Kinetic Theories of Geodesic Acoustic Mode in Toroidal Plasmas

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Abstract. Geodesic Acoustic Modes (GAM) are known to constitute a continuous spectrum due to radial inhomogeneities. The existence of a singular layer causes GAM to mode convert to short wavelength kinetic GAM (KGAM) via finite ion Larmor radius (FLR) and finite guiding-center drift-orbit-width (FOW) effects. The dispersion relation of GAM/KGAM with FLR/FOW as well as parallel electric field contributions is derived to demonstrate the mode conversion to KGAM and propagation in the lower-temperature and/or higher-q region. Corresponding collisionless damping of GAM/KGAM excited in the large q region, including higher-order harmonics of ion transit resonances, has been investigated and the analytical expression for the damping rate agrees well with numerical results in its validity regime. Excitation of energetic-particle-induced GAM (EGAM) by velocity space anisotropy is also investigated taking into account the coupling to the GAM continuous spectrum. The response of energetic particles is studied nonperturbatively, and both local and nonlocal EGAM dispersion relations are derived assuming a single pitch-angle slowing-down energetic particle equilibrium distribution function. For a sharply localized energetic particle (EP) source, it is shown that the EGAM mode is self-trapped where the EP drive is strongest, with an exponentially small damping due to tunneling coupling to outward propagating KGAM. While for a broadly distributed EP source, it is shown that the EGAM will be heavily continuum damped due to the strong coupling to GAM continuous spectrum.

1. GAM continuous spectrum and collisionless damping

Geodesic Acoustic Modes (GAMs) [1] are toroidally symmetric normal modes unique to toroidal plasmas, and the mode structure is also nearly poloidally symmetric. They exist since the charge separation effect, due to ion radial magnetic drift associated with geodesic curvature, causes a finite parallel a.c. electric field ($\propto T_e/T_i$) and a perturbed ion diamagnetic current to ensure quasi-neutrality via electron and ion dynamic responses, respectively. GAMs have received much attention in magnetic fusion plasma due to their potentially important roles in regulating drift waves, and, hence, transports via nonlinear interactions.

GAM can be described by the magnetic flux surface averaged quasineutrality condition, which reads

$$\partial_r \left(\overline{\delta J_r}\right) = 0, \tag{1}$$

where δJ_r is the fluctuating radial current and $\overline{(\cdots)}$ denotes magnetic flux surface averaging. Here, we consider a large aspect-ratio axisymmetric Tokamak with straight field line flux coordinates (r, θ, ξ), and the equilibrium magnetic filed is given by $\mathbf{B}_0 = B_0(\mathbf{e}_\xi/(1+\epsilon\cos\theta) + (\epsilon/q)\mathbf{e}_\theta)$, where, ξ and θ are respectively, toroidal and poloidal angle-like flux coordinates of the torus. δJ_r is made up of polarization current and the perturbed diamagnetic current due to density accumulation in poloidal direction. The wave equation of GAM, at the lowest order, is

$$\partial_r \left[n_0(r)\omega\left(1 - \omega_G^2(r)/\omega^2\right)\right]\partial_r\overline{\delta\phi} = 0, \tag{2}$$

with ω_G the lowest order GAM frequency. Equation (2) is identical to that describing shear Alfvén wave (SAW) resonance [2] and, thus, it demonstrates that GAM also constitutes a continuous spectrum described by $\omega = \omega_G(r)$ [3]. Fluctuations of GAM continuous spectrum consist of singular structures, and will decay asymptotically in time as $\propto (1/t)\exp(-i\omega_G(r)t)$ [4]. The corresponding radial wavenumber increases with time as

$$|k_r| \equiv |\partial_r \exp(-i\omega_G t)/\exp(-i\omega_G t)| \simeq |(d\omega_G(r)/dr)t|. \tag{3}$$

Equation (3) can be viewed as a physical manifestation of phase mixing, $k_r \to \infty$ as $t \to \infty$, so the fluid approximation will break down and kinetic effects, such as FLR effects, must be taken into account and regularize the singularity [5]. The GAM wave equation will become

$$\partial_r \left[n_0(r)\omega\left(1 - \left(\omega_G^2(r)/\omega^2\right)\left(1 + \alpha k_r^2\rho_L^2\right)\right)\right]\partial_r\overline{\delta\phi} = 0, \tag{4}$$

where ρ_L is the ion Larmor radius. This equation describes the mode conversion of GAM to short wave-length kinetic GAM (KGAM) at $\omega_G(r_c) = \omega$, analogy to SAW mode conversion to its kinetic counterpart (kinetic Alfvén wave) [5]. α is positive for typical plasma parameters, so the resulting KGAM will propagate in the lower GAM continuum frequency region, i.e., lower temperature and/or higher q region, which is usually outward. The dependence of the sign of α on the plasma parameters, such as T_e/T_i and q, are discussed in [3].

