Field-Reversed Configuration (FRC) Equilibrium and Stability

D. C. Barnes,[a] E. V. Belova,[b] R. C. Davidson,[b] A. Ishida,[c] H. Ji,[b] R. Milroy,[d] P. Parks,[e] D. Ryutov,[f] M. Schaffer,[e] L. Steinhauer,[d] M. Yamada[b]

e-mail contact of first author: dbarnes@lanl.gov

[a] Los Alamos National Laboratory, US,
[b] Princeton Plasma Physics Laboratory, US,
[c] Niigata University, Japan,
[d] University of Washington, US,
[e] General Atomics, US,
[f] Lawrence Livermore National Laboratory

Abstract: New results elucidating FRC equilibrium and stability are reported. Both prolate and oblate configurations are considered. For prolate FRC's, macrostability is calculated incuding effects beyond static MHD. Hall MHD gives a stability condition similar to an empirically-based limit. At finite Ti, gyroviscosity and resonant ion effects are included. Full 3D kinetic-ion calculations show resonant ion destabilization and nonlinear saturation *via* wave-particle trapping. Advances in two-fluid equilibrium with flow are reported. A Fourier-Beltrami state with two modes has approximate equipartition between flow and magnetic energy, and may explain experimentally observed features. Effects of mirror-trapped electrons near the field line ends is considered in both collisional and collisionless regimes, and found to reduce growth rates significantly for elongation of 5 or greater. Oblate FRC's are shown to be interchange stable for moderate separatrix pressure and to be accessible to formation by spheromak merging. The interchange stability condition for doublet shaped FRC's is also reported.

1. Introduction and Motivation

The FRC (Field-Reversed Configuration) [1] is a poloidal-only **B,** high-β compact torus. Typical FRC's are very elongated, and macroscopic stability is inferred experimentally. For usual parameters, macroscopic modes are not observed in static situations. Even when large deformations are produced in a dynamic situation (*e.g.* during translation) the axisymmetric state is restored. It seems that plasma instabilities, if present at all, are associated with observed enhanced transport. Further, some observations [2] suggest that the FRC state is a preferred state realized by relaxation from a range of initial states.

A zero gyroradius FRC is subject to virulent MHD instabilities, with growth on the axial transit time. If these modes were operative, confinement would scale with $\tau_A = z_s/v_A$, where z_s is the ½ length of the (separatrix of the) configuration and v_A is the Alfven speed computed with peak B_m and peak n_m. In contrast, it is observed that $\tau_E \sim C\, R^2/\rho_i$ [3], where $R = r_s/\sqrt{2}$ is the major radius (r_s is the maximum separatrix radius), and ρ_i is the ion gyroradius, again using B_m, greatly exceeding the τ_A scaling. Thus, MHD modes are stabilized in the experimental regime, apparently by sufficient kinetic effects. A key parameter from observations is the normalized Hall or gyroradius parameter $G = S^*/E$, where $S^* = r_s/d_i$ with d_i the collisionless ion skin depth computed with n_m, and $E = z_s/r_s$ the elongation. Radial pressure balance shows that S^* is also proportional to the number of ρ_i between R and r_s . Well behaved FRC experiments usually operate with $G < 3.5$ [4], providing an empirical macrostability limit, and instabilities may and occasionally have been observed when this condition is violated.

These empirical laws almost uniquely determine the reactor extrapolation in terms of a single system parameter, the peak pressure P. Earlier work [5] assumed compression by a converging conducting wall with kinetic energy (KE) comparable to the final plasma thermal energy. In a steady-state system ($P \sim 1000$ atmospheres), these results still apply with thermal energy ½ of KE. Results show that a steady state FRC requires a quite large and energetic system, over 100*m* of length and 100 *GJ* of energy. While such a regime is not of great economic interest, if a different technology can increase *P* to the megabar regime, an attractively small system is indicated. The proposed MTF (Magnetized Target Fusion) approach operates at these extreme pulsed pressures. In an intermediate regime ($P \sim 3\text{x}10^4$ atmospheres), there exists the attractive option of rotating liquid metal walls in a system like LINUS [6]. Because confinement is only dependent on radial dimensions, while energy increases linearly with length (or *E*), the advantage of increasing *G* is evident. Thus, it becomes quite important to precisely quantify physical limits of *G*.

