Effect of the Toroidal Asymmetry on the Structure of TAE Modes in Stellarators

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Abstract. It was found recently with the use of the ballooning formalism that the toroidal asymmetry of the stellarator magnetic configuration can significantly change the structure of Torodicity-induced Alfvén Eigenmodes (TAE) with high toroidal mode numbers (n), producing singular (stepwise) wave functions [Yakovenko, Yu.V., et al., Energetic Particles in Magnetic Confinement Systems (Proc. 10th IAEA Tech. Mtg, Kloster Seeon, 2007), IAEA, Vienna (2008), CD-ROM file Iv.03]. In this work, the effect of the toroidal asymmetry on TAE-modes with finite n in stellarators is studied with the code BOA-en. A sequence of TAE-modes with almost the same frequency and localization but with different n is considered. The asymmetry consecutively couples the modes in the sequence. When the magnitude of the asymmetry is sufficiently large, the modes are strongly mixed (that is, the contributions of separate TAE-modes to the resulting wave functions are comparable). The mixed wave functions are strongly anharmonic in the toroidal direction, having a tendency to localization of the wave energy along certain field lines. They cannot be characterized by a certain number n; their periodicity in the toroidal direction is determined by the periodicity of the magnetic configuration harmonic that couples the individual TAE-modes. Strongly anharmonic TAE-modes can appear most easily in stellarators with an even number of the field periods (N). In particular, one can expect that the TAE-modes that will appear near several rational flux surfaces in the 2-period stellarator QPS may turn out to be strongly anharmonic. In the devices with odd N, coupling between TAE-modes with different n is weaker, and Ellipticity-induced Alfvén Eigenmodes (EAE) may become strongly anharmonic more easily than the TAE-modes.

1. Introduction

Instabilities of Alfvén eigenmodes (AE) are often observed in tokamaks and stellarators [1–3]. In spite of considerable similarities between AEs in tokamaks and AEs in stellarators, the lack of the axial symmetry in the stellarators results in important differences between them. First of all, new gaps appear in the stellarator Alfvén continua, with new types of eigenmodes (the so-called mirror-induced and helicity-induced Alfvén eigenmodes, MAE and HAE) residing in these gaps [4–6]. These eigenmodes are typically trapped in certain sectors of the plasma cross section ("waveguides") due to the interference of magnetic configuration Fourier harmonics with similar period lengths along the field lines [7] (it seems possible that this interference can explain some features observed in earlier numerical simulations [8]). The resonances responsible for the destabilization of AEs in stellarators are also more various than those in tokamaks [9].

It was found recently [10] with the use of the ballooning formalism that even weak toroidal asymmetry can significantly change the properties of the Torodicity-induced Alfvén Eigenmodes (TAE), which are often observed in tokamaks and stellarators. The reason for this lies in the fact that the TAE modes are infinitely degenerate in the local approximation: For each TAE mode, there is an infinite set of modes with approximately the same frequency and location. The mode numbers of these modes satisfy the relationship

$$\frac{2m_1 - 1}{2n_1} = \frac{2m_2 - 1}{2n_2} = \frac{2m_3 - 1}{2n_3} = \cdots = \iota_*^{-1}, \tag{1}$$

where ι_* is the rotational transform at the point near which the modes are localized, and the poloidal and toroidal mode numbers of the two main Fourier harmonics of the j-th mode are $(m, n) = (m_j - 1, n_j)$ and $(m, n) = (m_j, n_j)$, respectively. Toroidal asymmetry couples these modes, producing a narrow band of continuous spectrum. The corresponding solutions tend to become localized along field lines. Thus, the asymmetry affects the structure of the TAE modes in the same fashion as it affects that of the ballooning modes [11]. Then a question arises: to what extent this result obtained in the framework of the ballooning formalism is valid for modes with low n? The aim of this work is to answer this question by studying numerically the effect of the asymmetry on TAE-modes with finite n.