It is pointed out in [3] that the mode frequency and mode structure, described by equation (4), can only be solved when there exists a source term, which can be due to coupling to an antenna or drive by drift wave turbulence or energetic particles. When drive by free energy source is considered, the drive threshold in terms of drift wave turbulence intensity or energetic particle density depends crucially on the GAM collisionless damping rate; so, it is desirable to have an analytical expression of GAM damping, valid over broad range of Tokamak parameters.

In the small ion drift orbit width limit, i.e., with $k_r\rho_L q \ll 1$, the dispersion relation of GAM can be readily obtained from the degeneracy argument [6] of GAM and SAW beta-induced eigenmode (SAW-BAE) [7], and the resulting dispersion relation coincides with the corresponding expressions of the other authors in their validity limits [8].

The GAM collisionless damping rate derived from the small drift orbit limit decreases with q, so GAM tends to exist in the edge region of Tokamak to minimize ion collisionless damping [8]. However, we need to relax the $k_r\rho_L q \ll 1$ assumption for the nonlinear excitation favors short KGAM radial wavelengths [3], and derive the corresponding expression that are valid in this regime. A very compact expression for GAM dispersion relation can be derived for $1/q^2 \ll k_r\rho_L \ll 1$, i.e., large drift orbit limit for resonant ions [9]:

$$\begin{aligned} D_r &= b\left\{1 - \left(\frac{7}{4} + \tau\right)\frac{v_{it}^2}{\omega^2 R_0^2} + b\frac{v_{it}^2}{\omega^2 R_0^2}\left(\frac{31}{16} + \frac{9}{4}\tau + \tau^2 - \frac{v_{it}^4}{\omega^4 q^2 R_0^4}\left(\frac{23}{8} + 2\tau + \frac{\tau^2}{2}\right)\right)\right. \\ &\quad \left. - b\frac{v_{it}^4}{\omega^4 R_0^4}\left(\frac{747}{32} + \frac{481}{32}\tau + \frac{35}{8}\tau^2 + \frac{1}{2}\tau^3\right)\right\}, \qquad (5) \\ D_i &= \sqrt{2}\frac{\omega}{|\omega|}\exp\{-\omega R_0/(\sqrt{2b}v_i)\}\left[1 + \frac{k_r\rho_i v_i}{\omega R_0} + \frac{2bv_i^2}{\omega^2 R_0^2}\left(1 + \frac{5}{4}\tau + \tau^2\right)\right. \\ &\quad \left. -2b + \frac{1}{24}\left(\frac{\omega^2 R_0^2}{4b^2 q^2 v_{it}^2} - \frac{\sqrt{2b}\omega R_0}{b^2 q^2 v_{it}}\right)\right], \qquad (6) \end{aligned}$$

where $b \equiv k_r^2\rho_L^2/2$. In equation (6), all the high order ion transit resonance contribution to the GAM collisionless damping are included, and the GAM/KGAM collisionless damping rates from equations (5) and (6) are in excellent agreement with TEMPEXT numerical simulations in the common validity regime of the two approaches [10].

2. GAM excitation by energetic particles

As in the case of SAW continuum, from which EPM are excited when wave-particle resonant drive exceeds continuum damping, EPs can also excite modes from the GAM continuous spectrum. That EPs can drive GAMs has been observed in recent experiments [11] and theoretical stability properties of this energetic-particle-induced GAM (EGAM) were analyzed in [12].

A question that raises naturally from the SAW-GAM analogy is " what is the contribution of the GAM continuous spectrum in the excitation of EGAM?" In this work, we consider EGAM excited via the transit frequency resonance with EPs in the small magnetic drift orbit limit, taking into account the coupling to GAM continuous spectrum [13, 14, 15].