The empirical scaling suggests three approaches toward FRC reactor conditions:

- Achieve a scientific understanding of the empirical laws and design for the indicated high P (in a pulsed system).
- Improve both stability and confinement in a steady-state elongated system.
- Find an alternative geometry or kinetic regime with improved stability and confinement (such as an oblate configuration).

All of these approaches are being pursued in the world FRC program. Because of the importance and ubiquity of the scaling observations, a key science element is linking these relations with a theoretical understanding. Here, we describe theoretical work related to the question of FRC equilibrium and (kinetic) stability. We do not consider confinement further here. Because of the prevalence of prolate experimental configurations, our main focus is placed there.

2. Prolate FRC Equilibrium

Rather than find FRC equilibria from a model $P(\psi)$ by adjusting a few parameters, it has emerged recently that capturing some FRC features requires an alternative approach. A natural approach is to order the *z* variation small, order $\varepsilon = 1/E$ compared to the transverse variations, thus requiring *uniformly* slow z variation._ In this case, (essentially) only one pressure profile is allowed by equilibrium. The profile depends on the vacuum boundary conditions and has been found for the experimentally relevant case of a uniform cylindrical flux-conserving wall [7]. In this case, $P' \approx c_0 + c_1\psi$. Further, the shape is very sensitive to the pressure profile, determined by order ε^2 corrections. Solutions have a similarity shape which uniformly elongates as ε decreases, and the MHD drive also decreases uniformly, leading to less unstable configurations. Finally, there are reasons to believe that these profiles are achieved experimentally. Besides providing good agreement with observed profiles and having the least MHD instability drive, marginal stability to tearing holds for these profiles. Thus, a different profile might relax toward the uniformly long state by relatively benign tearing activity, rearranging the flux to achieve the marginal profile. Additionally, observed anomalous FRC resistivity, might also be partly associated with tearing driven by weak deviations from this marginal state in addition to microinstabilities. These conjectures are the subject of continuing theoretical work.

Additional FRC features may be explained by a two-fluid equilibrium theory. Recent work [8] has established the formalism for finding stationary energy states (a pre-requisite for stability) with appropriate conserved quantities. These quantities are the electron and the ion helicities, independently, defined as:

$$K_e = \frac{1}{2\mu_0}\int d\mathbf{r}\mathbf{A}\bullet\nabla\times\mathbf{A}$$
$$K_i = \frac{1}{2\mu_0}\int d\mathbf{r}\left(\mathbf{A}+\frac{M}{e}\mathbf{u}\right)\bullet\nabla\times\left(\mathbf{A}+\frac{M}{e}\mathbf{u}\right) \tag{1}$$

where **A** is vector potential and **u** flow velocity. The variational solution has two arbitrary constants and no arbitrary functions. An analytic solution is obtained for uniform density. The plasma "condenses" to either single-mode or double-mode states. In compact toroids these are analyzed by a Fourier-Beltrami expansion [9]. Analytic solutions are found. Double-mode states always have lower ordered energy (higher disordered energy) than single-mode states. Double-mode states appear in two branches, one with higher energy and one with lower. The latter is energetically favorable, but has exceedingly fine structure of the flow. As such it is an unlikely end state of a relaxation. The higher-energy branch states have significant toroidal and poloidal flows and an approximate equipartition between flow and magnetic energy. Ongoing work is developing details of these solutions for comparison to experimental observations.

3. Prolate FRC Stability

Recent FRC stability calculations have focused on issues related to the empirical stability limit given in the Introduction. Numerical calculations using the HYM [10,11] and FLEX [12] simulations codes have been performed with a range of equilibria and for varying G. A semi-analytic theory has also been obtained by applying the small ε expansion to the stability problem [13]. Results of these two studies have been compared. The general stability picture shows unstable MHD modes (for G large) with axial polarization (ξ_z dominant) at all modest toroidal mode numbers $n = 1,\ldots,E$. These modes are the small ε branch of the internal tilt, which is known to be unstable for prolate compact toruses. In the relevant regime ($G \sim 1$ – 10) Hall and FLR effects strongly affect these modes. The simplest theories [13,14] show reactive stabilization for $G < G_{crit}$, where the limit depends on model details, and is in semi-quantitative agreement with the empirical picture.