2. Model of TAE modes

We consider numerically a sequence of TAE modes with the mode numbers satisfying Eq. (1) in a low-shear configuration similar to that of Wendelstein-line stellarators. It is known that in the case of low shear, $\hat{s} \ll \epsilon_t$, where $\hat{s}$ is the magnetic shear and ϵ_t is the relative magnitude of the toroidal harmonic of the magnetic field, a TAE mode in a tokamak essentially consists of two Fourier harmonics, $(m, n) = (m_j - 1, n_j)$ and (m_j, n_j), toroidal coupling with harmonics with other values of m being negligible [12]. This fact simplifies our consideration, giving us a possibility to consider only two Fourier harmonics for each n; however, we believe that our main conclusions are valid for high-shear configurations, too. When the axial symmetry is broken, the modes become coupled via toroidally asymmetric Fourier harmonics of equilibrium quantities (the flux surface shape and the magnetic field). The equilibrium harmonic $\propto \exp(i\mu\theta - i\nu N\phi)$ (here N is the number of the field periods, θ and ϕ are the poloidal and toroidal coordinates, respectively) can couple harmonics of two modes satisfying Eq. (1) provided that

$$\frac{\nu N}{\mu + s} = \frac{2n_j}{2m_j - 1} = \iota_* \tag{2}$$

with $s = -1, 0$ or 1. When N is even, coupling is achieved via harmonics with $\nu = 1$, which are typically much larger than the harmonics with larger ν; when N is odd, coupling is possible only via harmonics with even ν, i.e., $\nu = 2$ and larger. To achieve a most pronounced effect of the toroidal asymmetry, we took $N = 4$, like in the conceptual 4-period Helias reactor HSR4/18 [13], and the rotational transform ι that passes through the value of $4/5$, like in the low-ι configuration of Wendelstein 7-X (W7-X) [14]. In this configuration, there is a sequence of TAE modes with the mode numbers $(m, n) = (2, 2)\&(3, 2)$, $(7, 6)\&(8, 6)$, $(12, 10)\&(13, 10)$, etc., which are consecutively coupled, chiefly via the metric tensor harmonic $\propto \cos(\mu\theta - \nu N\phi)$ with $(\mu, \nu) = (4, 1)$. This harmonic (which characterizes the helical quadrangularity of the plasma cross section) proves to be rather large in realistic Wendelstein configurations.

To describe the modes, we use the following equation system derived by means of Fourier decomposition from the equation of Alfvén oscillations in low-pressure plasmas [5]:

$$\frac{1}{r}\frac{d}{dr}\left[(\lambda - k_l^2)r\frac{d\Phi_l}{dr}\right] - \frac{m_l^2}{r^2}(\lambda - k_l^2)\Phi_l - \frac{k_l}{r}(rk_l')'\Phi_l + \hat{A}_{l,l-1}\Phi_{l-1} + \hat{A}_{l,l+1}\Phi_{l+1} = 0, \quad (3)$$

Here Φ_1, Φ_2, Φ_3, ... is as a sequence of Fourier harmonics of the wave potential, r is the radial coordinate defined as $r = (2\psi/B_0)^{1/2}$, ψ is the toroidal magnetic flux, $\lambda = \omega^2 R^2/v_A^2$, ω is the mode frequency, v_A is the average Alfvén velocity at the magnetic axis, R is the major radius of the plasma, $k_l = m_l\iota - n_l$ is the normalized longitudinal wave number. The mode numbers (m_l, n_l) of the harmonic Φ_l are given by $(m_{2p-1}, n_{2p-1}) = (5p-3, 4p-2)$, $(m_{2p}, n_{2p}) = (5p-2, 4p-2)$. Thus, each pair of the $2p-1$-th and $2p$-th harmonics of the sequence can form TAE modes near the point r_* at which $\iota(r_*) = \iota_* = 4/5$ (i.e., $k_{2p-1} = -k_{2p}$), whereas $(m_{2p+1}, n_{2p+1}) - (m_{2p}, n_{2p}) = (4, N)$. The operators coupling the harmonics are given by

$$\hat{A}_{lj}\Phi_j = \frac{1}{2r}\frac{d}{dr}\left[(\lambda\epsilon_{c(1,0)} - k_l k_j \epsilon_{g(1,0)})r\frac{d\Phi_j}{dr}\right] - \frac{k_l}{2r}(r\epsilon_{g(1,0)}k_j')'\Phi_j + \frac{1}{2}\epsilon_{g(1,0)}D_{jl}\iota'\frac{d\Phi_j}{dr} \quad (4)$$

for $l = 2p-1$, $j = 2p$ or $l = 2p$, $j = 2p-1$ and by

$$\hat{A}_{lj}\Phi_j = \frac{1}{2r}\frac{d}{dr}\left[\epsilon_{g(4,1)}(\lambda - k_l k_j)r\frac{d\Phi_j}{dr}\right] - \frac{k_l}{2r}(r\epsilon_{g(4,1)}k_j')'\Phi_j + \frac{1}{2}\epsilon_{g(4,1)}D_{jl}\iota'\frac{d\Phi_j}{dr} \quad (5)$$

