Theory for Neoclassical Toroidal Plasma Viscosity in Tokamaks

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Abstract. Error fields and resistive magnetohydrodynamic (MHD) modes are ubiquitous in real tokamaks. They break the toroidal symmetry in $|\boldsymbol{B}|$ in tokamaks. Here, $\boldsymbol{B}$ is the magnetic field. The broken toroidal symmetry leads to enhanced neoclassical toroidal plasma viscosity and consequently the rate of the toroidal flow damping. The neoclassical toroidal plasma viscosity also results in a steady state toroidal plasma flow even without toroidal momentum sources. The same physics mechanisms and phenomena are also applicable and can be observed in the vicinity of the magnetic islands. All these physics consequences are of interests to tokamak devices such as International Thermonuclear Experimental Reactor (ITER). ITER is expected to have low toroidal rotation. Thus, understanding viscosity becomes crucial in predicting rotation in ITER.

1 Introduction

Real tokamaks have error fields and magnetohydrodynamic (MHD) modes that break toroidal symmetry[1-3]. There are two mechanisms that break the symmetry on the perturbed magnetic surface: one is the perturbed field itself [1,2] and the other results from the distortion of the magnetic surface due to the perturbed field [3,4]. The broken symmetry enhances the toroidal plasma viscosity and the rate of the toroidal flow damping. It also results in a steady state toroidal plasma flow [3,4].

A comprehensive theory has been developed to extend the existing theory of the neoclassical toroidal plasma viscosity [3,4] to the low collisionality regimes relevant to tokamak experiments [5]. The theory extends the stellarator transport theory [6,7] to include multiple perturbed modes. Specifically, we have extended the theory by solving the bounce averaged drift kinetic equation in the low collisionality regimes to obtain various asymptotic limits: besides the $1/\nu$ regime, the collisional boundary layer $\sqrt{\nu}$ regime, the superbanana plateau regime, the collisionless detrapping/retrapping regime, and the superbanana regime. Here, ν is the collision frequency. The transport fluxes in these regimes can be categorized as mainly due to resonant and non-resonant particles. The resonant transport fluxes in the superbababa plateau regime and the superbanana regime involve the resonance between the $\boldsymbol{E}\times\boldsymbol{B}$ drift and the ∇B drift. Here, $\boldsymbol{E}$ is the electric field and $\boldsymbol{B}$ is the magnetic field. The resonance results in enhanced particle and energy losses for high energy particles. The non-resonant transport fluxes in the collisional boundary layer $\sqrt{\nu}$ regime and the collisionless detrapping/retrapping regime involve mainly the low energy particles in the vicinity of the trapped and circulating boundary. When collision frequency is high enough to destroy boundary layer physics, the non-resonant transport fluxes enter the 1/ν regime. All these fluxes are nonlinear function of the radial electric field except the ones in the 1/ν regime. For the resonant transport fluxes, they decrease exponentially as the appropriately defined $\boldsymbol{E}\times\boldsymbol{B}$ Mach number exceeding unity, similar to the nonlinear plasma viscosity [8] that is responsible for the L-H transitions observed in tokamaks and stellarators. For the non-resonant transport fluxes, they decrease algebraically when the respective Mach numbers are larger than one. Qualitative results of the theory can be understood from the random walk argument. An approximate analytic expression that joins all these asymptotic limits has been constructed for modeling purposes [5] and is in good agreement with the numerical results [9]. This expression can also be used to determine the steady state toroidal plasma flow. The theory has been tested in NSTX [10], JET [11], MAST [12] and DIII-D [13].

It is known that the neoclassical toroidal plasma viscosity results in a steady state toroidal rotation even without momentum sources. The same phenomenon has also been observed in stellarator experiments [14]. It can be used to control toroidal plasma rotation in fusion grade tokamak experiments such as ITER. However, enhanced energy loss that comes with the enhanced momentum loss should impose a constraint on such control schemes by limiting the magnitudes of the error fields and the mode numbers. The heat fluxes in the theory together with the tolerable energy loss can be used to calculate maximum values of the error fields that can be used in such schemes.

The theory is extended to include the finite gradient B drift effects on the boundary layer analysis, the boundary effects on the superbanana plateau resonance, and to the region in the vicinity of a magnetic island.

2 Theory for Neoclassical Toroidal Plasma Viscosity

We refine the theory for the neoclassical toroidal plasma viscosity to include the finite gradient B drift effects in the boundary layer analysis, and the boundary effects on superbanana plateau resonance.