Several complications to this picture emerge, however. First, finite temperature (ion and/or electron) effects interfere with reactive stabilization of the fundamental modes by introducing other branches which become unstable as G is decreased. Second, the reactively stabilized modes form a slow-mode/fast-mode pair, with the slow-mode a negative energy wave. Individual ions have frequencies which resonate with this slow-mode, and resulting Landau damping causes destabilization. Much of our work has thus focused on the question of whether linearly or nonlinearly stable FRC states exist, and how they avoid or obscure effects of these residual instabilities.

Recent HYM simulations have studied the uniformly long equilibria obtained in Ref. 6, and compared them to those obtained with a earlier different $P(\psi)$. These results confirm MHD predictions of growth rate and mode structure. Namely, elongation scaling holds only for the special profile with $\gamma \sim \varepsilon$ and a similarity of mode structure. Other profiles give equilibria which become "racetrack" with all variation concentrated at the ends, with modes also concentrating there and γ saturating (not decreasing) beyond some modest value of E.

Including Hall effects, γ is found to decrease only moderately as G is decreased. In contrast to the simple reactive theory, stabilization is not observed. Instead, a different structure appears as the most unstable mode with γ about ½ the MHD value (Fig. 1). In the kinetic regime (small G), the mode rotates oppositely to the current, that is in the electron diamagnetic direction, rather than in the ion direction as predicted by the reactive theories and observed at large G.

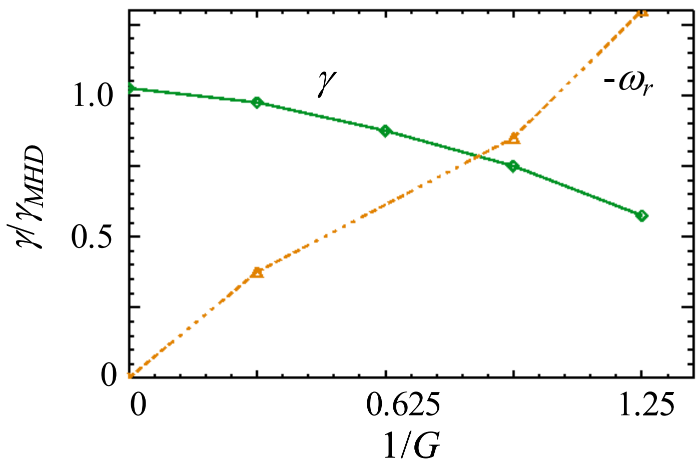


Figure 1: Growth rate and negative real frequency from HMHD calculations

To understand these results, the previous small ε stability theory has been extended to include all effects of finite temperature [15]. There are several terms [14,16] previously neglected, so that earlier results apply only in case both fluids are isothermal ($\Gamma_e = 1 = \Gamma_i$).

In the general case, remarkably, the analysis proceeds in a similar manner, and the previous structure is recovered. Namely, following Ishida, *et al.* a symmetric quadratic form is obtained. Only derivatives along the field direction survive to leading order, and it is possible to eliminate the transverse displacement components, which appear only through $D = \nabla\bullet\xi$, by minimizing with respect to D. The result is of the same form as obtained earlier

$$\frac{1}{2}\int d\mathbf{r}\; n_e(\omega-\omega^*)^2\left[1-l\,\theta_i\left(1+\frac{P}{B^2}\right)\right]|\xi|^2 = \frac{1}{2}\int d\mathbf{r}\left(A\left|\mathbf{B}\bullet\nabla\xi\right|^2 + F|\xi|^2\right) \qquad (2)$$

where ω is the mode frequency, $\omega^* = -n\,P'/Gn_e$ is the diagmagnetic frequency, ξ is the axial component of the electron displacement, n_e the number density, $\theta_{i,e} = T_{i,e}/(T_i+T_e)$, and l is a switch for including/excluding FLR effects. All quantities in (2) are normalized as in Ref. 12, except that r_s has been used as the unit of length and B_m the unit of B to simplify some coefficients. The inertia A and restoring force F are given by

