High-frequency waves in a relativistic Plasma waveguide

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An analysis of the space-charge wave and a transverse magnetic modes in a completely filled cylindrical metallic waveguide is presented. Dispersion relation for four families of relativistic electromagnetic and electrostatic modes are derived. A nonlinear wave equation is derived in a form which includes the coupling of TM and TE modes due to the helical wiggler and finite axial magnetic field. Numerical solutions are presented to facilities the development of devices for generation of high-power electromagnetic radiation, charged particle acceleration, and other applications of plasma waveguides. The dependence of the cutoff frequencies, and dispersion curves of various modes on the γ (Lorenz factor) is studied in detail. Space-charge and cyclotron modes are found to be strongly dependent on γ . The dispersion curves of different modes with cyclotron frequency are illustrated for completely filled waveguide.

1. INTRODUCTION

The free-electron laser (FEL) makes use of the unstable interaction of a relativistic electron beam with a transverse wiggler magnetic field to generate coherent electromagnetic radiation. In experiments, megawatts to gigawatts of coherent radiation have been generated in the sub millimeter to millimeter wavelength range. Several FEL experiments operate at moderately high beam current and make use of a magnetic guide field $B_0\hat{\mathbf{e}}_z$ to steer the electron beam in the axial direction. A free electron laser of this type would employ a cylindrical metallic waveguide annular, partially or completely filled by the beam electron with the helical wiggler magnetic field and axial guide field then act in combination to affect the particle motion and determine the detailed properties of the FEL interaction. The operative mechanism would consist of parametric decay of an electrostatic pump wave (which is propagating in the beam frame) into a forward-scattered space charge wave and a backscattered transverse magnetic waveguide mode. Uddholm *et al.*[1] have investigated a study of nonrelativistic parametric excitation of a space charge wave and transverse waveguide modes in a beam filled waveguide with an axial magnetic field of arbitrary magnitude. A detailed nonrelativistic analysis of the high frequency eigenmodes of a coaxial waveguide is investigated by Maraghechi *et al.*[2]

This paper presents a study of the relativistic parametric excitation of space charge wave, cyclotron wave, EH and HE modes in a completely filled waveguide. This is a generalization of some part of the theory by Uddholm *et al.*

2. DERIVATION OF THE NONLINEAR WAVE EQUATION

We consider a relativistic electron beam which completely fills a cylindrical metallic waveguide of inner radius R , with a finite axial magnetic field $\mathbf{B}_0$. The beam passes through a helical wiggler. Our analysis will be carried out in the beam frame.

$$\frac{\partial n}{\partial t}+\nabla\cdot(n\mathbf{v})=0 \ , \qquad (1)$$

$$\frac{d}{dt}(\gamma m_0\mathbf{v})=-|e|[\mathbf{E}+\mathbf{v}\times\mathbf{B}], \qquad (2)$$

Where n is the particle number density, $\mathbf{v}$ the fluid velocity, e the electron charge, m the electron rest mass, $\mathbf{E}$ the electric field, and $\mathbf{B}$ the magnetic field. In the relativistic case $\gamma = \left[1 - v^2/c^2\right]^{-1/2}$ is not constant and

$$\frac{d\gamma}{dt} = \frac{-|e|}{m_0 c^2}(\mathbf{v}.\mathbf{E}), \tag{3}$$

so, equation (2) change to

$$\frac{d\mathbf{v}}{dt} = \frac{-|e|}{m_0\gamma}\left[\mathbf{E} + \mathbf{v}\times\mathbf{B} - \left(\frac{\mathbf{v}\,.\,\mathbf{E}}{c^2}\right)\mathbf{v}\right]. \tag{4}$$

The electromagnetic fields will satisfy Maxwell's equations. In the lab-frame, the unperturbed quantities n_0, $\mathbf{E}_0 = 0$ are assumed to be uniform and constant. Also, the magnetic field and velocity in the lab-frame are

$$\mathbf{B}_0 = B_0\hat{\mathbf{z}} + B_w\left[\hat{\mathbf{x}}\cos(k_w z) + \hat{\mathbf{y}}\sin(k_w z)\right] \tag{5}$$

$$\mathbf{v}_0 = v_{\parallel}\hat{\mathbf{z}} + \frac{\Omega_w v_{\parallel}}{\Omega_0 - \gamma k_w v_{\parallel}}\left[\hat{\mathbf{x}}\cos(k_w z) + \hat{\mathbf{y}}\sin(k_w z)\right]$$

where B_w is the amplitude of wiggler, k_w is the wave number of wiggler, and $\Omega_w = eB_w/m_0$, $\Omega_0 = eB_0/m_0$. With the perturbation of this unperturbed state denoted with a tilde