2.1 Magnetic Coordinates and Magnetic Field Spectrum

We adopt Hamada coordinates here [15]. The contravariant representation for the magnetic field $\boldsymbol{B}$ in these coordinates is $\boldsymbol{B} = \psi'\nabla V\times\nabla\theta - \chi'\nabla V\times\nabla\zeta$, where θ is the poloidal angle, ζ is the toroidal angle, $\chi' = \boldsymbol{B}\bullet\nabla\theta$, $\psi' = \boldsymbol{B}\bullet\nabla\zeta$, and V is the volume enclosed inside the magnetic surface. The inverse Jacobian is $\nabla V\times\nabla\theta\bullet\nabla\zeta = 1$. For doubly periodic toroidal plasmas, the magnetic field spectrum can be expressed as

$$B = B_0\left(1-\varepsilon\cos\theta\right) - B_0\sum\nolimits_{m,n}[b_{mnc}\cos(m\theta-n\zeta)+b_{mns}\sin(m\theta-n\zeta)], \qquad (1)$$

where B_0 is the strength of the magnetic field on the magnetic axis, m is the poloidal mode number, n is the toroidal mode number, ε is the amplitude of the $\cos\theta$ mode, b_{mnc} and b_{mns} are the Fourier amplitudes for the (m,n) mode. We are interested in cases where the magnitudes of perturbed field amplitudes b_{mnc} and b_{mns} are small so that there are no new classes of trapped particles besides those trapped in the equilibrium magnetic field variation $\left(\varepsilon\cos\theta\right)$. For shaped equilibria, ε is not necessarily equal to r/R, where r is the minor radius and R is the major radius. The spectrum in Eq.(1) can be expressed as B/B_0

$=(1-\varepsilon\cos\theta)$ - $\sum_n\left[A_n(\theta)\cos n\zeta_0 + B_n(\theta)\sin n\zeta_0\right]$, where is $\zeta_0 = q\theta$ -ζ, q is the safety factor, $A_n(\theta)=\sum_m\{b_{mnc}\cos[(m-nq)\theta]+b_{mns}\sin[(m-nq)\theta]\}$, and $B_n(\theta)$ $=\sum_m\{-b_{mnc}\sin[(m-nq)\theta]+b_{mns}\cos[(m-nq)\theta]\}$.

2.2 Linear Bounce Averaged Drift Kinetic Equation

We are interested in the collisionality regime where the effective collision frequency ν/ε for trapped particles, i.e., bananas, is less than their bounce frequency $v_t\sqrt{\varepsilon}/(Rq)$. Here, v_t = $\sqrt{2T/M}$ is the thermal speed of particles, T is the temperature, and M is the mass of the species. In this collisionality regime, the dominant physics for the transport fluxes caused by the broken toroidal symmetry in B are bananas wobbling off the magnetic surface to form drift orbits, which have a typical width, e.g., superbananas, of the order of the local minor radius. Thus, they cause significant transport losses over the conventional transport losses in ideal axisymmetruc tokamaks [16,17]. To develop theory for the bananas wobbling off the magnetic surface, it is nature to bounce average the drift kinetic equation. The linear version of the bounce averaged drift kinetic equation is [3,4,6,7]

$$\left\langle \mathbf{v}_d\bullet\nabla\zeta_0\right\rangle_b \frac{\partial f_{10}}{\partial\zeta_0} + \left\langle \mathbf{v}_d\bullet\nabla V\right\rangle_b \frac{\partial f_M}{\partial V} = \left\langle C(f_{10})\right\rangle_b, \qquad (2)$$