$$\begin{aligned}
A &= \frac{(1-\bar\omega)\Gamma + \bar\omega q c}{\Gamma + \left(1+\bar\omega+2\bar\theta\bar\omega\right)c + \bar\omega\left(1+\bar\theta\bar\omega\right)c^2}\\
F &= A\,P_e'' - 2\,P_e'^2\left(1+\bar\theta\bar\omega\right)\\
&\quad\times\frac{\Gamma\left(1-\Gamma\bar\omega+\bar\theta\bar\omega\right)+2\Gamma\bar\omega(1-\bar\omega)c+\bar\omega^2 q\,c^2}{P\left[\Gamma+\left(1+\bar\omega+2\bar\theta\bar\omega\right)c+\bar\omega\left(1+\bar\theta\bar\omega\right)c^2\right]^2}\\
q &= \Gamma - \bar\omega\left[\left(1+\bar\theta\right)^2 - \Gamma\bar\theta\right]
\end{aligned} \qquad (3)$$

where $\bar\omega = \omega^*/\omega$, $\Gamma = \theta_e\Gamma_e + \theta_i\Gamma_i$, $\bar\theta = (\Gamma_i - 1)\theta_i$, $c = B_z^2/P$, and P_e the external pressure as in Ref. 12, with the indicated derivatives as d/dz. Notice that when $\Gamma_e = 1 = \Gamma_i$, (2) reduces to the previous form except for an overall factor of $1-\overline{\varpi}$. Since ω^* is a flux function, the resulting field line o.d.e. is the same as previously solved.

Eigenvalues and eigenfunctions are obtained by integrating the field-line o.d.e. resulting from (2) using a shooting method. It is found that, as in the simpler theories, a threshold $G_{crit} \sim 0.8$ exists and that this value is insensitive to the thermodynamic parameters (θ_i, Γ_i, Γ_e). Thus, the modes found numerically either should disappear at very large elongation, where the present analysis may be applied, or they are not perturbations of the axial displacement branch, as assumed in this analysis. Including FLR effects in the asymptotic theory [$l = 1$ in (2)] greatly expands the stable region. A threshold of $G_{crit} \sim 8$ is found. Since resonant ion effects destabilize these reactively stabilized modes, it is expected that the actual threshold would be lower, perhaps in reasonable agreement with the experimental value of 3.5.

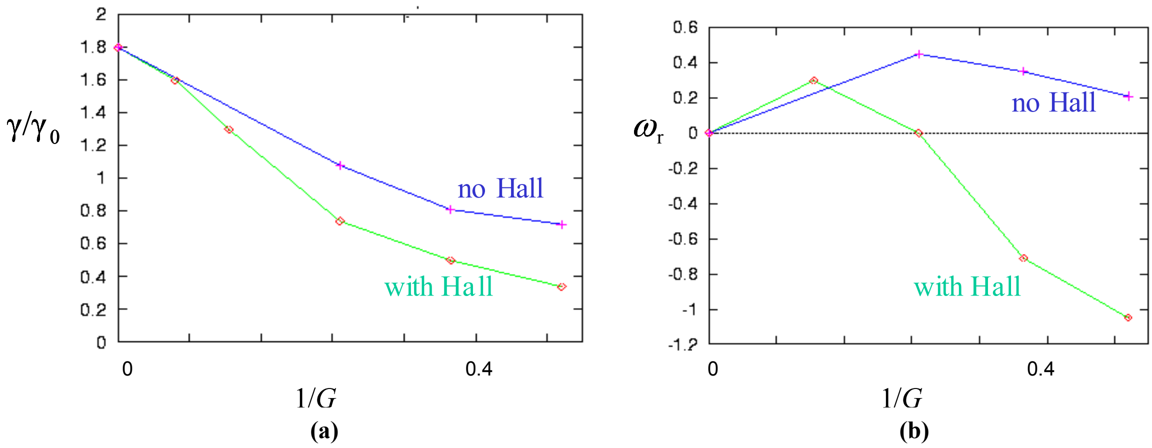
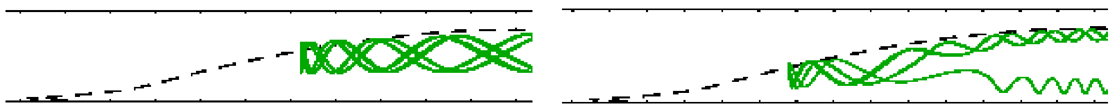


Figure 2: HYM calculations with and without Hall terms.: a) growth rates; b) real frequencies.