for $l = 2p+1$, $j = 2p$ or $l = 2p$, $j = 2p+1$, where $D_{jl} = m_j n_l - m_l n_j$, the coupling parameters $\epsilon_{g(\mu,\nu)}$ and $\epsilon_{c(\mu,\nu)}$ characterize the normalized Fourier coefficients of the metric tensor component $g^{\psi\psi} = |\nabla\psi|^2$ and the magnetic field:

$$h_{g,c} = 1 + \frac{1}{2}\sum_{\mu,\nu=-\infty}^{\infty} \epsilon_{g,c(\mu,\nu)} \exp(i\mu\theta - i\nu N\phi), \quad (6)$$

$h_g \equiv g^{\psi\psi}/\langle g^{\psi\psi}\rangle$, $h_c \equiv h_g\langle B\rangle^4/B^4$. When writing these equations, we assumed for the sake of simplicity that the harmonics are coupled only due to the angular dependence of $g^{\psi\psi}$ and B and neglected the coupling due to other components of the metric tensor. In addition, we assumed that the plasma density is homogeneous to avoid as much as possible the difficulties associated with the consideration of modes that lie within the Alfvén continuum.

3. Numerical modelling

The new code BOA-en has been developed to solve Eqs. (3)–(5). This code is based on the code BOA-e [15], but it is more flexible and enables one to work with an arbitrary number of mode harmonics. For our calculations we took parabolic ι-profiles passing through $\iota = \iota_*$ at $r = r_* = 0.6a$: $\iota = \iota_* + \alpha[0.36 - (r/a)^2]$, where a is the minor radius of the plasma, α is a parameter determining the magnitude of the magnetic shear (which, in turn, determines the mode width). We took this parameter in the interval of $0.08 \leq \alpha \leq 0.25$. The coupling parameter $\epsilon_{g(4,1)}$ was taken in the form $\epsilon_{g(4,1)} = c(r/a)^2$ with $0 \leq c \leq 0.65$, so that $\epsilon_{g(4,1)}(r = 0.6a) \leq 0.23$, which approximately corresponds to the magnitude of this parameter in W7-X. The coupling parameters $\epsilon_{g(1,0)}$ and $\epsilon_{c(1,0)}$ were constant (for the same reason as we assumed the plasma density to be constant), which does not seem important. Namely, the radial dependences of $\epsilon_{g(1,0)}$ and $\epsilon_{c(1,0)}$ were the same as in W7-X; their magnitude, however, was larger and corresponded approximately to a stellarator with the aspect ratio of HSR4/18 and $\beta \sim 10\%$.

When the $(4,1)$-harmonic (the toroidal asymmetry) is switched off, the code finds a set of TAE-modes harmonic in ϕ (i.e., they depend on ϕ as $\exp(in\phi)$). Since the shear is small, the code finds a set of multiple modes for each n (see Fig. 1, where results of calculations for $\alpha = 0.08$ and $l = 3, \ldots, 8$ are presented). The frequencies of the modes with different n are close (at least, those of the modes that lie sufficiently far from the gap boundaries); still, they are different due to non-locality effects. The shapes of the eigenfunctions with close frequencies look similar, their width decreasing with n. As the coupling $(4,1)$-harmonic of the metric tensor appears, the eigenfunctions become anharmonic in ϕ (i.e., their Fourier spectrum contains satellites with different numbers n), which is accompanied by "repulsion" of the frequencies of similar modes. When the increase of the distance between the mode frequencies exceeds their initial distance (which is the case for realistic values of the $(4,1)$-harmonic), the eigenfunctions undergo a qualitative change (even though this change takes place gradually): The contributions of harmonics with different n becomes comparable (see Fig. 2). In fact, such modes can no longer be characterized by a certain toroidal mode number n. We will refer to such eigenfunctions as strongly mixed or strongly anharmonic. The dependence of the wave function on ϕ at the outer circumference of the plasma ($\theta = 0$) is shown in Fig. 3 for two eigenmodes at three radial locations. One can observe that the interference of modes with the toroidal numbers 6, 10, and 14 has produced wave functions that have the period equal to π in the toroidal direction and, at first glance, look like modes with the dominant harmonic $n = 2$. However, this impression is false; harmonics with $n = 2$ were not included to this calculation. The wave functions are almost stepwise in ϕ; as the relative phases of the harmonics weakly change along the magnetic field lines, this means that the electric field of the mode is localized at certain field lines.

