Explicit threshold of the toroidal ITG mode instability

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Abstract

The explicit stability threshold of the toroidal ITG mode instability is analytically derived using the standard reactive fluid model. It is shown that in the peak density region, the threshold gets significant smaller due to finite ion Larmor radius effects, and the marginal unstable modes acquire finite wavelengths.

1. Introduction

Low frequency electrostatic turbulence driven by spatial gradients is believed to be the main source of anomalous transport in magnetically confined fusion plasmas [1, 2]. During the last years, a significant number of both theoretical and numerical investigations in plasma dynamics has been focused on the effects related with the development of the ion temperature gradient (ITG) mode instability [3]. This is due to the successful interpretations of various experimental trends – related to the observed levels of turbulent transport in tokamak plasmas – which are based on the dynamics of the ITG mode instability.

The derivation of a general ITG model can be obtained either from a low–frequency expansion of the general fluid equations [4] based on the drift velocity ordering, or by using as a starting point the nonlinear gyrokinetic equation as in Ref. [5]. For the development of the ITG instability, the ion temperature gradient is necessary to be combined with other effects: The magnetic curvature, the parallel incompressibility or the presence of impurity species [6] are main examples of such additional effects.

However, in a toroidal system the instability is mainly driven by the magnetic field curvature [7] and it is termed toroidal ITG mode instability. The associated marginal instability threshold has been determined and analyzed in numerous works and in a variety of interacting physical processes, such as negative magnetic shear, electron trapping, or ion Landau damping. For instance, we may refer to the analysis based on the advanced fluid model [8]– in Ref. [9] or more recently in Ref. [10]. However, to our knowledge, the relevant investigations have derived the marginal stability threshold without taking explicitly into account the finite ion Larmor radius (FLR) effects. As a result, the derived thresholds either depend on the wavenumbers of the marginally unstable modes, or correspond – as we will show here – to larger values than the actual instability threshold. The resulting inaccuracy is reflected on the asymptotic behavior of the conventional stability curve $\eta_i(\epsilon_n, \tau) \to \infty$ when $\tau\epsilon_n \to 0$ (e.g. Ref. [8], p.125).

2. ITG threshold

In what follows, the standard advanced reactive fluid model by Weiland [8] is adopted and the linear stability properties of the toroidal ITG driven modes are investigated by keeping rigorously the FLR terms. Effects attributed to parallel ion dynamics, magnetic shear, electron particle trapping, Landau damping or finite beta effects are omitted. By using a low frequency

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expansion based on the standard drift velocity ordering, the ion continuity and the ion temperature equations – which describe the dynamics of the ITG modes, can be written in the following normalized form [8, 11], respectively

$$\frac{\partial n_i}{\partial t} - \left[\frac{\partial}{\partial t} - \tau\left(1+\eta_i\right)\frac{\partial}{\partial y}\right]\nabla_\perp^2\phi + \frac{\partial \phi}{\partial y} - \epsilon_n\frac{\partial}{\partial y}\left[\phi + \tau\left(n_i + T_i\right)\right] = 0, \tag{1}$$

and

$$\frac{\partial T_i}{\partial t} - \frac{5}{3}\tau\epsilon_n\frac{\partial T_i}{\partial y} + \left(\eta_i - \frac{2}{3}\right)\frac{\partial \phi}{\partial y} - \frac{2}{3}\frac{\partial n_i}{\partial t} = 0. \tag{2}$$

The length and the time scales have been normalized with respect to $\rho_s = c_s/\omega_{ci}$ and L_n/c_s, respectively, where $c_s^2 = T_e/m_i$ is the ion sound velocity defined at electron temperature, $\omega_{ci} = eB_0/m_ic$ is the ion gyrofrequency, and $L_g^{-1} = -d\ln g(r)/dr$ describes the inverse characteristic scalelength of inhomogeneity, along the radial direction, of the plasma parameter $g(r)$. The electrostatic potential has been normalized by $\phi = e\delta\phi/T_e\ L_n/\rho_s$, the perturbed density by $n = \delta n/n_0\ L_n/\rho_s$, and the perturbed ion temperature by $T_i = \delta T_i/T_{i0}\ L_n/\rho_s$. Furthermore, the curvature of the magnetic field lines R and the ion temperature inhomogeneity scale length L_{T_i} are given in terms of the plasma inhomogeneity scale length L_n by $\epsilon_n = 2L_n/R$ and $\eta_i = L_n/L_{T_i}$, respectively. Furthermore, τ denotes the ratio of ion to electron temperature, $\tau = T_i/T_e$.