The effect of FLR is also seen from particle ion HYM calculations which include or exclude the Hall terms. Figure 2 shows that the FLR effects are mostly responsible for stabilization, while the Hall effects are very important in determining the mode rotation and structure.

Finally, HYM has extensively addressed the issue of resonant ion effects. Recently several encouraging results have been obtained. Diagnostics have been developed which distinguish stochastic ion orbits from integrable ones. The latter are those most likely to be associated with resonant effects, although weakly stochastic orbits (with slowly varying orbit frequencies) still contribute significantly to resonant effects. Results have been obtained which indicate nonlinear stabilization. Under appropriate initial conditions, resonant ions are accelerated and decelerated to flatten f and a long-lived FRC is produced [17].

The effects of resonant ions are also found to be sensitive to the details of the equilibrium. There is a general argument which indicates that uniform elongation will cause all small orbit ions to be stochastic, with correlation time comparable to transit time. Slow z variation implies weak curvature, along the length of each field line. Of course, the integral of the curvature must give a turning angle of 2π over the entire length. This means that the curvature is almost entirely concentrated at the ends of each line. Since the field is also weak there, $G \sim 1$ implies that the local gyroradius is larger than the radius of curvature, leading to a loss of adiabatic invariants (e.g. μ) at each transit of the line. In this way, all small orbit ions will have a magnetized motion along the lines, with a pseudo-random μ changing with each transit. This has already been reported by Glasser and Cohen [18] for the Solovev (Hill's vortex) equilibrium, which is a uniformly elongated case with different vacuum boundary conditions. The exception is large orbit ions which do not sample the end region by virtue of having pitch angle such that z velocity is small. Figure 3 shows ½ the poloidal plane of two ion orbits. Histories for these two orbits show that the orbit in (a) is integrable, while the one in (b) is strongly stochastic, consistent with the picture advanced previously.



(a) **(b)**

Figure 3: Comparison of integrable orbit (left hand) with stochastic orbit (right hand). One-half of each orbit is shown and reflection symmetry holds.

A comparison of long equilibria with *uniformly* slow z variation with a comparable case with racetrack ends (*i.e.* one using the special profile with one using a smooth, but different profile) further illustrates this point. In Fig. 4 a comparison of comparable E cases with and without the uniformly long profile shows that the particle distribution in frequencies is wider in racetrack configurations, indicating more resonant particles, larger growth rates, and increased difficulties in avoiding strong resonant ion effects. Much additional work on ion orbit distributions remains for the future. For example, internal islands, which are likely to appear near the marginally tearing stable low B region may lead to an additional decrease of integrable ions.

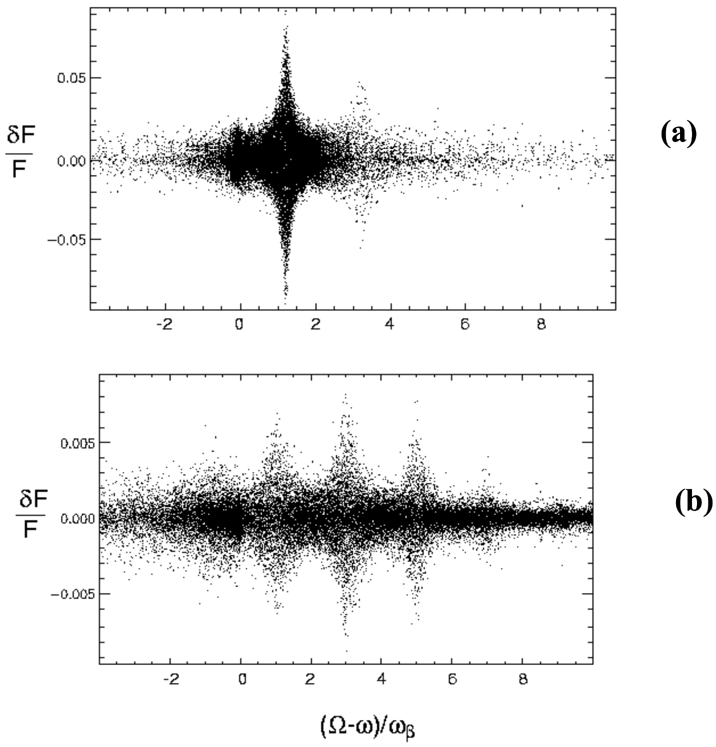


Figure 4: Ion response vs. frequencies.