where $f_{10}(V,\zeta_0,E,k)$ is the perturbed distribution function, f_M is the Maxwellian distribution function, E = $M\mathrm{v}^2/2$ is the energy, v is the speed, k^2= $\left[E-\mu B_0(1-\varepsilon)\right]/(2\mu B_0\varepsilon)$ is the pitch angle parameter, μ = $M\mathrm{v}_\perp^2/(2B)$, $\mathrm{v}_\perp$ is the speed of the particles perpendicular to $\boldsymbol{B}$, for low β plasmas the drift velocity $\mathbf{v}_d$ [16] is $\mathbf{v}_d$= -$\mathrm{v}_\parallel \boldsymbol{n}\times\nabla(\mathrm{v}_\parallel/\Omega)$, $\boldsymbol{n}$ = $\boldsymbol{B}/B$, Ω = $eB/(Mc)$, c is the speed of light, $\mathrm{v}_\parallel$ is the speed of the particles parallel to $\boldsymbol{B}$, β is the ratio of the thermal energy to the magnetic field energy, $C(f_{10})$ is the Coulomb collision operator, the bounce averaging operation in Eq.(2) is defined as $\langle A\rangle_b$= $(\oint d\theta AB/|\mathrm{v}_\parallel|)/(\oint d\theta B/|\mathrm{v}_\parallel|)$, $\oint$ = $\int_{-\theta_t}^{\theta_t}$, and $\pm\theta_t$ are the turning points of the trapped particles where $|\mathrm{v}_\parallel|$ = 0. Note that the bounce averaging operation is performed in between turning points of the trapped particles because only the terms that are even functions of $\mathrm{v}_\parallel$ survive the original bouncing averaging operation $\langle A\rangle_b$= $\sum_\sigma\left(\oint d\theta AB/|\mathrm{v}_\parallel|\right)/\left(\oint d\theta B/|\mathrm{v}_\parallel|\right)$, where $\sigma = \mathrm{v}_\parallel/|\mathrm{v}_\parallel|$ denotes the sign of $\mathrm{v}_\parallel$. The pitch angle parameter k^2 separates circulating particles with k^2 > 1 from trapped particles, i.e. bananas, with k^2 < 1. Note that the notation E without an argument k denotes particle energy and with an argument k represents the complete elliptic integral of the second kind. The explicit expressions for bounce averaged toroidal drift speed $\left\langle \mathbf{v}_d\bullet\nabla\zeta_0\right\rangle_b$, radial drift speed $\left\langle \mathbf{v}_d\bullet\nabla V\right\rangle_b$, and collision operator $\left\langle C(f_{10})\right\rangle_b$ are

$$\left\langle \mathbf{v}_d\bullet\nabla\zeta_0\right\rangle_b = \frac{c\Phi'}{\chi'} - \frac{c\mu B_0}{e\chi'}\varepsilon'\left[\frac{2E(k)}{K(k)}-1\right],$$

$$\left\langle \mathbf{v}_d\bullet\nabla V\right\rangle_b = \frac{c\mu B_0}{e\chi'}\frac{1}{4K(k)}\sum_n\oint d\theta\left[A_n(\theta)(-n\sin n\zeta_0)+B_n(\theta)(n\cos n\zeta_0)\right]\Big/\sqrt{k^2-\sin^2(\theta/2)},$$

and

$$\langle C(f_{01})\rangle_b = \frac{\nu_D}{\varepsilon K(k)}\frac{\partial}{\partial k^2}\left\{\left[E(k)-(1-k^2)K(k)\right]\frac{\partial f_{01}}{\partial k^2}\right\}.$$

The prime denotes d/dV in section 2. The deflection collision frequency ν_D is defined in [18]. The curvature drift is neglected by invoking large aspect ratio expansion. Thus, we treat $\mu B_0 \approx E$. We use the pitch angle scattering operator because it has an enhancement factor of $1/\varepsilon$.

2.3 Finite Gradient *B* Drift Effects on Boundary Layer Analysis

The gradient B drift speed in the toroidal direction is often neglected in the transport theory for non-axisymmetric large aspect ratio tori except in the superbanana plateau and superbanana regimes because the typical magnitude of the $\boldsymbol{E}\times\boldsymbol{B}$ drift is larger than that of the gradient B drift for thermal particles. However, to improve the accuracy of the theory, we include the finite gradient B drift effects in the boundary layer analysis.

Following the procedure developed in Ref.[19], the solution for Eq.(2) is

$$f_{10} = -\frac{c\mu B_0}{e\chi'\langle \mathbf{v}_d\bullet\nabla\zeta_0\rangle_b}\frac{\partial f_M}{\partial V}\frac{1}{4K(k)}\times$$

$$\Bigg\{\sum_n\oint\frac{d\theta}{\sqrt{k^2-\sin^2\left(\frac{\theta}{2}\right)}}\left[A_n(\theta)\left(1-e^{-\sqrt{|n|}y}\cos\sqrt{|n|}y\right)+\sigma_\omega B_n(\theta)e^{-\sqrt{|n|}y}\sin\sqrt{|n|}y\right]\cos n\zeta_0\ +$$