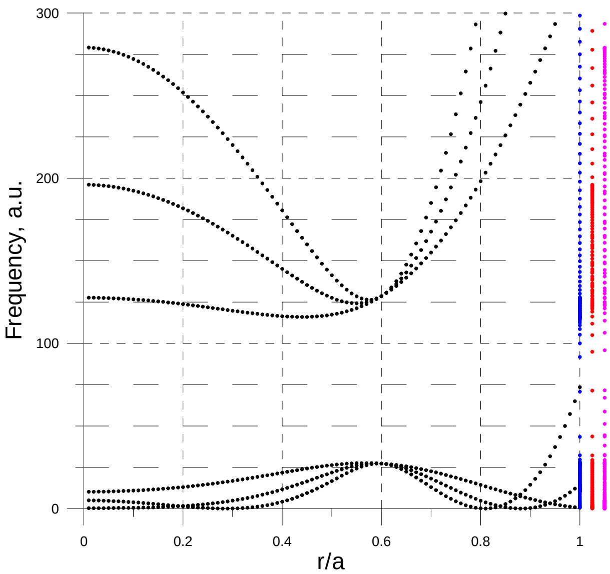

FIG. 1. The Alfvén continuum and the mode spectrum of TAE-modes with $n = 6$, $n = 10$, and $n = 14$ in the case of toroidal symmetry for $\alpha = 0.08$. Black dots, the continuum; red dots, the spectrum of modes with $n = 6$; blue dots, with $n = 10$; magenta dots, with $n = 14$.

4. Discussion

Our calculations has shown that toroidal asymmetry of the configuration can completely change the structure of TAE-modes. They can no longer be characterized by a definite toroidal mode number n, being a mixture of modes with different n. It follows that from Eq. (2) that the period of the mixed wave functions in the toroidal direction equals $4\pi/(\nu N)$. The shapes of the wave functions is strongly anharmonic and has the tendency

