

Nonlinear equilibria, stability and generation of zonal structures in toroidal plasmas

L. Chen 1) and F. Zonca 2)

1) Dept. of Physics and Astronomy, Univ. of California, Irvine CA 92697-4575, U.S.A.

2) Associazione Euratom-ENEA sulla Fusione, C.P. 65 - I-00044 - Frascati, Rome, Italy

e-mail contact of main author: liuchen@uci.edu

Abstract. The long-lived saturated zonal flow (ZF) structures, spontaneously generated by drift-wave (DW) turbulence, are viewed as generators of nonlinear equilibria. Here, we derive a nonlinear evolution equation for the zonal response, valid for arbitrary wavelengths, which gives the temporal evolution of the zonal structures, whose time-asymptotic behavior corresponds to the nonlinear equilibrium. Its corresponding stability, meanwhile, determines the nature of the ZF instability and the nonlinear up-shift of turbulent transport thresholds. On a shorter-time scale, the temporal evolution of the zonal response describes the DW-ZF generation and the regulation of the DW intensity by the ZFs. As application, we present a kinetic stability analysis of zonal structures in the ITG case and estimate the nonlinear up-shift of the threshold for ITG driven turbulent transport.

1. Introduction

The crucial role played by zonal flows [1] in regulating the saturation level of drift wave turbulence and ultimately of turbulent transport (see, e.g., Ref. [2, 3, 4]), has brought significant attention on determining the amount of zonal flows which can be spontaneously generated by the turbulence itself before zonal flows become unstable as well due to Kelvin Helmholtz (KH)-like mode excitations [5, 6]. In this framework, drift waves (DW) are the “primary” instability and spontaneously generate zonal flows (ZF), the “secondary”, which can be limited in amplitude by the onset of “tertiary” KH-like modes [5, 6]. The “tertiary” instability has been proposed as explanation for the nonlinear up-shift of the critical Ion Temperature Gradient (ITG) driven turbulence threshold [7].

In this work, we propose that the long lived saturated ZF structures, spontaneously generated by DW turbulence, can be considered as generators of *neighboring nonlinear equilibria* [8]. In the present theoretical framework, we compute the general form of these neighboring nonlinear equilibria in terms of the zonal flow structures as well as of the characteristics of the “primary” DW turbulence. We adopt the four-wave modulation interaction model for describing the ZF-DW interplay [9] and, hence, solve the nonlinear gyrokinetic equation [10] for the toroidally and poloidally symmetric particle response to ZFs and DW turbulence. The solution is obtained for an arbitrary wavelength ordering and, thus, can generally describe both ITG as well as Trapped Electron Mode (TEM) regimes. The derived nonlinear evolution equation for the zonal response consistently describes the temporal evolution of the zonal structures, whose time-asymptotic behavior corresponds to the nonlinear equilibria. The stability of the nonlinear equilibria, meanwhile, determines the nature of the “tertiary” instability regime and the nonlinear up-shift of critical thresholds [7]. On a shorter time-scale, the temporal evolution of the zonal response describes the DW-ZF generation and the regulation of the DW intensity by the ZFs [2, 3, 4]. In the long wavelength limit, the DW turbulence drive for ZFs reduces to

the well-known form of Reynolds stress [1].

A specific application of this theoretical framework is given in the present work. Employing the time asymptotic response of the zonal structures as the nonlinear equilibria, we have carried out a kinetic stability analysis of the “tertiary” KH-like modes in the ITG case and determined the threshold condition for their excitation, i.e. the maximum allowable level of zonal flows spontaneously generated by drift wave turbulence. In other words, our work represents extensions of previous studies [5, 6], which are limited to fluid descriptions of the “tertiary” modes and therefore valid in the strongly unstable domain. More specifically, we analyze the stability properties of the nonlinear plasma equilibria in the presence of ZFs with respect to low frequency drift waves with $|k_\perp| \gg |k_z|$, $2\pi/|k_\perp|$ and $2\pi/|k_z|$ being, respectively, drift wave and zonal flow wavelengths. In toroidal geometry, we demonstrate the crucial role played by trapped as well as barely circulating particles in setting the “tertiary” instability threshold, due to the fact that the characteristic frequency of the KH-like modes is smaller than the typical thermal ion bounce frequency. Meanwhile, the “tertiary” instability threshold determined via the kinetic stability analysis is lower than in the fluid description [5, 6]. Given this result, we suggest that the “tertiary” instability modes are more appropriately described as trapped ion ITG (TITG) rather than as KH-like modes. While the stability properties of the nonlinear zonal equilibria are given in terms of integral eigenmode equations [6], simple estimates for the threshold condition for the “tertiary” instability can be derived in the local limit. We also discuss how this instability condition can be translated into an estimate of the nonlinear up-shift of the critical threshold for the ITG turbulence driven transport, known as the “Dimits-shift” [7]. In fact, employing the time asymptotic response of the zonal structures as the nonlinear equilibria allows us to directly connect the starting reference equilibrium quantities to the nonlinear equilibrium features due to finite ZF amplitude as, e.g., radial modulations in the temperature profile [11]. With respect to these long lived zonal structures, the “tertiary” modes act as collisionless dissipation, i.e. their existence enhances the r.m.s. fluctuation level of the “primary” DW turbulence and of turbulent transport, which is strongly suppressed by ZFs for reference plasma equilibrium gradients below the “Dimits-shift” [7].

