1, \ \ \xi_r(1) = 0. \tag{17}$$

Then, the solution in the outer regions can be written as

$$\xi_L(r, t) = c_L(t)\xi_{out,L}(r), \tag{18}$$

and

$$\xi_R(r, t) = c_R(t)\xi_{out,R}(r). \tag{19}$$

The matching condition is that $\xi_r(r)$ is continuous and smooth at $r = r_L$ and $r = r_R$. From the above formalism the continuity condition is automatically satisfied at each matching point. The smoothness condition yields an algebraic equations for c_p $(p = L, R)$

$$A\begin{pmatrix} c_L^{n+1} \\ c_R^{n+1} \end{pmatrix} = \begin{pmatrix} (\xi_{in}^{n+1})'(r_L) \\ (\xi_{in}^{n+1})'(r_R) \end{pmatrix}, \tag{20}$$

where

$$A = \begin{pmatrix} \xi'_{out,L}(r_L) - \xi'_{in,L}(r_L) & -\xi'_{in,R}(r_L) \\ -\xi'_{in,L}(r_R) & \xi'_{out,R}(r_R) - \xi'_{in,R}(r_R) \end{pmatrix}, \tag{21}$$

which can be easily inverted $(' = d/dr)$. The matrix A is independent of time, and therefore, only the right-hand side of Eq.(20) should be updated for each time step.

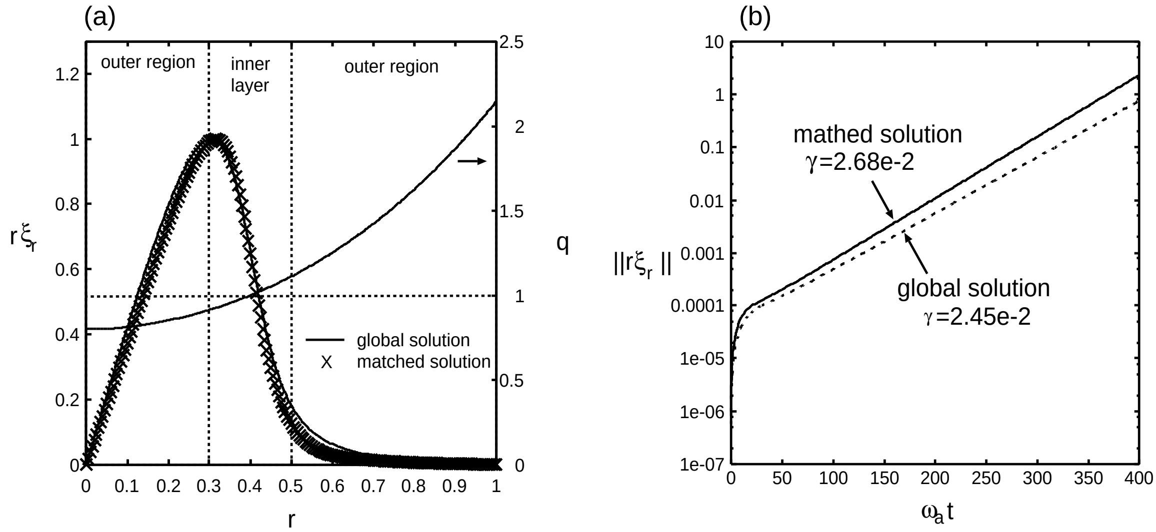

FIG. 1. (Left)Structure of $m = 1$ internal kink mode including poloidal rigid rotation. The safety factor profile is superimposed. The $q = 1$ surface is located at $r = 0.4$. The inner layer is set as $r \in (0.3, 0.5)$. The matched solution of the present scheme agrees well with the global one;(Right) Comparison of growth rates of $m = 1$ internal kink mode.

2.3. Application to m=1 internal mode with plasma rotation

We study the $m = 1$ internal mode to show the effectiveness of the new matching scheme. Application to external modes requires the matching condition at the plasma surface that the normal component of the magnetic field is continuous, and can be easily implemented. The following examples are calculated with a fine mesh size of 10^{-4} for a plasma column with its length $R_0 = 3.0$ and $\Delta t = 0.1\omega_a$ (ω_a: the Alfvén frequency at the plasma surface). Figure 1 shows the present scheme applied to $m = 1$ internal kink mode with $n = 1$ ($n = kR_0$). The safety factor profile is superimposed. The $q = 1$ surface is located at $r = 0.4$; the matching points are $r_L = 0.3$ and $r_R = 0.5$. In this case, we assume that the poloidal rotation is rigid as $v_\theta = r\Omega$, and that the rotation is Alfvénic, $\Omega = 0.076$ (v_θ is a few % of Alfvén speed).

The left part of Fig. 1 shows that the matched solution agrees well with the global solution [here, the term "global" indicates the solution obtained by solving the Frieman-Rotenberg equation in the whole region $r \in (0, 1)$,] and capture the $m = 1$ kink mode structure. The right part of Fig. 1 indicates the growth rates of the both solutions are very close.