$$\sum_n\oint\frac{d\theta}{\sqrt{k^2-\sin^2\left(\frac{\theta}{2}\right)}}\left[B_n(\theta)\left(1-e^{-\sqrt{|n|}y}\cos\sqrt{|n|}y\right)-\sigma_\omega A_n(\theta)e^{-\sqrt{|n|}y}\sin\sqrt{|n|}y\right]\sin n\zeta_0\Bigg\},\quad(3)$$

where $\sigma_\omega = \pm 1$ indicates the direction of the toroidal drift frequency $\langle \mathbf{v}_d\bullet\nabla\zeta_0\rangle_b$, y is the stretch variable defined as $y = \left(1-k^2\right)\left(\nu_D/\nu_t\right)^{-1/2}\left(\Delta k^2\right)^{-1}$, $\nu_{*d}=\left[\left(4\nu_t/\varepsilon\right)/\left|\langle \mathbf{v}_d\bullet\nabla\zeta_0\rangle_b\right|_{1-\Delta k^2}\right]$, and $\Delta k^2 = \left[\nu_{*d}/\ln\left(16/\sqrt{\nu_{*d}}\right)\right]^{1/2}$. The subscript $1-\Delta k^2$ indicates that the quantity is evaluated at the edge of the boundary layer, if the quantity diverges at $k^2=1$, and evaluated at $k^2=1$ if it converges. The collision frequency ν_t is defined as $\nu_i = \sqrt{2}\pi N_i e_i^4 \ln\Lambda/(M_i^{1/2}T_i^{3/2})$ for ions, and $\nu_e = \sqrt{2}\pi N_i Z_i^2 e_e^4\ln\Lambda/(M_e^{1/2}T_e^{3/2})$ for electrons, where Z_i is the charge number of ions. Using the distribution in Eq.(3) to calculate the flux surface averaged particle flux yields

$$\langle \Gamma\bullet\nabla V\rangle = -N\left(v_t^4/\pi^{3/2}\right)\left[M/(e\Phi')\right]^2\left(\nu_t/\sqrt{32\varepsilon}\right)\left[\eta_1\left(p'/p+e\Phi'/T\right)+\eta_2 T'/T\right],\quad(4)$$

where $\eta_j=(1/2)\int dx x^{5/2}\left(x-5/2\right)^{j-1}e^{-x}\left(\nu_D/\nu_t\right)\int_0^1 dk^2\left[E(k)-\left(1-k^2\right)K(k)\right]\sum_n\left(\bar{\alpha}_n^2+\bar{\beta}_n^2\right)$ for $j=1$-3,

$$\bar{\alpha}_n=\frac{\partial}{\partial k^2}\oint d\theta\frac{(-1/2)}{\sqrt{k^2-\sin^2(\theta/2)}}\frac{F_b}{K(k)}[A_n(\theta)(1-e^{-\sqrt{|n|}y}\cos\sqrt{|n|}y)+\sigma_\omega B_n(\theta)e^{-\sqrt{|n|}y}\sin\sqrt{|n|}y]$$

$$\bar{\beta}_n=\frac{\partial}{\partial k^2}\oint d\theta\frac{(-1/2)}{\sqrt{k^2-\sin^2(\theta/2)}}\frac{F_b}{K(k)}[B_n(\theta)(1-e^{-\sqrt{|n|}y}\cos\sqrt{|n|}y)-\sigma_\omega A_n(\theta)e^{-\sqrt{|n|}y}\sin\sqrt{|n|}y]$$

$F_b = \{1 - \sigma_{e\Phi'}(x/x_{\min})[2E(k)/K(k) - 1]\}^{-1}$, $x_{\min} = \left|2\Phi' e / \left(M v_t^2 \varepsilon'\right)\right|$, and $\sigma_{e\Phi'}$=1 if both e and Φ' have the same sign otherwise $\sigma_{e\Phi'}$=-1,. The flux surface averaged normalized heat flux $\langle \vec{q} \bullet \nabla V / T \rangle$ is the same as $\langle \Gamma \bullet \nabla V \rangle$ except η_1 is replaced by η_2 and η_2 by η_3.

2.4 Boundary Effects on Superbanana Plateau Resonance

It is customary to neglect the boundary effects on the superbanana plateau resonance [6,7,20]. However, when the resonance pitch angle is close to either k^2=1 or k^2= 0, the boundary conditions at these points become important [9] and the physics of the superbanana plateau resonance is modified.