2.1. Theoretical formulation

Energetic particle physics, are readily included in equation (1), one has

$$\frac{e}{T_e}(n_c+n_h)(\delta\phi-\overline{\delta\phi}) = -\frac{e}{T_c}n_c\delta\phi+\langle J_0(k_\perp\rho_{L,c})\delta H_{g,c}\rangle+\langle\frac{e}{m}\frac{\partial F_{0h}}{\partial E}\delta\phi+J_0(k_\perp\rho_{L,h})\delta H_{g,h}\rangle, \quad (7)$$

in which, n_c and n_h are respectively the equilibrium density of bulk ions and EPs, and $\langle\cdots\rangle$ indicates velocity space integration. δH_g is the nonadiabatic response of ions, which can be solved from the gyrokinetic equation

$$(\omega-\omega_d+i\omega_{tr}\partial_\theta)\delta H_g = -(e/m)(\partial F_{0s}/\partial E)J_0(k_\perp\rho_L)\omega\delta\phi, \quad (8)$$

where, $\omega_{tr}=v_\parallel/(qR_0)$ is the transit frequency, $\omega_d=\hat{\omega}_d\sin\theta=-k_r(v_\perp^2+2v_\parallel^2)/(2\Omega R_0)\sin\theta$ is the magnetic drift frequency associated with the geodesic curvature, $\Omega=eB/mc$ is the gyrofrequency and $J_0(k_r\rho_L)$ is the Bessel function accounting for the FLR effects.

To maximize the transit resonance drive, we adopt the optimal ordering for the GAM frequency $\omega\sim\omega_{tr,h}$ and, hence, $q^2\sim T_h/T_c$. Meanwhile, for the consistent treatment of small but finite EP magnetic drift orbit widths, we assume $k_r\rho_{d,h}\ll 1$, in which, $\rho_{d,h}$ is the drift orbit width of EPs. Adopting δ as the smallness expansion parameter in our asymptotic analysis, we then take $1/q\sim O(\delta^{1/2})$ and $k_r\rho_{d,h}\sim O(\delta^{1/2})$. We also assume $n_h/n_c\sim O(\delta)$, thus, the contribution of EPs and bulk ions will enter the dispersion relation at the same order.

In order to make further analytic progress, we take a single pitch-angle slowing-down equilibrium distribution for the EPs; i.e., $F_{0h}=c_0(r)\delta(\Lambda-\Lambda_0)H_E$, where $\delta(x)$ is the Dirac delta function, $\Lambda\equiv\mu/E$ is the pitch angle, $\mu=v_\perp^2/(2B)$ is the magnetic moment, $c_0(r)=\sqrt{2(1-\Lambda_0 B)}n_b(r)/(4\pi B\ln(E_b/E_c))$, $n_b(r)$ is the density of the EPs beam, E_b and E_c are, respectively, the EP birth and critical energies, and $H_E=\Theta(1-E/E_b)/(E^{3/2}+E_c^{3/2})$, with $\Theta(1-E/E_b)$ being the Heaviside step function.

Solving equation (7) order by order, we obtain the local dispersion relation of EGAM from the surface averaged quasi-neutrality condition as

$$\mathscr{E}_{EGAM} = -1+\frac{\omega_G^2}{\omega^2}+N_b\left[C\ln\left(1-\frac{\omega_{tr,b}^2}{\omega^2}\right)+\frac{\Lambda_0B(2-\Lambda_0B)^2}{(1-\Lambda_0B)^{5/2}}\frac{\omega_{tr,b}^2/\omega^2}{1-\omega_{tr,b}^2/\omega^2}\right]=0\,; \quad (9)$$

where $\omega_{tr,b} = \sqrt{2E_b(1-\Lambda_0 B)}/(qR_0)$, $N_b = \sqrt{1-\Lambda_0 B}q^2 n_b/(4\ln{(E_b/E_c)}n_c)$ and $C = (2-\Lambda_0 B)(-2+5\Lambda_0 B)/(2(1-\Lambda_0 B)^{5/2})$.