Considering quasi–neutral oscillations and assuming Boltzmann distribution for the electron density response, i.e. $n_e = n_i = \phi$, we linearize Eqs. (1) and (2) and by applying the usual Fourier expansion for the perturbed quantities, i.e $\tilde{g}(r,t) = \tilde{g}\exp(i\mathbf{k}_\perp \cdot \mathbf{r} - i\omega t)$, the dispersion relation for the toroidal ITG modes is derived. Here, $\mathbf{k}_\perp$ denotes the wave-vector perpendicular to the toroidal axis of the magnetic field and ω the frequency of the toroidal ITG mode. The solution of the dispersion relation leads to the determination of the real frequency ω_r and the growth rate γ_k of the mode, which are given by

$$\omega_r = \frac{k_y}{2(1+k_\perp^2)}\left\{1 - \epsilon_n - \frac{10}{3}\epsilon_n\tau - \left[\tau(1+\eta_i) + \frac{5}{3}\tau\epsilon_n\right]k_\perp^2\right\}, \tag{3}$$

and

$$\gamma_k = \frac{k_y}{2(1+k_\perp^2)}\sqrt{f(k_\perp^2)}, \tag{4}$$

respectively. The polynomial $f(k_\perp^2)$ under the square root of Eq. (4) can be written in the following suitable form,

$$f(k_\perp^2) = -\tau^2\left(1 + \eta_i - \frac{5}{3}\epsilon_n\right)^2 k_\perp^4 + 2\tau(1+\epsilon_n)(\eta_i - \eta_B)k_\perp^2 + 4\epsilon_n\tau(\eta_i - \eta_C). \tag{5}$$

The parameters η_B and η_C are given by

$$\eta_B = \frac{4\epsilon_n - 1}{\epsilon_n + 1} - \frac{5}{3}\frac{\epsilon_n^2}{\epsilon_n + 1}\left(1 - \frac{4}{3}\tau\right), \quad \eta_C = \frac{2}{3} + \frac{10}{9}\epsilon_n\tau + \frac{(\epsilon_n - 1)^2}{4\epsilon_n\tau}. \tag{6}$$

The development of the toroidal ITG instability requires the condition $f(k_\perp^2) > 0$ to be hold. As a consequence, unstable modes will be those with perpendicular wavenumber in the range defined by the roots of equation $f(k_\perp^2) = 0$, which are given by

$$k_{\perp\pm}^2 = \frac{(\epsilon_n + 1)(\eta_i - \eta_B) \pm \sqrt{D_\eta}}{\tau\left(1 + \eta_i - \frac{5}{3}\epsilon_n\right)^2}, \tag{7}$$

where

$$D_\eta = (1+\epsilon_n)^2(\eta_i - \eta_B)^2 + 4\epsilon_n\tau(\eta_i - \eta_C)\left(1 + \eta_i - \frac{5}{3}\epsilon_n\right)^2 . \tag{8}$$