2. Temporal evolution of zonal structures and generation of nonlinear equilibria

The theoretical framework of our analysis is that of nonlinear gyrokinetics; *i.e.* we decompose the fluctuating particle distribution function into adiabatic and nonadiabatic responses as [10]

$$\delta F_k = \frac{e}{m} \delta \phi_k \frac{\partial}{\partial \mathcal{E}} F_0 + \sum_{\mathbf{k}_\perp} \exp(-i\mathbf{k}_\perp \cdot \mathbf{v} \times \mathbf{b}/\omega_c) \delta \bar{H}_k, \quad (1)$$

where $\mathcal{E} = v^2/2$ is the energy per unit mass, the other notation is standard and the subscripts to indicate the plasma particle species have been dropped unless needed to avoid confusion. The nonadiabatic response of the gyrokinetic particle distribution function, $\delta \bar{H}_k$, is obtained from the nonlinear gyrokinetic equation [10]:

$$\begin{aligned} \left(\partial_t + v_\parallel \partial_\ell + i\omega_d \right)_k \delta \bar{H}_k &= i \frac{e}{m} Q F_0 J_0(\gamma) \delta \phi_k - \delta \bar{\mathbf{u}}_{Ek'} \cdot \nabla \delta \bar{H}_{k''}, \\ Q F_0 &= \omega_k \frac{\partial F_0}{\partial \mathcal{E}} + \mathbf{k} \cdot \frac{\mathbf{b} \times \nabla}{\omega_c} F_0, \\ \delta \bar{\mathbf{u}}_{Ek} &= i \frac{c}{B} (\mathbf{b} \times \mathbf{k}_\perp) J_0(\gamma) \delta \phi_k. \end{aligned} \quad (2)$$

Here, we have assumed fluctuations characterized by the scalar potential only, $\delta\phi_k$. Furthermore, $\partial_\ell = \mathbf{b} \cdot \nabla$, ω_d is the magnetic drift frequency, $\gamma = k_\perp v_\perp / \omega_c$, $\delta\bar{\mathbf{u}}_{Ek}$ is the nonlinear $\mathbf{E} \times \mathbf{B}$ drift and $\mathbf{k} = \mathbf{k}' + \mathbf{k}''$. To solve Eq. (2) for the nonlinear modification of the equilibrium due to the presence of zonal structures, we adopt the procedure of Ref. [12]. Thus, denoting $\delta\bar{H}_z$ as the ($m = 0, n = 0$) (zonal) nonadiabatic particle response, we have

$$\begin{aligned}\delta\bar{H}_z &= \exp(-iQ_z) H_z, \\ Q_z &= \frac{q}{(r/R_0)} k_z \left(\frac{v_\parallel}{\omega_c} - \frac{\bar{v}_\parallel}{\omega_c} \right),\end{aligned}\quad (3)$$