Note that, in equation (9), the first term of EP response (the logarithmic term) is the resonance drive at the EP transit frequency while the second term (the second term in the square bracket) determines how much the real frequency of local EGAM is lower than the local GAM continuum frequency and will not contribute to the drive of EGAM. From equation (9), the EGAM is locally unstable only when $C > 0$, which gives the local necessary instability condition

$$\Lambda_0 B > 2/5. \tag{10}$$

We note here that the local EGAM dispersion relation, equation (9), has two unstable branches. When $\omega_G < \omega_{tr,b}$, there is the GAM branch with a real frequency very close to the local GAM continuum frequency; when $\omega_G > \omega_{tr,b}$, we have the beam branch, with its real frequency very close to $\omega_{tr,b}$. In this work, we will focus on the beam branch; an EP-mode indeed.

Next we will look at the effect of GAM continuous spectrum on the excitation of EGAM. We will look at two different cases. First, we consider the EGAM driven by EP beam sharply localized away from the GAM resonance layer [16]; thus, we are looking at the case where the EGAM coupling to the GAM continuous spectrum is formally exponentially small [14]. Second, we will look at the case when the scale length of EP beam is comparable with (larger than) that of GAM continuous spectrum and thus, the limit where EGAM is strongly coupled to GAM continuous spectrum [3, 15]. We expect, that the two limits will yield different contributions from GAM continuous spectrum, since, in the second case, there may be singularity in EGAM mode structure that is formally eliminated in the first case by the small drift orbit assumption [14].

2.2. EGAM driven by localized energetic particles

Considering the exponentially small EGAM coupling to GAM continuous spectrum, we assume an EP beam localized about $r = r_b$, where the local GAM continuum frequency is larger than the beam transit frequency, and the beam characteristic spatial scale length L_b is much smaller than that of GAM continuous spectrum, $L_g \equiv |\omega_G^2(r)/(\partial\omega_G^2(r)/\partial r)|$; such that the beam is localized away from the GAM resonance layer r_c, where $\omega_G(r_c) = \omega \approx \omega_{tr,b}$. In our orderings, we have that FOW and FLR of EPs enter at $O(\delta^4)$; while FOW and FLR of bulk ions enter at $O(\delta^6)$ [14]. Thus, FOW/FLR effects are dominated by EPs inside the localization domain of the beam, while FOW/FLR effects of bulk ions take over away from the beam. In the following, we will solve the mode equations in both the inner ($|r - r_b| \lesssim L_b$) and outer region ($|r - r_b| \gg L_b$) up to the relevant orders of the asymptotic expansions, in order to take into account leading order FOW/FLR effects needed to solve the mode structures and frequencies of EGAM.

2.2.1. Global EGAM wave equation: localized solution

In the inner region, EGAM is bounded by the beam localization [16], and EP FLR/FOW effects dominate. Keeping up to $O(\delta^4)$ terms in equation (7), assuming vanishing current to the tokamak wall for we are searching a localized solution, and letting $\delta E = -\partial_r\overline{\delta\phi}$, we obtain the following "localized" GAM/EGAM eigenmode equation:

$$\left\{\partial_r\left(-\rho_{d,b}^2 N_b(r)H/2\right)\partial_r + \mathscr{E}_{EGAM} + F\right\}\delta E = 0. \tag{11}$$

where $\rho_{d,b} \equiv q\sqrt{2E_b}/\Omega_i$, H is an $O(1)$ function of $\Lambda_0 B_0$ describing EP FOW effects, and $F = F_1 + F_2$, denoting the local frequency shift. The expressions of F_1, F_2 and H are given in [14].

The characteristic scale of the mode is $\Delta \approx \sqrt{\rho_{d,b} L_b} \ll L_b \ll L_g$, so we can expand $N_b(r) \cong N_b(r_b)(1-(r-r_b)^2/L_b^2)$. We can further ignore the coupling to the GAM continuous spectrum, i.e., $\omega_G \approx \omega_G(r_b)$, and introduce $r - r_b = \xi z$; the mode equation becomes then

$$\left[\partial_z^2 - 2\xi^2(\mathscr{E}_{EGAM}(r_b) + F(r_b))/(\rho_{d,b}^2 N_b(r_b)H) - z^2\right]\delta E = 0, \tag{12}$$

where $\xi^4 = \rho_{d,b}^2 N_b(r_b) H L_b^2/(2(-1+\omega_G^2(r_b)/\omega^2+F_1))$ and causality constraint must be applied in determining ξ^2. Equation (12) is the Weber equation and its eigenvalue condition gives the following "localized" EGAM dispersion relation

$$-2\xi^2(\mathscr{E}_{EGAM}(r_b) + F(r_b))/(\rho_{d,b}^2 N_b(r_b)H) = 2l + 1, \quad l = 0, 1, 2, \cdots. \tag{13}$$