A sufficient condition for the development of the instability is these roots to be real and at least one of them to be positive. Inspection of Eqs. (7, 8) leads us to the conclusion that for $\eta_i < \eta_B, \eta_C$ the toroidal ITG instability can not take place since $k_{\perp\pm}^2 < 0$. On the other hand, instability certainly occurs when the condition $\eta_i > \eta_C$ holds, independently on the value of η_B. In this case, it is $k_{\perp -}^2 < 0$, and consequently the wavenumbers of the unstable toroidal ITG modes range as $0 < k_\perp^2 < k_{\perp +}^2$. The value η_C is the conventional threshold of the toroidal ITG instability [10, 11] obtained from Eq. (5) in the limit of negligible FLR effects. However, when FLR effects are taken into account, η_C becomes the marginal stability threshold *only* when $\eta_C < \eta_B$, and the marginal unstable mode has wavenumber $k_\perp = 0$. In the opposite case, i.e. when $\eta_B < \eta_C$, the threshold η_{th} is expected to be $\eta_B < \eta_{th} < \eta_C$. For $\epsilon_n < 1$, it is $\eta_C > \eta_B$ for any τ. The same inequality is valid also for $\epsilon_n > 1$ when τ belongs to the interval $0 < \tau < \tau_{CB}(\epsilon_n)$. The parameter $\tau_{CB}(\epsilon_n)$ is given by $\tau_{CB}(\epsilon_n) = 3\left[\epsilon_n - 1 + \sqrt{(\epsilon_n - 1)(7\epsilon_n - 3)/5}\right]/4\epsilon_n$, and defines a curve where $\eta_C = \eta_B$. It is evident now that η_C is the actual marginal stability threshold *only when* $\epsilon_n > 1$ *and* $\tau > \tau_{CB}$.

In what follows, we limit our analysis in the case $\eta_C > \eta_B$ and we seek for a threshold in the range $\eta_B < \eta_{th} < \eta_C$. Since necessary condition for the existence of the toroidal ITG instability is $D_\eta > 0$, the roots of equation $D_\eta = 0$ are appropriate candidates for the threshold. These are:

$$\eta_* = \frac{10}{9}\epsilon_n\tau + \frac{2}{3}, \quad \text{and} \quad \eta_\pm = \frac{5}{3}\epsilon_n - 1 - \frac{1}{2\tau} \pm \sqrt{\frac{10}{9}\epsilon_n^2 + \frac{1}{4\tau^2} + \frac{5}{3}\frac{\epsilon_n}{\tau}(1 - \epsilon_n)}. \tag{9}$$

A rigorous analysis of Eqs. (6, 9) shows that in the range of present interest, i.e. $\eta_C > \eta_B$, the derived roots η_*, $\eta_\pm$ obey the following ordering:

$$\begin{cases} \eta_- < \eta_+ < \eta_B < \eta_* & : \quad \text{when } \epsilon_n < 1 \text{ for any } \tau, \\ \eta_- < \eta_+ < \eta_B < \eta_* & : \quad \text{when } \epsilon_n > 1, \text{ for } \tau < \tau_*(\epsilon_n), \text{ and} \\ \eta_- < \eta_* < \eta_B < \eta_+ & : \quad \text{when } \epsilon_n > 1, \text{ for } \tau_*(\epsilon_n) < \tau < \tau_{CB}(\epsilon_n) \end{cases}$$

The parameter $\tau_*(\epsilon_n)$ is given by $\tau_*(\epsilon_n) = 3(1 - 1/\epsilon_n)/2$, and defines a curve where $\eta_*(\epsilon_n, \tau_*) = \eta_+(\epsilon_n, \tau_*) = \eta_B(\epsilon_n, \tau_*)$. From the analysis above, we conclude that the explicit threshold of the toroidal ITG instability is given by

$$\eta_{th}(\epsilon_n, \tau) = \begin{cases} \eta_* & \text{for } \epsilon_n < 1, \\ \eta_* & \text{for } \epsilon_n > 1 \text{ when } 0 < \tau < \tau_*, \\ \eta_+ & \text{for } \epsilon_n > 1 \text{ when } \tau_* < \tau < \tau_{CB}, \text{ and} \\ \eta_C & \text{for } \epsilon_n > 1 \text{ when } \tau > \tau_{CB}. \end{cases} \tag{10}$$

It is evident that FLR effects are destabilizing for values of ϵ_n smaller than unity, and reduce slightly the threshold in the flat density regime. These results are in qualitative agreement with the Nyquist analysis of the full gyrokinetic dispersion relation in Ref. [5], where the authors claimed that critical condition for the instability to take place is $\eta_i > 2/3$. It becomes well understood now that for $\eta_i < \eta_C$ the growth of the toroidal ITG modes is attributed to FLR effects and not to the parallel compressibility as was erroneously believed so far [8, 10].