where $\mathbf{B} \cdot \nabla H_z = 0$, Q_z indicates the generator of coordinate transformation from particle guiding-center to magnetic drift/banana-center, q is the safety factor, $k_z = (-i\partial_r)$ is the radial wave vector of nonlinear equilibrium profile changes, as denoted by the subscript z , which stands for zonal and $\overline{(\dots)} = \int (d\ell/v_\parallel) (\dots) / \int (d\ell/v_\parallel)$, ℓ being the arc length along \mathbf{B} . For simplicity, we also considered a high aspect-ratio tokamak equilibrium, $R_0/r \gg 1$, with major radius R_0 and circular shifted magnetic flux surfaces; thus, the cyclotron frequency is $\omega_c = eB_0/(mc)$, with B_0 the magnetic field on axis. Assuming that the particle equilibrium distribution function is a local Maxwellian, Eq. (2) is then reduced to the following evolution equation for the zonal structures:

$$\begin{aligned}\frac{\partial}{\partial t} H_z &= \frac{e}{T} F_0 \left[e^{iQ_z} J_0(\gamma_z) \frac{\partial}{\partial t} \delta\phi_z \right] - \sum_{\mathbf{k}_z = \mathbf{k}' + \mathbf{k}''} \overline{[e^{iQ_z} \delta\bar{\mathbf{u}}_{Ek'} \cdot \nabla \delta\bar{H}_{k''}]}, \\ &= \frac{e}{T} F_0 \left[e^{iQ_z} J_0(\gamma_z) \frac{\partial}{\partial t} \delta\phi_z \right] + \sum_{\mathbf{k}_z = \mathbf{k}' + \mathbf{k}''} i \frac{c}{B_0} k'_\theta \frac{\partial}{\partial r} \overline{[e^{iQ_z} J_0(\gamma') \delta\phi_{k'} \delta\bar{H}_{k''}]},\end{aligned}\quad (4)$$

with $k'_\phi = -k''_\phi$, $k'_\theta = -k''_\theta$. Equation (4) is very general and describes the DW-ZF generation and the regulation of the DW intensity by the ZFs [2, 3, 4] as well as the formation of the nonlinear equilibria due to the zonal structures on the long time scales. Note also that Eq. (4) does not assume any specific wavelength ordering for neither DWs nor zonal structures: thus, the second term on the right hand side (RHS) reduces to the well-known form of Reynolds stress [1] only in the long wavelength limit. Once H_z is known, using Eqs. (1) and (3), one has

$$\overline{\delta F_z} = -\frac{e}{T} F_0 \delta\phi_z + J_0(\gamma_z) \Gamma_{dz}^* H_z, \quad (5)$$

for the total bounce/transit averaged zonal structure response by direct construction, with $\Gamma_{dz} = \overline{\exp(iQ_z)}$.

As a simple illustration of Eq. (4) on ITG nonlinear dynamics, let us consider the fluid approximation for ions, $|\partial_t| \approx |\omega| \gg |v_\parallel \partial_\ell|, |\omega_d|, |\delta\bar{\mathbf{u}}_{Ek'} \cdot \nabla| \approx \gamma_\ell, \gamma_{nl}$, where γ_ℓ and γ_{nl} are, respectively, the inverse linear and nonlinear time scales. Thus, for the k th normal mode of ITG we can assume, at the lowest order,

$$\delta\bar{H}_{i,k} \simeq \frac{e}{T_i} F_{i,0} \left(1 - \frac{\omega_{*i}}{\omega} \right)_k J_0(\gamma) \delta\phi_k, \quad (6)$$

where $\omega_{*i} = \omega_{*ni} + \omega_{*Ti}$ ($m_i \mathcal{E}/T_i - 3/2$), $\omega_{*ni} = T_i \mathbf{k} \times \mathbf{B}/B \cdot \nabla n_i / (n_i m_i \omega_{ci})$ and $\omega_{*Ti} = \mathbf{k} \times \mathbf{B}/B \cdot \nabla T_i / (m_i \omega_{ci})$. Substituting back into Eq. (4) and using Eq. (5), we readily obtain

$$\overline{\delta F_{i,z}} = -\frac{e}{T_i} F_{i,0} \left\{ \left(1 - J_0^2(\gamma_z) |\Gamma_{dz}|^2 \right) \delta\phi_z + J_0(\gamma_z) |\Gamma_{dz}|^2 \left[\alpha_0 - \alpha_1 \left(\frac{m_i v^2}{2T_i} - \frac{3}{2} \right) \right] A_z \right\},$$

$$\left[\alpha_0 - \alpha_1 \left(\frac{m_i v^2}{2T_i} - \frac{3}{2} \right) \right] A_z = \sum_{\mathbf{k}_z = \mathbf{k}' + \mathbf{k}''} \left(1 - \frac{\omega_{*i}}{\omega} \right)_{k''} \overline{\int_0^\infty dt J_0(\gamma'') \bar{\mathbf{u}}_{Ek'} \cdot \nabla \delta \phi_{k''}} . \quad (7)$$