Here, l is the radial eigenmode number. Meanwhile, the radial electric field is $\delta E \propto H_l((r - r_b)/\xi)\exp(-(r - r_b)^2/(2\xi^2))$, with H_l being Hermite polynomials. Different eigenstates have close real frequencies, while their growth rates decrease with the eigenmode number due to the sharp localization of the EPs.

2.2.2. Global EGAM wave equation: nonlocal solution and coupling to the GAM continuous spectrum

In the outer region where EPs fade away, contributions from EPs are negligible and the dispersiveness is dominated by thermal ions. For this reason, we need to keep terms up to $O(k_r^4 \rho_{L,t}^4)$ to include the nonlocal physics via thermal ion FLR/FOW effects. The corresponding eigenmode equation of EGAM/GAM is then given by

$$[\partial_r^2 + 2\left(1 - \omega_G^2(r)/\omega^2 - F_1 - F_3\right)/(\rho_{L,t}^2 G)]\delta E = 0. \tag{14}$$

with G denoting the FOW/FLR effects of bulk ions, F_3 the higher order terms to be added for consistency to the expression of F in equation (11) and the expressions of G and F_3, are given in [14].

This equation describes KGAM propagating in the $r > r_c$ region with the mode structure near $r = r_c$ given by Airy Functions. The scale length of KGAM is readily shown to be given by $L \approx \rho_{L,t}^{2/3} L_g^{1/3}$. Thus, the characteristic EGAM scale-length varies across the radial domain, unless we introduce the auxiliary ordering $L_b/L_g \approx \delta^{5/2}$ that would yield constant $k_r^2 \rho_{L,t}^2 \approx \delta^3$ across the entire domain.

In the following, we shall apply WKB analysis [17] and match solutions of equations (12) and (14) in the intermediate region and derive the global eigenmode dispersion relation of the nonlocal EGAM driven by sharply localized EPs. For the complete wave propagation and absorption physics in the region where the EP beam decays away, we need to keep systematically terms up to $O(k_r^4 \rho_{L,t}^4)$ to account for FOW/FLR effects of bulk ions and local wave frequency shift. Taking into account the above considerations and noting that the typical scale length of δE is $L \approx \sqrt{\rho_{d,b} L_b} \ll L_b, L_g$, equations (11) and (14) can then be combined into the following eigenmode equation

$$\left[\partial_r^2 + Q(r)\right]\delta E = 0, \tag{15}$$

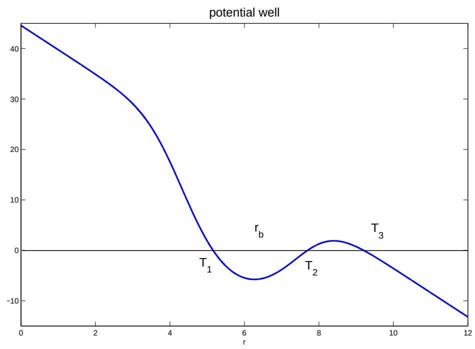


FIG. 1. : Potential well: $-Q$ vs r/L_b.

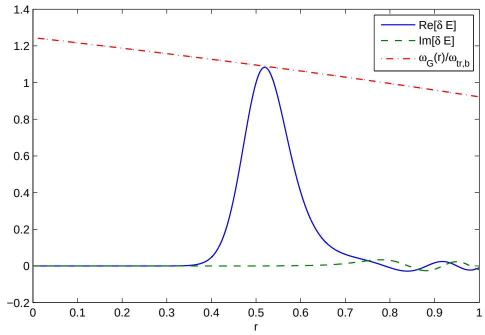


FIG. 2. : Sharply distributed EP: global mode structure.