Figure 2 shows the relative error of the matched solution against inner layer width Δr. As is expected, the error reduces as Δr gets large. This figure also indicates that even for large poloidal rotation (a few % of Alfvén speed), the sufficient width of the inner layer is $\Delta r \sim 0.2$. These results demonstrate the effectiveness of the present scheme. The initial value problem of the Frieman-Rotenberg equation can be solved only in the inner layer, resulting in substantial save of the CPU time.

3. Development of RWM analysis code for a tokamak configuration

3.1. Application of two-dimensional Newcomb equation for RWM analysis

We have developed a two-dimensional code for RWMs analysis assuming no rotation. In this case, the electro-magnetic interaction between plasma and resistive wall is described by the

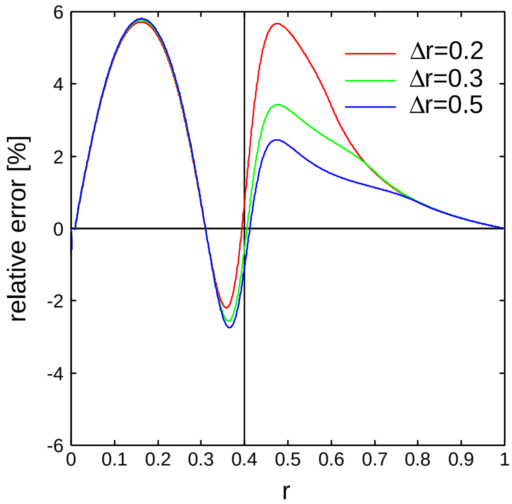

FIG. 2. Relative error of the matched solution to the global solution against rotation speed. Even for large plasma rotation (a few % of Alfvén speed), the error is in the order of several %.

extended energy principle[11]

$$\delta W = \delta W_p + \delta W_{IV} + \delta W_{OV} + D_W, \tag{22}$$

where δW_p is the change of potential energy in the plasma region, δW_{IV} the perturbed vacuum energy between the plasma surface and the resistive wall, δW_{OV} the perturbed vacuum energy outside the resistive wall, and D_W is the energy dissipation inside the resistive wall. The plasma potential energy can be evaluated by the normal components of the displacement vector at the plasma surface via the property of the Newcomb equation[12]. By using this property, we have adapted the MARG2D code[8], which solves the two-dimensional Newcomb equation, for RWMs analysis.

3.2. Benchmark against NMA (Normal Mode Approach) code

The codes for both vacuum regions and resistive wall dynamics have been combined with MARG2D into a code RWMaC (Resistive Wall Mode analysis Code). To verify RWMaC, we have benchmarked it with the NMA (Normal Mode Approach) code[11], which has the same theoretical foundation. In benchmarking we use an analytic equilibrium (so called Solov'ev equilibrium), shown in Fig. 3, to remove the error stemming from numerical equilibria. Figure 4 shows the eigenfunctions inside the plasma and the eddy current pattern induced by unstable RWM for this equilibrium. The eddy current pattern, localizing near the weak field side, is similar to one induced by the unstable external kink modes.

We have benchmarked RWMaC with NMA by studying the growth rate of RWM by changing the position of the resistive wall. Figure 5 shows how the growth rate depends on the wall position. When the wall is sufficiently close to the plasma surface to stabilize the ideal external kink mode deeply, the both results agree quite well. When the wall approaches to the marginal position, the growth rate of RWM tends to diverge for both codes. This divergence property has been theoretically studied[13]. To study this aspect, we plot the inverse of the growth rate vs. the wall position. The figure shows that RWMaC and NMA well capture this divergence property. Consequently, it can be concluded that RWMaC has been verified.

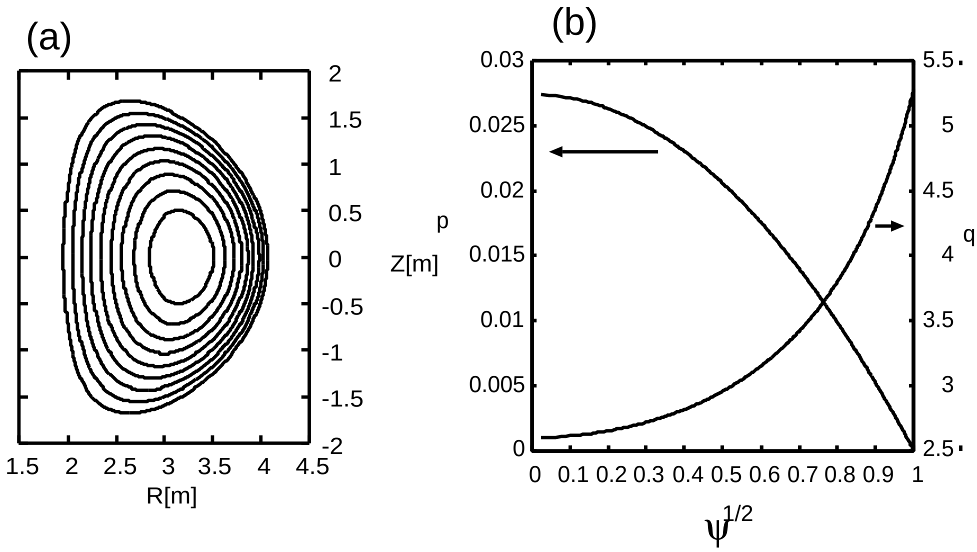

FIG. 3. (Left)Contour of magnetic flux for benchmarked Solov'ev equilibrium; (Right)The pressure and safety factor profiles