All results to this point have treated the electrons as a fluid. Additional stabilizing effects may arise from individual electron motion [19]. Near the FRC ends the magnetic field is weak, so that almost all particles residing there are mirror-trapped (along the field lines). On the other hand the ∇B and curvature drifts near the FRC ends are fast (because of a weak magnetic field and large field line curvature). Therefore, the drift frequency Ω_d of trapped particles may exceed the instability growth rate γ and therefore make electron response different from the fluid (MHD) response. For the equilibria where the X-point structure is just the cusp structure, one has the following estimate for Ω_d in the vicinity of the

X-point $\Omega_d \approx (T_e / eB_m r_s^2)(\Psi / \delta\Psi)$, where $\delta\Psi$ is the flux between the separatrix and flux surface we are considering, and Ψ is the flux between the separatrix and the O-point. Assuming that the normalized MHD growth rate is order unity, we find that non-MHD electron response takes place at the surfaces where $\delta\Psi / \Psi < E / \bar{S}$. where E is the FRC elongation, $\bar{S} \equiv r_s / \rho_i *$, and $\rho_i * \equiv c\sqrt{2m_i T_e} / eB_m$ is the ion gyroradius evaluated for the electron temperature. Note that effects we are discussing are very different from FLR effects. This can most clearly be seen from considering a limiting case where T_i=0 and FLR effects are absent, whereas the effect of non-MHD electron response remains large.

For illustration purpose, we discuss in more detail the limiting case of T_i=0. In such a situation, the ion response is purely MHD (i.e., the **E**×**B** drift, plus polarization drift). The response of the majority of electrons is adiabatic, because their parallel transit time is much shorter than the instability e-folding time, and yields the standard MHD contribution to the eigenmode equation. However, the electrons trapped near the ends produce an absolutely different response, leading to an apparent build-up of a net charge in every flux-tube. As the quasineutrality constraint prohibits such effect, in reality this means that perturbations vanish near the ends of FRC. In terms of the MHD analysis, this forces imposing line-tying constraints near the ends therefore prohibiting any non-axisymmetric displacements. This adds a substantial stabilizing factor to tilt modes.

As equation $\delta\Psi / \Psi < E / \bar{S}$ shows, even at $\bar{S}$ as high as 20 or 30, of interest for some reactor designs, and a moderate elongation of 5 or 6, the outer shell of FRC (*δΨ/δΨ<0.25*) will experience substantial stabilization. How these effects interfere with the effects of stability theory developed in the earlier part of this paper (especially with the effects of ion non-adiabaticity at T_i~T_e) is a subject of future work.

4. Oblate FRC Equilibrium and Stability

Conventional FRCs are highly prolate, but, as already noted, for diffusive transport, τ_E increases with the square of the shortest distance between the hot core and the cool edge [20], and near-spherical geometry would yield the lowest power, most compact FRC fusion reactors. Therefore, we are investigating short, oblate and doublet FRCs. Oblate FRCs do not have a very weak minimum of $|B|$ on a magnetic surface, except for surfaces near the separatrix, so they have fewer stochastic ion orbits than elongated FRCs. Numerical studies [11] show stabilization of the current-driven tilt and shift modes in oblate FRCs by a close fitting conducting shell, even in the reactor-relevant, large–s, MHD fluid limit. In contrast, oblate FRCs without a nearby wall are not