where $Q(r) = (2(\mathscr{E}_{EGAM} + F + F_3))/(-\rho^2_{d,b}N_b(r)H - \rho^2_{L,t}G)$. Equation (15) has the WKB solution

$$\delta E = (A_1 \exp(i\int \sqrt{Q(r)}dr) + B_1 \exp(-i\int \sqrt{Q(r)}dr))/Q^{1/4}(r). \tag{16}$$

The function $Q(r)$ has three regular turning points (zeros) T_1, T_2 and T_3. T_1 and T_2 are the turning points pair due to the localization effect of EPs and form a bound state as we have discussed for equation (12). T_3 is the turning point for mode conversion to KGAM, beyond which the mode propagates outward, as noted in the discussion following equation (14). The structure of the potential well ($-Q(r)$) as well as the positions of the turning points are qualitatively illustrated in Fig. *1.* .

The corresponding WKB dispersion relation of the eigenmode described by equation (15) can then be straightforwardly derived via asymptotic matching of the WKB solutions, equation (16), across the turning points and is given by

$$e^{2iW_1} = (e^{2iW_2} + 1)/(e^{2iW_2} - 1); \tag{17}$$

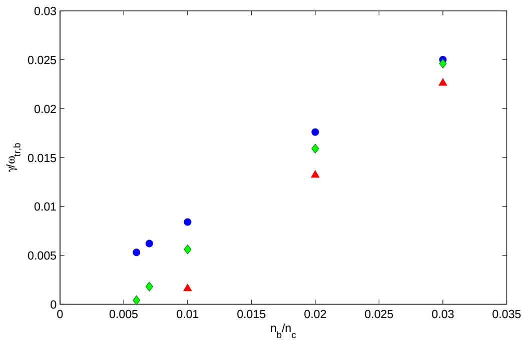
where $W_1 = \int_{T_1}^{T_2} \sqrt{Q(r)}dr$ and $W_2 = \int_{T_2}^{T_3} \sqrt{Q(r)}dr$. The tunneling coefficient e^{2iW_2} is formally exponentially small, and the WKB eigenmode dispersion relation of EGAM becomes approximately

$$W_1 = (l + 1/2)\pi - ie^{2iW_2}, \qquad \text{l=0,1,2,}\cdots. \tag{18}$$

Equation (18) is, of course, the well-known Bohr-Sommerfeld quantization condition including the tunneling coupling to outgoing KGAM. Near marginal stability,

$$\gamma = -W_{1i}/(\partial W_{1r}/\partial\omega_r) - e^{2iW_2}/(\partial W_{1r}/\partial\omega_r); \tag{19}$$

expressing the mode excitation when the EP resonant drive exceeds the tunneling-convective damping, and ω_r is solved from $W_{1r}(\omega_r) = 0$, where W_{1r} and W_{1i} are, respectively, the real and imaginary parts of W_1 [14]. The mode structure of EGAM from numerical solution of equation (15) (cf. Fig. *2.*) shows mode trapping around the radially position of strongest EP drive and that there is an exponentially small tunneling of the electric field to an outward propagating KGAM at the resonant layer with the GAM continuous spectrum, which is very similar to the DIII-D observations by Nazikian et al [18]. Meanwhile, the EGAM threshold



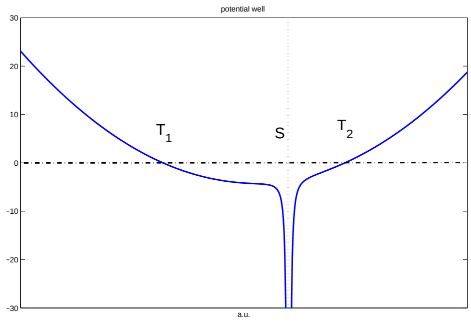


FIG. 3. : Sharply distributed EP: EGAM excitation threshold increases with decreasing L_G.

FIG. 4. : Broadly distributed EP: potential well.

condition, due to non-local coupling to KGAM, is expected to increase for decreasing L_g, and is shown numerically in Fig. *3*. with $L_3 < L_2 < L_1 = \infty$.