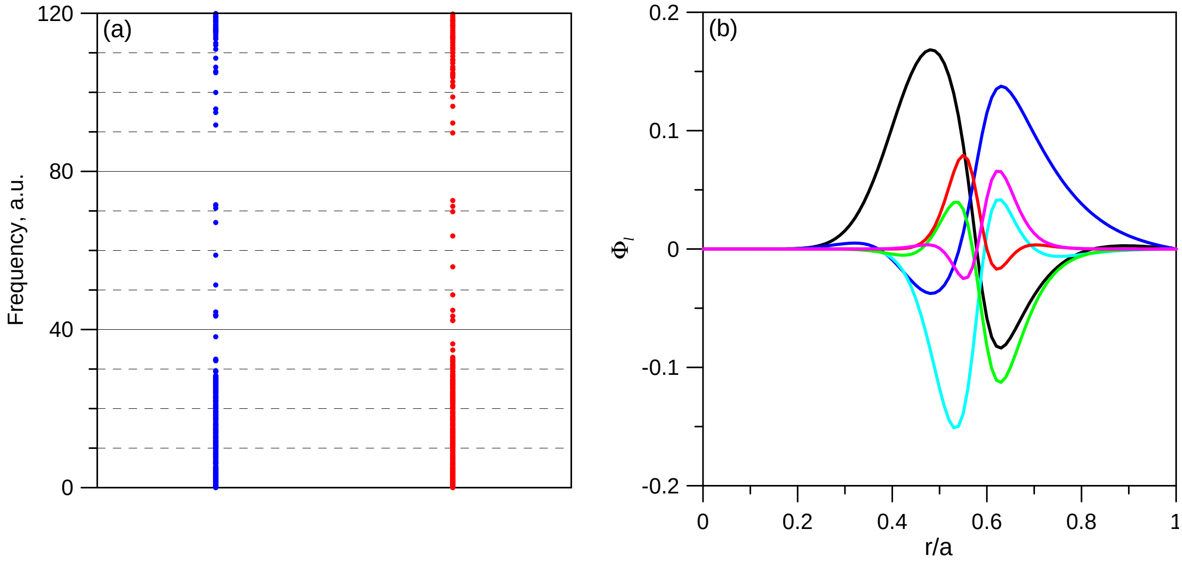

FIG. 2. Results of calculations for harmonics with $l = 3, \ldots, 8$ *in the presence of the* $(4, 1)$*-harmonic. (a), comparison of the frequency spectrum in the axisymmetric case (*$c = 0$*, blue dots) and with a non-zero* $(4, 1)$*-harmonic (*$c = 0.65$*, red dots). (b), the radial structure of harmonics of the eigenmode with the frequency of 89.7 a.u.; black line corresponds to* $l = 3$*,* $(m, n) = (7, 6)$*; blue,* $l = 4$*,* $(m, n) = (8, 6)$*; cyan,* $l = 5$*,* $(m, n) = (12, 10)$*; green,* $l = 6$*,* $(m, n) = (13, 10)$*; red,* $l = 7$*,* $(m, n) = (17, 14)$*; magenta,* $l = 8$*,* $(m, n) = (18, 14)$*.*

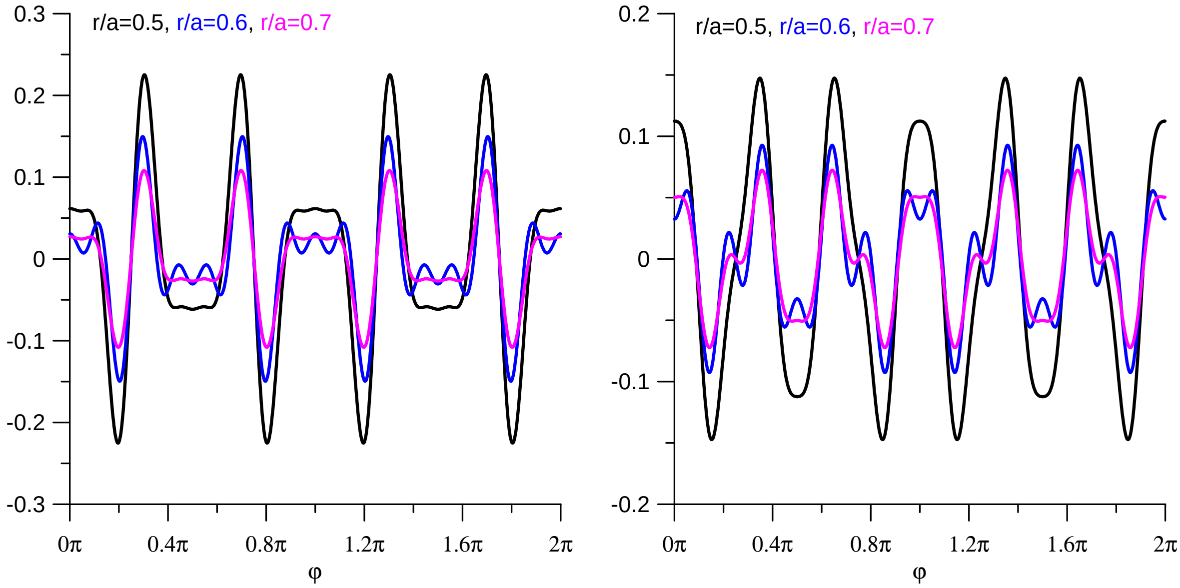

FIG. 3. The toroidal structure of the eigenmodes at $\theta = 0$*. Left panel, the frequency of 89.7 a.u.; right panel, the frequency of 92.2 a.u.*

to become stepwise. The Ellipticity-induced Alfvén Eigenmodes (EAE) can also be coupled by the toroidal asymmetry and form strongly anharmonic structures. Let us consider a sequence of EAE-modes, each consisting of the harmonics with the numbers (m_j, n_j) and $(m_j - 2, n_j)$. Similarly to Eq. (2), one write the following condition that these modes can be coupled by the configuration harmonic (μ, ν):

$$\frac{\nu N}{\mu + s} = \frac{n_j}{m_j - 1} = \iota_* \tag{7}$$

with $s = 0$ or ± 2. One can show that the anharmonic waves appearing due to mixing of these harmonics have the toroidal period of $2\pi/(\nu N)$.