Note that Eq. (7) and, more generally, Eq. (5) directly connect the nonlinear equilibria due to the zonal structures with the original plasma equilibrium via the time history of the DW turbulence. In the simplest toroidal ITG case, with $\omega \simeq \omega_d(3 + 1/\tau)$ and $\tau = T_i/T_e$, we can estimate $\alpha_0 \simeq 1 - \omega_{*ni}/\omega \simeq 1 - (R_0/L_n)/(3 + 1/\tau)$ and $\alpha_1 \simeq \omega_{*Ti}/\omega \simeq (R_0/L_T)/(3 + 1/\tau)$. Imposing

$$\delta n_{i,z} = \langle \langle \delta F_{i,z} \rangle \rangle = 0 \quad (8)$$

for consistency, we readily find

$$\chi_i \delta \phi_z = -\alpha_0 A_z . \quad (9)$$

Here, we have used the identity

$$\langle \langle \dots \rangle \rangle = \left(\oint (d\ell/B) \right)^{-1} \left(\oint (d\ell/B) \langle \dots \rangle \right) = 2\pi \left(\oint (d\ell/B) \right)^{-1} \sum_{\text{sgn}(v_{\parallel}) = \pm} \int d\mu d\mathcal{E} \tau_b \overline{\langle \dots \rangle} , \quad (10)$$

with $\langle \dots \rangle$ indicating velocity space integration and where $\mu = v_{\perp}^2/(2B)$ and τ_b is the particle bounce or transit time. Moreover,

$$\chi_i = \langle \langle (F_0/n_i) \left(1 - J_0^2(\gamma_z) |\Gamma_{dz}|^2 \right) \rangle \rangle \simeq 1.6q^2 (R_0/r)^{1/2} k_z^2 \rho_i^2 \quad (11)$$

in the long wavelength limit [12], ρ_i denoting the ion Larmor radius. Meanwhile, the (flux surface averaged) zonal temperature fluctuation is given by

$$\frac{\delta T_{i,z}}{T_i} = \left\langle \left\langle \frac{2m_i \mathcal{E} \delta F_{i,z}}{3T_i n_i} \right\rangle \right\rangle = -\frac{5}{3} \chi_i \frac{e}{T_i} \delta \phi_z + (\alpha_1 - \alpha_0) \frac{e}{T_i} A_z = -\left(\frac{\alpha_1}{\alpha_0} + \frac{2}{3} \right) \chi_i \frac{e}{T_i} \delta \phi_z . \quad (12)$$

According to Eq. (12), we expect that nonlinear zonal structures in the ITG case give

$$\delta T_{i,z}/(e\delta\phi_z) \simeq -(\alpha_1/\alpha_0 + 2/3)\chi_i < 0 , \quad (13)$$

corresponding to temperature and potential fluctuations in anti-phase, consistently with numerical simulation results [11].

In terms of $\delta T_{i,z}$, the time asymptotic solution of Eq. (4), describing the formation of zonal structures in the nonlinear equilibrium in the presence of zonal flows, can be written as

$$H_{i,z} = \frac{e}{T_i} \delta \phi_z F_{i,0} + \left[\frac{m_i \mathcal{E}}{T_i} \frac{(\alpha_1/\alpha_0 + \Delta(\lambda)/\chi_i)}{(\alpha_1/\alpha_0 + 2/3)} - \frac{3}{2} \right] \frac{\delta T_{i,z}}{T_i} F_{i,0} . \quad (14)$$

Here, we have taken the long wavelength limit and we have introduced the notation $\Delta(\lambda)$, with $\lambda = \mu/\mathcal{E}$, defined such $\Gamma_{dz} = 1 - \Delta(\lambda)m_i\mathcal{E}/T_i$ and satisfying $\langle \langle (F_{i,0}/n_i)\Delta(\lambda)m_i\mathcal{E}/T_i \rangle \rangle = \chi_i/2$. In the next Section, we assume Eq. (14) as nonlinear equilibrium paradigm for computing the nonlinear up-shift for ITG turbulent transport.