2.3. EGAM driven by broadly distributed energetic particles source

We consider EGAM excited by a radially broadly distributed EP source (i.e., $L_b \geq L_g$), such that the resonant point with the GAM continuous spectrum falls within the EP localization region. The EGAM can then be expected to couple strongly with the GAM continuum. In this case, the small orbit assumption employed in the localized energetic particle source limit will no longer be valid across the whole domain. With $k_r\rho_{d,h} \gtrsim 1$, the eigenmode equation of EGAM becomes in general an integro-differential equation. Here, we will examine its $k_r\rho_{d,h} \ll 1$ and $k_r\rho_{d,h} \gg 1$ limits, employ the Pade's approximation and derive a corresponding differential eigenmode equation valid asymptotically in both limits.

For $k_r\rho_{d,h} \ll 1$ and a broadly distributed EP source, the EP FOW/FLR effects dominate across the whole radially domain; as we showed in the localized EP source case. So we can ignore the FLR/FOW effects of bulk thermal ions and obtain:

$$\mathscr{E}_c = -1 + \omega_G^2/\omega^2 + F_1, \tag{20}$$
$$\mathscr{E}_h = \mathscr{E}_{EGAM} + F_2 + k_r^2\rho_{d,b}^2 N_b H/2 - \mathscr{E}_c \equiv \mathscr{E}_{h0}(1 + \mathscr{A} k_r^2\rho_{d,b}^2); \tag{21}$$

where, $\mathscr{E}_{h0} \equiv \mathscr{E}_{EGAM} + 1 - \omega_G^2/\omega^2 + F_2$ and $\mathscr{A} \equiv N_b H/(2\mathscr{E}_{h0})$. Meanwhile, at the resonant point of GAM continuous spectrum, $|k_r\rho_{d,b}| \to \infty$, the response of the EPs to the radial electric field becomes adiabatic; i.e.,

$$\delta n_E = -(e^2/m)\left\langle \partial F_{0E}/\partial E \right\rangle \overline{\delta\phi}. \tag{22}$$

However, we may still employ the small drift orbit approximation for thermal ions; since the FLR/FOW of thermal ions will remove the singularity at the resonant point. Combining equations (20), (21) and (22), we find the following Pade-approximation WKB dispersion relation of EGAM [15]:

$$k_r^2\mathscr{E}_c + k_r^2\mathscr{E}_{h0}/(1 - \mathscr{A} k_r^2\rho_{d,b}^2) = 0, \tag{23}$$

and the corresponding eigenmode equation becomes:

$$\partial_r[\mathscr{A}\rho_{d,b}^2\partial_r(-1 + \omega_G^2/\omega^2 + F_1)\partial_r + \mathscr{E}_E + F_1 + F_2]\partial_r\overline{\delta\phi} = 0. \tag{24}$$

Equation (24) recovers properly the GAM/EGAM eigenmode equation in the $k_r \rho_{d,b} \ll 1$ limit. At the resonant point of GAM continuous spectrum, however, it only qualitatively describes the response of EPs.

As noted for equation (4), the solution of an equation in the form of (24) is fully determined only when the non-homogeneous problem is solved in the presence of a source term. Here, we will ignore the coupling to the boundary, and try to find an internally localized solution. Written in the standard WKB form [like what we did in equation 15], equation (24) contains two regular turning points at $(\mathscr{E}_E + F_1 + F_2)/(\mathscr{A} * \rho^2_{d,b}\mathscr{E}_c) + (\partial_r \mathscr{E}_c)^2/(4\mathscr{E}_c^2) \simeq 0$ and a second order singular turning point at $\mathscr{E}_c = 0$; so equation (24) describes a localized solution trapped by a potential well, with the positions of the zeros and poles given in Fig. *4.* , in which T_1 and T_2 are, the two regular turning points and S is the second order singular turning point. At the resonant point of GAM continuous spectrum with $\mathscr{E}_c(S) = 0$, $\mathscr{E}_c(S) \simeq \partial_r \mathscr{E}_c * (r - S)$, equation (24) will yield the solution $\delta\phi \propto \ln(\sqrt{r - S})$, and $k_r \to \infty$ at S. The singularity in the mode structure, thus, suggests the continuum damping of EGAM at S due to the coupling to the GAM continuous spectrum. If we further include the FLR of bulk thermal ions in equation (24), then it will also have the physics of removing the singularity by bulk ion FLR and mode conversion to KGAM.

Acknowledgments

This work is supported by the Euratom Communities under the contract of Association between EURATOM/ENEA, USDoE GRANTS and the National Basic Research Program of China.

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