The flux surface averaged particle flux resulting from the modified superbanana plateau resonance becomes

$$\langle \Gamma \bullet \nabla V \rangle = -\frac{\pi}{4} C_p N \frac{v_t^2}{\pi^{3/2}} \sqrt{2\varepsilon} \frac{2}{\varepsilon'} \frac{cM}{|e|\chi'} \sum_n |n| \left(\alpha_n^2 + \beta_n^2\right) K(k_0) \left[\eta_1 \left(\frac{p'}{p} + \frac{e\Phi'}{T}\right) + \eta_2 \frac{T'}{T}\right], \quad (5)$$

where C_p= 0.1667 ≈ 1/6 when the resonant pitch angle k_0^2, defined as the resonance pitch angle at which the $\boldsymbol{E}\times\boldsymbol{B}$ drift speed cancels the gradient B drift speed, is close 1, C_p= 0.25 when resonance occurs at k_0^2= 0, $\eta_1 = \Gamma(5/2)$, $\eta_2 = \Gamma(7/2) - (5/2)\Gamma(5/2)$ and Γ is the gamma function. The parameter $\alpha_n = [1/4K(\kappa)] \int_{-\theta_t}^{\theta_t} d\theta\, A_n(\theta) \Big/ \sqrt{k^2 - \sin^2(\theta/2)}$, and $\beta_n = [1/4K(\kappa)] \int_{-\theta_t}^{\theta_t} d\theta\, B_n(\theta) \Big/ \sqrt{k^2 - \sin^2(\theta/2)}$. When k_0^2 is close to unity, we evaluate $K(k_0)$ at the edge of the resonance layer, thus, $K(k_0) \approx (1/2)\ln\left(16/\hat{v}^{1/3}\right)$, where $\hat{v} = \left\{(2\nu_t/\varepsilon) \Big/ \left[cMv_t^2 |\varepsilon'| |n| / (2|e|\chi')\right]\right\}$. When k_0^2 is close to zero, $K(k_0) \approx \pi/2$. The normalized heat flux $\langle q \bullet \nabla V / T \rangle$ has the same form as $\langle \Gamma \bullet \nabla V \rangle$ except η_1 is replaced by η_2 and η_2 by $\eta_3 = \Gamma(9/2) - 5\Gamma(7/2) + (25/4)\Gamma(5/2)$.

2.5 Flux-Force Relation

The transport fluxes calculated here are related to either the toroidal or the poloidal component of the viscous forces in Hamada coordinates [21]:

$$\langle \Gamma \bullet \nabla V \rangle = -\frac{c}{e\chi'\psi'} \langle \boldsymbol{B}_p \bullet \nabla \bullet \boldsymbol{\pi} \rangle = \frac{c}{e\chi'\psi'} \langle \boldsymbol{B}_t \bullet \nabla \bullet \boldsymbol{\pi} \rangle, \quad (6)$$

where $\boldsymbol{B}_p = -\chi' \nabla V \times \nabla \zeta$, $\boldsymbol{B}_t = \psi' \nabla V \times \nabla \theta$, and $\boldsymbol{\pi}$ is the Chew-Goldberger-Low viscous tensor. Note that the charge e in Eq.(6) is species dependent. The subscript that denotes plasma species is suppressed in Eq.(6). Thus, they can be used in the toroidal momentum evolution equation to model toroidal flow or the radial electric field.

3 Neoclassical Toroidal Plasma Viscosity in the Vicinity of a Magnetic Island

The theory for the neoclassical toroidal plasma viscosity in the vicinity of the magnetic island has been developed in Ref.[4] and the results have been used to calculate the island rotation frequency [22]. The theory is extended to include the boundary layer effects here. We adopt the representation of $\boldsymbol{B} = I\nabla\zeta + \nabla\zeta\times\nabla(\psi+\delta\psi)$ in section 3, where $I = RB_t$, B_t is the toroidal magnetic field, ψ is the equilibrium poloidal flux function, $\delta\psi = \tilde{\psi}\cos m\alpha$ is the perturbed poloidal flux resulting from the presence of the magnetic island with an amplitude $\tilde{\psi}$, $\alpha = \theta - \zeta/q_s$ is the helical angle, $q_s = m/n$ is q at the resonant radius $r = r_s$. The $|\boldsymbol{B}|$ on the island magnetic surface is [4]

$$\frac{B}{B_0} = 1 - \left[\frac{r_s}{R} \pm \frac{r_w}{R}\sqrt{\overline{\Psi} + \cos m\alpha}\right]\cos\theta, \tag{7}$$

in the vicinity of a magnetic island, where $\overline{\Psi} = -\Psi/\tilde{\psi}$, Ψ is the helical flux function, $r_w = \sqrt{2q_s^2\tilde{\psi}/(q_s' B_0 r_s)}$ is proportional the width of the island, and $q_s' = dq/dr\,|_{r_s}$. In the vicinity of the magnetic island, the parameter $\bar{k}^2 = 2/(1+\overline{\Psi})$ delineates the regions outside, where $\bar{k}^2 \leq 1$, and inside, where $\bar{k}^2 \geq 1$, the separatrix at which $\bar{k}^2 = 1$.