$$n = n_0 + \tilde{n}\ , \tag{6}$$

$$\mathbf{v} = \mathbf{v}_0 + \tilde{\mathbf{v}}\ , \tag{7}$$

$$\mathbf{E} = \tilde{\mathbf{E}}\ , \tag{8}$$

$$\mathbf{B} = \mathbf{B}_0 + \tilde{\mathbf{B}}\ . \tag{9}$$

Expansion of γ is

$$\gamma = \gamma_0 + \frac{\mathbf{v}_0\cdot\tilde{\mathbf{v}}}{c^2}\gamma_0^{\,3} + \frac{\tilde{\mathbf{v}}^2}{c^2}\gamma_0^{\,3} + \frac{3}{2}\frac{(\mathbf{v}_0\cdot\tilde{\mathbf{v}})(\mathbf{v}_0\cdot\tilde{\mathbf{v}})}{c^4}\gamma_0^{\,5} \tag{10}$$

where $\gamma_0 = \left[1 - v_0^2/c^2\right]^{-1/2}$, $\gamma_{\parallel} = \left[1 - v_{\parallel}^2/c^2\right]^{-1/2}$. In the beam frame the linear equation of motion will be

$$\gamma_b\frac{\partial\tilde{\mathbf{v}}}{\partial t} + \gamma_b(\mathbf{v}_0\cdot\nabla)\tilde{\mathbf{v}} + \gamma_b(\tilde{\mathbf{v}}\cdot\nabla)\mathbf{v}_0 + \frac{\gamma_b^{\,3}}{c^2}(\mathbf{v}_0\cdot\tilde{\mathbf{v}})\frac{\partial\mathbf{v}_0}{\partial t} + \frac{\gamma_b^{\,3}}{c^2}(\mathbf{v}_0\cdot\tilde{\mathbf{v}})(\mathbf{v}_0\cdot\nabla)\mathbf{v}_0 =$$

$$\frac{e}{m_0}\left[\tilde{\mathbf{E}} + \mathbf{v}_0\times\tilde{\mathbf{B}} + \tilde{\mathbf{v}}\times\mathbf{B}_0 - \frac{(\mathbf{v}_0\cdot\mathbf{E}_0)}{c^2}\tilde{\mathbf{v}} - \frac{(\mathbf{v}_0\cdot\tilde{\mathbf{E}})}{c^2}\mathbf{v}_0 - \frac{(\tilde{\mathbf{v}}\cdot\mathbf{E}_0)}{c^2}\mathbf{v}_0\right] \tag{11}$$

where $\gamma_b = \gamma_0/\gamma_{\parallel}$ is the Lorentz factor in the beam frame. For the space charge mode we shall assume that the electrostatic approximation is valid. This is correct if the wavenumber in the axial direction is sufficiently high $k >> \omega/c$. The non-linear wave equation for the electrostatic potential $\tilde{\varphi}$ $(\tilde{\mathbf{E}} = -\nabla\tilde{\varphi})$ is then obtained as

$$\left\{\gamma_b{}^3 \frac{\partial^2}{\partial t^2}\left[\left(\frac{\partial^2}{\partial t^2}+\frac{\omega_p{}^2}{\gamma_b}\right)\frac{\partial^2}{\partial z^2}+\left(\frac{\partial^2}{\partial t^2}+\frac{\omega_p{}^2}{\gamma_b{}^3}\right)\nabla_\perp^2\right]+\gamma_b(\Omega_0+\alpha)^2\left[\left(\frac{\partial^2}{\partial t^2}+\frac{\omega_p{}^2}{\gamma_b}\right)\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial t^2}\nabla_\perp^2\right]\right\}\tilde{\varphi}$$

$$=-\left\{\gamma_b{}^2\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial z}+(\Omega_0+\alpha)^2\frac{\partial}{\partial z}\right\}\times\left\{\frac{en_0}{\varepsilon_0}\left(\mathbf{\Omega}_w\times\tilde{\mathbf{v}}_\perp\right)_z+\frac{en_0}{\varepsilon_0}\gamma_b(\mathbf{v}_0\cdot\nabla)\tilde{\mathbf{v}}_z-\omega_p{}^2\left(\mathbf{v}_0\times\tilde{\mathbf{B}}\right)\cdot\mathbf{z}\right\}$$

$$-\gamma_b{}^2\frac{\partial^2}{\partial t^2}(\nabla_\perp\cdot\mathbf{A}_\perp)-\gamma_b(\Omega_0+\alpha)\frac{\partial}{\partial t}\nabla_\perp\cdot(\mathbf{A}_\perp\times\mathbf{z}) \qquad (12)$$

Where

$$\mathbf{A}_\perp=\frac{en_0}{\varepsilon_0}\left(\mathbf{\Omega}_w\times\tilde{\mathbf{v}}\right)_\perp+\frac{en_0}{\varepsilon_0}\gamma_b(\mathbf{v}_0\cdot\nabla)\tilde{\mathbf{v}}_\perp+\frac{en_0}{\varepsilon_0}\gamma_b(\tilde{\mathbf{v}}\cdot\nabla)\mathbf{v}_{0\perp}$$

$$+\frac{en_0}{\varepsilon_0}\frac{\gamma_b{}^3}{c^2}(\mathbf{v}_0\cdot\tilde{\mathbf{v}})(\mathbf{v}_0\cdot\nabla)\mathbf{v}_{0\perp}-\omega_p{}^2\left(\mathbf{v}_0\times\tilde{\mathbf{B}}\right)-\frac{e}{\varepsilon_0}\left[\gamma_b\frac{\partial}{\partial t}+[\Omega_0+\alpha](\mathbf{z}\times)\right]\tilde{n}\,\mathbf{v}_{0\perp}$$