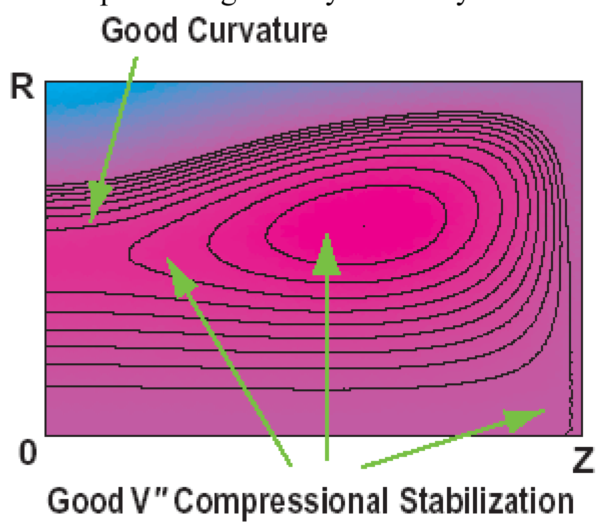


Figure 5: Computed doublet FRC equilibrium.

simultaneously stable to both modes [11]. Peaked pressure profiles are favorable to compressional stabilization of both n = 1 and high-n pressure-driven interchange modes [21]. Doublet FRCs might be especially interesting, because they have regions of favorable average magnetic curvature. Interchange stability calculations were obtained for a family of equilibria with a quadratic pressure profile (Fig. 5), $P = (P_0 - P_{sep})\psi^2 + P_{sep}$. The neighborhood of the O–point (ψ =1), is the region of least stability, and we derived an analytical expression for the marginal stability condition. It was found that doublets are more stable than standard FRCs; oblate doublets are more stable than elongated doublets; and more indentation improves stability. Therefore, doublets require less pressure at the separatrix (ψ = 0) than standard FRCs to achieve stability, and may be less vulnerable to transport induced by interchange turbulence.

5. Summary and Conclusions

Recent numerical and analytical work has advanced the study of FRC kinetic stability. A combination of uniformly slow z variation with proper internal structure and finite Te effects may show a regime of true linear stability and lead to a quantitative, verifiable picture of stability limits consistent with empirical scaling observations.

6. References

[1] TUSZEWSKI, M., *Nucl. Fusion* **28** (1988) 2033.
[2] ONO, Y., *et. al*., *Nucl. Fusion* **39** (1999) 2001.
[3] HOFFMAN, A. L., SLOUGH, J. T., *Nuclear Fusion* **33** (1993) 27.
[4] TUSZEWSKI, M, Private Comm.
[5] SIEMON, R. E., *et al.*, Proc. 12th International Toki Conference on Plasma Physics and Controlled Nuclear Fusion, Toki-city, Japan (December, 2001).
[6] ROBSON, A. E., in Megagauss Physics and Technology (TURCHI, P. J., Ed.), Plenum Press, New York (1980), 425.
[7] BARNES, D. C., *Phys. Plasmas* **8** (2001) 4856.
[8] STEINHAUER, L. C., *Phys. Plasmas* **9** (2002) 3767.
[9] YOSHIDA, Z. MAHAJAN S., *Phys. Rev. Lett.* **88** (2002) 095001/1.
[10] BELOVA, E. V., *Phys. Plasmas* **7** (2000) 4996.
[11] BELOVA, E. V., *Phys. Plasmas* **8** (2000) 1267.
[12] MILROY, R. D., *et al.*, *Phys. Fluids* **B1** (1989) 1225.
[13] BARNES, D. C., *Phys. Plasmas* **9** (2002) 560.
[14] ISHIDA, A., *et al., Phys. Fluids* **31** (1988) 3024.
[15] BARNES, D. C., in prepartion (2002).
[16] PEARLSTEIN, L.D., FREIDBERG, J. P., *Phys. Fluids* **21** (1978) 1218.
[17] BELOVA, E. V. *et al.*, Plasma Phys. and Controlled Nuclear Research, Proceedings of the 18th International Conference, Sorrento 2000 (IAEA, Vienna, 2000).
[18] GLASSER, A. H., COHEN, S. A., *Phys. Plasmas* **9** (2002) 2093.
[19] RYUTOV, D. D., "Stabilization of FRC by the charge-uncovering effect." Paper 1BO3, 2000 Sherwood Fusion Theory Meeting, Los Angeles, March 27-29, 2000.
[20] AUERBACH, S.P., CONDIT, W.C., *Nucl. Fusion* **21** (1981) 927.
[21] SPARKS, L., *et al.*, *Phys. Fluids* **23** (1980) 611.

Work supported by US Department of Energy, and by the Los Alamos National Laboratory.