3. Nonlinear up-shift for ITG turbulent transport

As stated in the Introduction, the stability of the nonlinear equilibria computed from Eq. (5) determines the nature of the ‘‘tertiary’’ instability regime and the nonlinear up-shift of critical

thresholds [7]. As an application, we analyze the ITG case, whose nonlinear equilibrium is readily obtained from Eqs. (5) and (14); i.e.,

$$\overline{\delta F_{i,z}} = \left[\frac{m_i \mathcal{E}}{T_i} \frac{(\alpha_1/\alpha_0 + 2\Delta(\lambda)/\chi_i)}{(\alpha_1/\alpha_0 + 2/3)} - \frac{3}{2} \right] \frac{\delta T_{i,z}}{T_i} F_{i,0} . \quad (15)$$

Meanwhile, $\delta \bar{H}_{e,k} = -(e/T_e)\delta\phi_z$, which exactly cancels the electron adiabatic response. Thus, electron do not contribute to forming zonal structures in the ITG case; $\delta F_{e,z} = 0$.

Here, we present a kinetic stability analysis of the nonlinear equilibrium, given by Eq. (15), with respect to low frequency DW, $|\omega\tau_b| \ll 1$, with $|k_\perp| \gg |k_z|$ [5, 6]. The frequency ordering is consistent with previous investigations of zonal structure stability in the presence of K–H modes [5, 6]. Meanwhile, the condition on the wave-vector is given by the requirement that modes are localized within the potential wells associated with the zonal structures. The quasi-neutrality condition describing the DW dispersion relation for one ion species with $n_i = n_e$ reads

$$\left(1 + \frac{T_i}{T_e}\right) \delta\phi_k = \frac{T_i}{n_i e} \langle J_0(\gamma_k) \delta \bar{H}_{i,k} \rangle_s , \quad (16)$$

where we have assume adiabatic electron response. The nonadiabatic ion response, $\delta \bar{H}_{i,k}$, is obtained from the nonlinear gyrokinetic equation, Eq. (2) [10]. Similarly to Eq. (3), we can define

$$\delta \bar{H}_k = e^{-iQ_k} \delta H_{bk} , \quad (17)$$

where Q_k indicates the generator of coordinate transformation from particle guiding-center to magnetic drift/banana-center and we have dropped the ‘‘ion’’ subscript i for simplicity. The bounce averaged response is then readily obtained in the form [8]

$$\delta H_{bk} = \left[1 + \frac{\omega_* + \omega_{*T}^{nl} - \bar{\omega}_d}{\bar{\omega}_d + \omega_z - \omega} \right] \overline{[J_0(\gamma_k) e^{iQ_k} \delta\phi_k]} \frac{e}{T_i} F_0 \quad (18)$$

$$\omega_{*T}^{nl} = \left[\frac{m_i \mathcal{E}}{T_i} \frac{(\alpha_1/\alpha_0 + \Delta(\lambda)/\chi_i)}{(\alpha_1/\alpha_0 + 2/3)} - \frac{3}{2} \right] J_0(\gamma_z) \frac{ck_\theta}{eB_0} \frac{\partial}{\partial r} \delta T_{i,z} , \quad (19)$$

$$\omega_z = \frac{c}{B_0} k_\theta J_0(\gamma_z) \frac{\partial}{\partial r} \delta\phi_z , \quad (20)$$

where ω_{*T}^{nl} is the nonlinear diamagnetic frequency associated with the zonal structures, while ω_z is the $\mathbf{E} \times \mathbf{B}$ frequency shift due to ZF. By direct substitution, the quasi-neutrality condition becomes [8]

$$\left(1 + \frac{T_i}{T_e}\right) \delta\phi_k = 4\pi \int d\mu d\mathcal{E} \frac{B}{|v_\parallel|} \frac{F_0}{n_i} \left\{ J_0(\gamma_k) e^{-iQ_k} \left[1 + \frac{\omega_* + \omega_{*T}^{nl}}{\bar{\omega}_d + \omega_z - \omega} \right] \overline{[J_0(\gamma_k) e^{iQ_k} \delta\phi_k]} \right\} , \quad (21)$$

where we have used the fact that $|\bar{\omega}_d| \ll |\omega_*|$. The solution of the above integral equation exists since it is possible to construct a variational principle [6, 8]