3. Dispersion Relation and Results

Ignoring non-linear terms (right hand side) and assuming harmonic oscillations of the form

$$\tilde{\varphi}=\tilde{\varphi}(r)\exp[i(kz+l\theta-\omega t)] \qquad (13)$$

where r is the radial distance measured from the axis of the waveguide, k is the axial wavenumber, l is an integer, θ is the azimuthally angle and ω is the angular frequency, we can write (12) in the form

$$\nabla_\perp^2\tilde{\varphi}=-\mathrm{K}^2\tilde{\varphi} \qquad (14)$$

where

$$\mathrm{K}^2=\frac{\left(\gamma_b\omega^2-\omega_p{}^2\right)\left[(\Omega_0+\alpha)^2-\gamma_b{}^2\omega^2\right]}{\omega^2\left[\omega_p{}^2+\gamma_b(\Omega_0+\alpha)^2-\gamma_b{}^3\omega^2\right]}k^2 \qquad (15)$$

This equation is different from nonrelativistic case. Correction factor of cyclotron frequency α is

$$\alpha=\frac{\gamma_b{}^5-\gamma_b{}^3+\gamma_b{}^2-1}{\gamma_b{}^3}\omega_1+\frac{1-\gamma_b{}^2}{\gamma_b{}^2}\Omega_0 \qquad (16)$$

where $\omega_1=\gamma_\parallel k_w v_\parallel$. In the nonrelativistic case $\gamma_0=\gamma_\parallel \;\to\; \gamma_b=1$ and $\alpha=0$, so the relativistic dispersion relation and nonrelativistic dispersion relation are the same.

For the fully cylindrical metallic waveguide, the solutions to (14) are subject to the boundary condition that the potential must vanish at the metal-plasma interface. Thus

$$\tilde{\varphi}=\hat{\varphi}J_l(\mathrm{K}r)\exp[i(kz+l\theta-\omega t)] \qquad (17)$$

and

$$K^2 = \frac{p_{l\nu}^2}{R^2} = \frac{\left(\gamma_b\omega^2 - \omega_p{}^2\right)\left[(\Omega_0+\alpha)^2 - \gamma_b{}^2\omega^2\right]}{\omega^2\left[\omega_p{}^2 + \gamma_b(\Omega_0+\alpha)^2 - \gamma_b{}^3\omega^2\right]}k^2 \tag{18}$$

where $p_{l\nu}$ is the νth zero of the Bessel function J_l. The relativistic dispersion relation for space charge waves is given by (18).

Figure 1 shows the correction factor α as a function of normalized cyclotron frequency. In the low Ω_0 and for group II in the sufficiently high Ω_0, the correction factor is near zero (the same as nonrelativistic case).

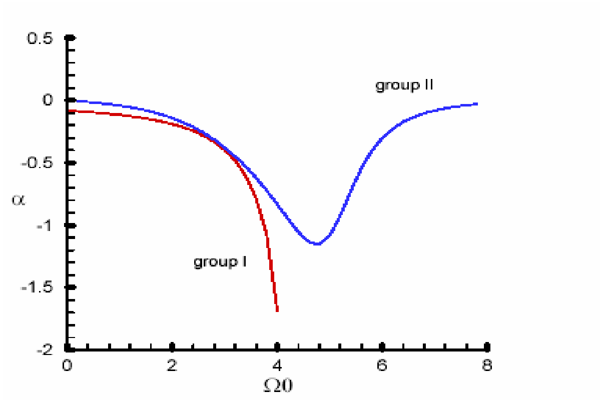

Fig. 1: Correction factor α as a function of normalized cyclotron frequency

First and second modes of space – charge for wave number $k = 5$ and wiggler amplitude $B_w = 1kG$ is shown in figure 2. Second modes are lower than first modes. For group II, in the Ω_0 sufficiently high, the relativistic frequency is very close to nonrelativistic case.

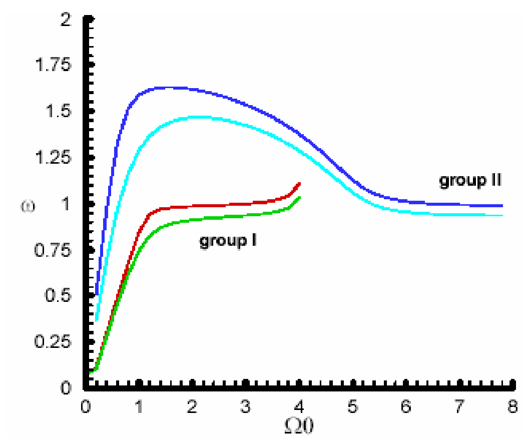

Fig. 2: First and second of space-charge modes for wavenumber $k = 5.0$ and $B_w = 1kG$

References

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2- B. Maraghechi, B. Farokhi, and J. E. Willett; Physics of Plasmas, **6**, 3778, (1999)