Furthermore, there always exist a critical wavenumber $k_{\perp m}$ where a maximum growth occurs for given conditions. This wavenumber can be determined by the condition $d\gamma_k/dk_\perp^2 = 0$,

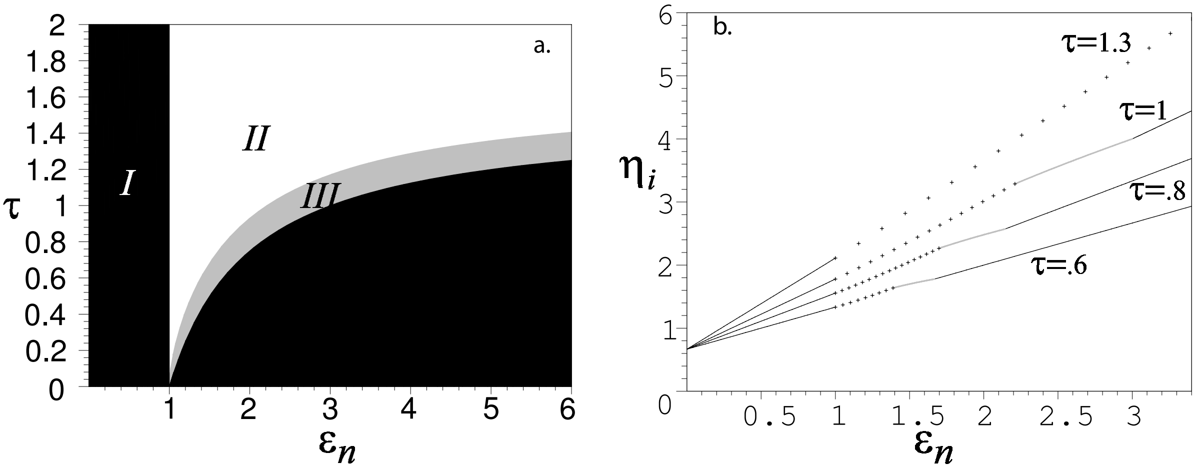

Figure 1: a. Instability threshold in the plane (ϵ_n, τ). The exact marginal stability threshold is given by $\eta_*(\epsilon_n, \tau)$ in the black regions, by $\eta_C(\epsilon_n, \tau)$ in the white region, and by $\eta_+(\epsilon_n, \tau)$ in the gray one. b. Marginal stability curves $\eta_i(\epsilon_n)$ for different values of τ. The black solid parts of each curve corresponds to η_*, the dotted parts to η_C, and the grey solid parts to η_+.

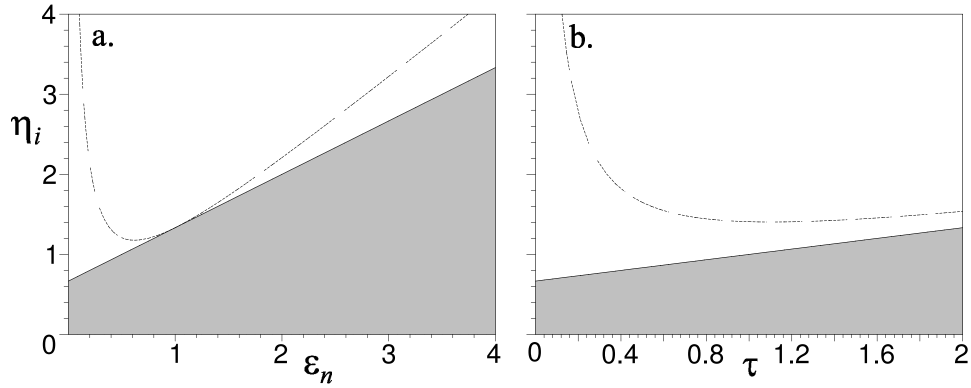

Figure 2: The marginal stability curve η_i: a. versus ϵ_n, for $\tau = 0.6$ and b. versus τ, for $\epsilon_n = 0.3$. The white and the gray areas denote the unstable and stable regions respectively, as defined by the exact threshold. The dashed line represents the threshold η_C which is valid in absence of FLR effects.