$$V(\delta\phi_k^*, \delta\phi_k) = \frac{N(\delta\phi_k^*, \delta\phi_k)}{D(\delta\phi_k^*, \delta\phi_k)} , \quad (22)$$

with $N(\delta\phi_k^*, \delta\phi_k)$ and $D(\delta\phi_k^*, \delta\phi_k)$ bilinear forms in $\delta\phi_k^*$, $\delta\phi_k$ and $V(\delta\phi_k^*, \delta\phi_k)$ characterized by an extremum for $N(\delta\phi_k^*, \delta\phi_k) = D(\delta\phi_k^*, \delta\phi_k)$ if the quasi-neutrality condition, Eq. (21), is satisfied. In Eq. (22)

$$\begin{aligned} N(\delta\phi_k^*, \delta\phi_k) &= 4\pi \int d\mu d\mathcal{E} \frac{F_0}{n_i} \left(\int \frac{d\ell}{|v_{\parallel}|} / \oint \frac{d\ell}{B} \right) \left[\overline{|\delta\phi_k|^2} - \overline{|J_0(\gamma_k) e^{iQ_k} \delta\phi_k|^2} \right] , \\ D(\delta\phi_k^*, \delta\phi_k) &= 4\pi \int d\mu d\mathcal{E} \frac{F_0}{n_i} \left(\int \frac{d\ell}{|v_{\parallel}|} / \oint \frac{d\ell}{B} \right) \frac{\omega_* + \omega_{*T}^{n\ell}}{\bar{\omega}_d + \omega_z - \omega} \overline{|J_0(\gamma_k) e^{iQ_k} \delta\phi_k|^2} . \end{aligned} \quad (23)$$

As it is readily verified from Eq. (18), the contribution of circulating particles to the non-adiabatic response δH_{bk} is negligible and the main effect in Eq. (21) comes from trapped and barely circulating ions.

On the basis of Eq. (21), finite orbit widths of trapped as well as barely circulating particles give the radial dispersion of the waves, while the radial profile of ω_z and $\partial_r \delta T_{i,z}$ provide the radial potential well. The condition for bound states to exist is that the mode be localized near a minimum of the potential well at radial positions where, in the long wavelength limit [5]

$$\partial_r \omega_z = \partial_r^2 \delta\phi_z = \partial_r^2 \delta T_{i,z} = 0 . \quad (24)$$

Note that all these conditions are equivalent because of Eq. (12). The radial scale of the modes is obtained by balancing radial dispersiveness and potential well variations near its minimum: thus [8]

$$|k_r|^2 \rho_{bi}^2 \approx |k_z|^2 \Delta r^2 \approx |k_z/k_r|^2 , \quad (25)$$

where $\rho_{bi} = q\rho_i(R_0/r)^{1/2}$ is the trapped ion banana width. As a consequence, $|k_r|\rho_{bi} \approx |k_z\rho_{bi}|^{1/2} \ll 1$, consistently with $|k_z/k_r| \ll 1$ and the long wavelength limit assumed here.

A simple estimate of the mode excitation threshold can be obtained in the radial local limit. In this way, $\overline{|J_0(\gamma_k) e^{iQ_k} \delta\phi_k|} \rightarrow \delta\phi_k$ on the right hand side (RHS) of Eq. (21), while ω_z and $\omega_{*T}^{n\ell}$ assume their value at the mode radial location given by Eq. (24). Considering also that, typically, $\alpha_1/\alpha_0 \gg 1$, the marginal stability condition is [8]

$$\omega = \omega_z + \frac{3}{2} \bar{\omega}_{dT_i} , \quad (26)$$

where we have approximated $\bar{\omega}_d$ with the deeply trapped ion precession frequency, i.e., $\bar{\omega}_d = (m_i \mathcal{E}/T_i) \bar{\omega}_{dT_i}$. The mode described by the dispersion relation Eq. (26) is radially localized at the position given by Eq. (24), has a typical radial width given by Eq. (25) and belongs to the DW branch. Meanwhile, it is destabilized by the ion temperature gradient due to zonal structures, superimposed to the original plasma profiles. Thus, this mode is a trapped ion temperature gradient driven mode (TITG) and has a critical destabilization threshold given by [8]

$$\left(1 + \frac{T_i}{T_e}\right) \delta\phi_m(r) < \left(\frac{2r}{R_0}\right)^{1/2} \int_0^\infty \frac{d\kappa}{\kappa^2} W(\kappa) C_m(\kappa) \left[1 - \frac{R_0}{T_i} \frac{\partial}{\partial r} (T_i + \delta T_{i,z})\right] \sum_l C_l(\kappa) \delta\phi_l(r) . \quad (27)$$