The fact that the behaviour of the mode wave function becomes stepwise due to the toroidal asymmetry of the magnetic configuration agrees with analysis based on the ballooning formalism [10, 11]. The local theory [12] predicts that the frequencies of TAE-modes localized at the same radius do not depend on n to main order. In agreement with this, our calculations show that the frequencies of the TAE-modes of the considered sequence quickly converge. At the same time, one can show the coupling intensity does not decrease with n. This gives us grounds to conclude that all further modes in the sequence will be also mixed to anharmonic state. Therefore, one can expect that including more toroidal harmonics into our calculations would make the stepwise structure of the wave functions even more pronounced. This means that in practice the shape of the wave functions is determined to a large extent by non-ideal factors; one can expect that these factors smoothen the stepwise functions, affecting primarily high-n harmonics.

The strong anharmonicity of TAE-modes should manifest itself especially easily in stellarators with even N. In this case, according to Eq. (2), TAE-modes with different n can be coupled via equilibrium harmonics with $\nu = 1$, which are typically much larger than the harmonics with higher ν. In particular, one can suppose that strong toroidal anharmonicity of TAE- and EAE-modes may be observed in the 2-period stellarator QPS [16]. Depending on the plasma current and pressure, ι in QPS varies between 0.17 and 0.37. Equation (2) shows the TAE-mode located at $\iota_* = 2/7$ and consisting of the harmonics with $(m, n) = (3, 1)$ and $(4, 1)$) can become anharmonic due to the configuration harmonic with $(\mu, \nu) = (6, 1)$. The same can happen with the TAE-mode located at $\iota_* = 2/9$ and consisting of the harmonics with $(m, n) = (4, 1)$ and $(5, 1)$) due to the configuration harmonic with $(\mu, \nu) = (8, 1)$. Since the aspect ratio of the device is rather small, one can expect that the $(6, 1)$ harmonic of the cross section shape is not too weak to cause coupling of TAE-modes with different n, although detailed calculations are required to reach a definite conclusion. The EAE-mode located at $\iota = 1/3$ and consisting of the harmonics with $(m, n) = (2, 1)$ and $(4, 1)$) can become strongly anharmonic due to the cross section shape harmonic with $(\mu, \nu) = (4, 1)$, which is certainly not weak.

Wendelstein 7-X has an odd number of periods, $N = 5$. In this case, according to Eq. (2), coupling between TAE-modes is possible only via configuration harmonics with $\nu = 2$ or larger, which are rather weak in W7-X. However, EAE-modes with $n = 5$ can become anharmonic due to configuration harmonics with $\nu = 1$. Such eigenmodes can appear at $\iota_* = 5/6$. This value of ι is within the planned range of ι in W7-X, although it is avoided in all standard vacuum configurations [14].

5. Conclusions

Our calculations have shown that coupling of TAE modes having different n but located in the vicinity of the same radial position, which takes place in stellarators due to toroidal asymmetry of the magnetic configuration, can produce strongly anharmonic eigenmodes. It should be emphasized that although the parameters of the stellarator configuration considered here were chosen to facilitate the observation of coupling between TAE-modes, this configuration does not seem unrealistic. The amplitudes of eigenmode harmonics with different n are of the same order; therefore, one cannot attribute a definite toroidal mode number to these eigenmodes. The period of the anharmonic eigenmodes in ϕ is determined by the periodicity of the configuration harmonic causing the coupling; it equals $4\pi/(\nu N)$ for the TAE-modes and $2\pi/(\nu N)$ for the EAE-modes. The toroidal structure of the wave functions of the strongly anharmonic eigenmodes exhibits a tendency to stepwise behaviour, which agrees with previous results obtained in the framework of the ballooning formalism. This means that the shape of the strongly anharmonic wave functions is determined by non-ideal factors (finite ion Larmor radius effects, resistivity etc.). The amplitude of a strongly anharmonic eigenmode strongly depends on ϕ, which might be of importance for magnetic diagnostics (the Mirnov coils positioned in different toroidal positions could show different mode amplitudes).

Strongly anharmonic TAE-modes can appear most easily in devices with even number of periods. When N is even, coupling of TAE-modes can occur via the configuration harmonics with $\nu = 1$, which are typically larger than the harmonics with larger ν. In particular, TAE-modes and EAE-modes with low mode numbers in QPS (where $N = 2$) may turn out to be anharmonic. EAE-modes can be coupled by configuration harmonics with $\nu = 1$ even when N is odd. For example, the EAE-modes arising near $\iota = 5/6$ may prove to be anharmonic (if this value of ι appears in W7-X).

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