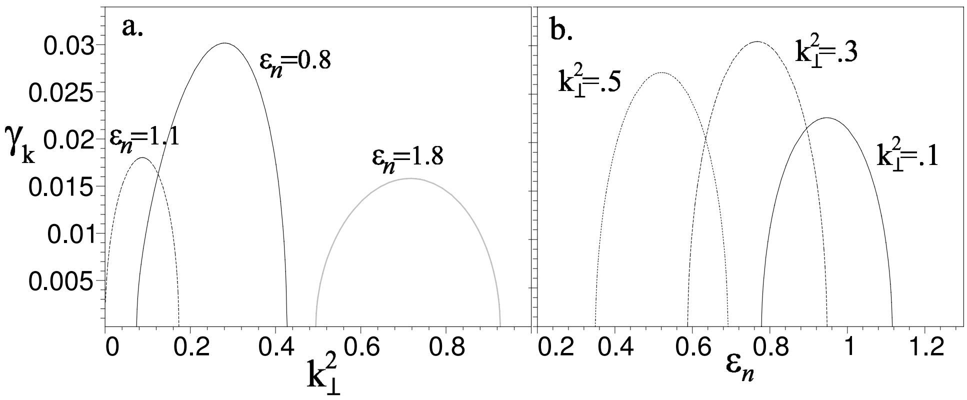

Figure 3: The normalized growth rate γ_k of the toroidal ITG instability, for purely poloidal propagation ($k_x = 0$) and $\tau = 0.8$: a. versus $k_\perp^2$, for (i) $\epsilon_n = 0.8$, $\eta_i = 1.005\eta_*$ (Region I in Fig. 1a), (ii) $\epsilon_n = 1.1$, $\eta_i = 1.005\eta_C$ (Region II in Fig. 1a), and (iii) $\epsilon_n = 1.8$, $\eta_i = 1.0002\eta_+$ (Region III in Fig. 1a), b. versus ϵ_n for three distinct ITG modes; $k_\perp^2 = 0.1,\ 0.3$ and 0.5 for $\eta_i = 1.005\eta_*$.

which leads to a cubic equation. Close to the marginal stability conditions, i.e. $f(k_{\perp m}^2) \simeq 0$, we determine for $k_{\perp m}$:

$$k_{\perp m}^2 \simeq \begin{cases} \frac{(1+\epsilon_n)(\eta_i-\eta_B)}{\tau\left(1+\eta_i-\frac{5}{3}\epsilon_n\right)^2}, \\ \frac{2}{3}\frac{\eta_i-\eta_B+\epsilon_n(\eta_C-\eta_B)-\sqrt{[\eta_i-\eta_B+\epsilon_n(\eta_C-\eta_B)]^2-3\tau\left(1+\eta_i-\frac{5}{3}\epsilon_n\right)^2}}{\tau\left(1+\eta_i-\frac{5}{3}\epsilon_n\right)^2}. \end{cases} \tag{11}$$

The first solution corresponds to the wavenumber of the most unstable mode when $\eta_B < \eta_{th} \simeq \eta_i < \eta_C$ while the second solution is valid for $k_\perp^2 \ll 1$ and corresponds to the wavenumber of the most unstable mode when $\eta_i \simeq \eta_{th} = \eta_C < \eta_B$.

3. Summary

In this work, the explicit marginal stability threshold for the development of the toroidal ITG instability was derived in the frame of the standard advance reactive fluid model. It was shown that FLR effects can decrease significantly the marginal instability threshold and the associated marginally unstable modes acquire finite wavelengths. These results predict that a significant activity of toroidal ITG turbulence can be present at regions of peaked plasma density, such as the plasma edge, modifying the confinement in the hot ion mode regime of tokamak operation.

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