Here, we have assumed modes in the form

$$\delta\phi_k = e^{in\phi} \sum_m e^{-im\theta} \delta\phi_m(r) . \quad (28)$$

We have also introduced the notation $\kappa^2 = 2(r/R_0)\lambda/[1 - (1 - r/R_0)\lambda]$, where $\kappa^2 < 1$ [$0 \leq \lambda < (1 - r/R_0)$] indicates circulating particles, while trapped particles have $\kappa^2 > 1$ [$(1 - r/R_0) < \lambda \leq (1 + r/R_0)$]. In this way

$$\overline{\delta\phi_k} = e^{in(\phi - q\theta)} \sum_l C_l(\kappa) \delta\phi_l(r) , \quad (29)$$

$$C_l(\kappa) = \frac{1}{2\pi W(\kappa)} \oint d\theta \frac{\cos(nq - l)\theta}{(1 - \kappa^2 \sin^2(\theta/2))^{1/2}} , \quad (30)$$

with $W(\kappa) = (2/\pi)\mathbb{K}(\kappa)$ for circulating particles ($\kappa^2 < 1$), while $W(\kappa) = 2/(\pi\kappa)\mathbb{K}(1/\kappa)$ for trapped particles ($\kappa^2 > 1$), \mathbb{K} indicating the complete elliptic integral of the first kind. For modes localized near a rational surface $q = m/n$, consistent with Eq. (24), so that only one poloidal component dominates the RHS of Eq. (27), we readily derive the critical destabilization threshold for the TITG mode as

$$-\frac{R_0}{T_i} \frac{\partial}{\partial r} (T_i + \delta T_{i,z})_c = \frac{(1 + T_i/T_e)}{(2r/R_0)^{1/2} \Lambda_m} , \quad (31)$$

where

$$\Lambda_m = \int_0^\infty \frac{d\kappa}{\kappa^2} W(\kappa) C_m^2(\kappa) = O(1) . \quad (32)$$

Equation (31) is the condition for the nonlinear equilibrium in the presence of the zonal structures to become unstable for TITG modes. Thus, it can be associated with the ‘‘Dimits-shift’’ [7]. Equations (7) and (12), meanwhile, connect the properties of nonlinear equilibrium and zonal structures with those of the original plasma equilibrium and the dynamic evolution of DW turbulence. Note that Eq. (31) has been derived with some simplifying assumptions to explicitly formulate that in a closed form. More generally, both TITG mode frequency at marginal stability, Eq. (26), as well as the instability threshold, Eq. (31), should be obtained from a direct solution of the quasi-neutrality condition, Eq.(21).

4. Summary and Discussions

The nonlinear evolution equation for the zonal response, derived in the present work, has been used to determine the form of the nonlinear equilibria generated by ITG-ZF interactions. The stability analysis of these equilibria with respect to trapped ion ITG (TITG) modes determines the nonlinear up-shift of the critical threshold for ITG turbulent transport, which we propose as interpretation of the ‘‘Dimits-shift’’ [7]. The tertiary TITG modes act as collisionless dissipation on the long lived zonal structures characterizing the nonlinear equilibria. Thus, their excitation is expected to regulate zonal structures and ZFs, thereby enhancing the r.m.s. fluctuation level of the primary ITG turbulence and of turbulent transport.

The present approach is derived for arbitrary wavelengths, although the applications, reported here, are restricted to the long-wavelength limit. The application of the four-wave modulation interaction model [9] in the present theoretical framework makes it possible to further investigate the zonal structures of nonlinear equilibria at arbitrary mode numbers and to generalize the four-wave modulational instability paradigm of Chen et al. [9] to arbitrary wavelengths. For example, it is possible to derive the dispersion relation for the spontaneous excitation of

ZFs by Collisionless TEM (CTEM) and to demonstrate the significant roles played by trapped electrons in the short-wavelength $k_{\perp}\rho_i = O(1)$ regime. Detailed analyses of these issues are beyond the scope of the present work and will be reported elsewhere.

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