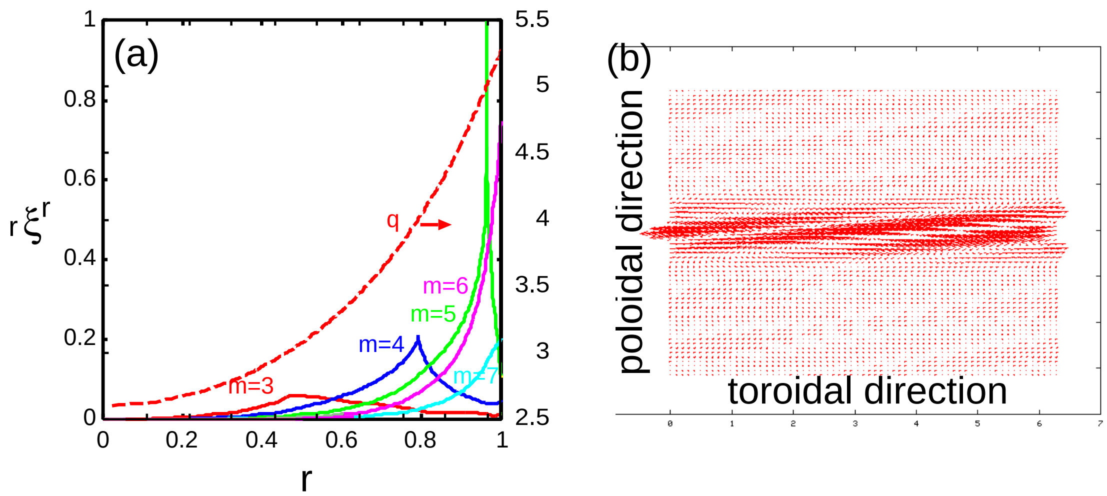

FIG. 4. (Left) Structures of eigenfunctions inside the plasma. The m=5,6 poloidal harmonics are dominant; (Right) Eddy current pattern on the resistive wall induced by unstable RWM. The current localizes at the weak field side, similar with the external kink modes.

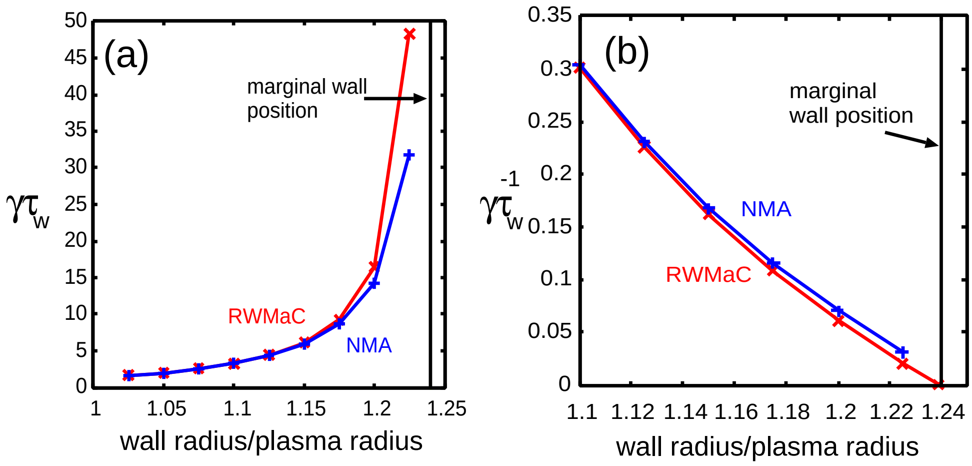

FIG. 5. (Left) The RWM growth rate vs. the wall position. When the wall is sufficiently close to the plasma surface to stabilize the ideal external kink mode deeply, the both results agree quite well. (Right) Inverse of the growth rate vs. the wall position. RWMaC and NMA capture the divergence property of the growth rate when approaching the marginal wall position.

4. Summary

To study the rotation effects on RWM stability, we have proposed a new matching scheme. The new scheme evades some obstacles in MHD stability analysis: for non-self-adjointness due to rotation, we take initial value approach, and for singular property of Newcomb equation we use inner layers with finite width. The numerical experiments demonstrated the effectiveness of the present scheme. It is expected that CPU time will be substantially saved when the new scheme is applied to the two-dimensional tokamak configuration. For the two-dimensional code, we have developed a RWM analysis Code RWMaC, which assumes still no rotation. RWMaC has been successfully benchmarked against NMA code, and will implement the proposed matching scheme in the near future.

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