The linear bounce averaged drift kinetic equation for the island magnetic geometry $(\Psi, \alpha, \theta, E, \mu)$, using constant-$\psi$ approximation, is, for trapped particles,

$$\left\langle \mathbf{v}_d \bullet \nabla\alpha \right\rangle_b \frac{\partial f_{01}}{\partial\alpha} + \left\langle \mathbf{v}_d \bullet \nabla\Psi \right\rangle_b \frac{\partial f_M}{\partial\Psi} = \left\langle C(f_{01}) \right\rangle_b, \tag{8}$$

When $\nu/\varepsilon >> \left\langle \mathbf{v}_d \bullet \nabla\alpha \right\rangle_b$, the solution of Eq.(8) yields the well-known 1/ν scaling of the transport fluxes [4,6,7]. Here, we are interested in the limit where $\nu/\varepsilon << \left\langle \mathbf{v}_d \bullet \nabla\alpha \right\rangle_b \sim \mathbf{v}_E \bullet \nabla\alpha$ with $\mathbf{v}_E$, the $\boldsymbol{E}\times\boldsymbol{B}$ drift velocity by neglecting the ∇B, and the curvature drift velocity. When the ∇B drift velocity, and the curvature drift velocity are comparable to the $\boldsymbol{E}\times\boldsymbol{B}$ drift velocity, the solution of Eq.(8) yields superbanana plateau regime and superbanana regime [6,7,21,23]. We will address the physics of these two regimes separately. The appropriate expressions for the components of the drift velocity are

$$\left\langle \mathbf{v}_d \bullet \nabla\alpha \right\rangle_b \approx \frac{Ic\Phi'}{B^2}\boldsymbol{B}\bullet\nabla\theta\,\frac{\partial\Psi}{\partial\psi}, \tag{9}$$

and, in a large aspect ratio tokamak,

$$\left\langle \mathbf{v}_d \bullet \nabla\Psi \right\rangle_b \approx \frac{I\boldsymbol{B}\bullet\nabla\theta}{B\Omega}\frac{\partial\Psi}{\partial\psi}\frac{\mathrm{v}^2}{2}\frac{\partial\Delta}{\partial\alpha}\left[\frac{2E(k)}{K(k)} - 1\right], \tag{10}$$

where $\Delta = \pm(r_w/R)\left(\overline{\Psi} + \cos m\alpha\right)^{1/2}$. In section 3, prime denotes $\partial/\partial\Psi$.

The solution of the perturbed distribution outside the island is,

$$f_{010} = \mp\frac{B}{c\Phi'\Omega}\frac{\mathrm{v}^2}{2}\left[\frac{2E(k)}{K(k)} - 1\right]\frac{\partial f_M}{\partial\Psi}\frac{r_w}{R}\sqrt{\overline{\Psi}+1}\,\frac{2\pi}{K(\bar{k})}\times$$

$$\sum_{n=1}^{\infty}\frac{\mathrm{q}^n}{1+\mathrm{q}^{2n}}\left[\left(1 - e^{-\sqrt{n}y}\cos\sqrt{n}y\right)\cos n\eta - e^{-\sqrt{n}y}\sin\sqrt{n}y\sin n\eta\right], \tag{11}$$

where $\mathrm{q} = e^{-\pi K'/K}$, $K'(\bar{k}) = K\left(\sqrt{1-\bar{k}^2}\right)$, $\eta = \pi u/K(\bar{k})$, $u = \int_0^{\varphi} dx/\sqrt{1-\bar{k}^2\sin^2 x}$, $\varphi = m\alpha/2$, the stretch variable y is defined as

$$\mathrm{y} = \left(1-k^2\right)\left[\frac{(Ic\,|\Phi'|/B)(\hat{\boldsymbol{n}}\bullet\nabla\theta)\left(q_s' r_w/q_s\right)m\sqrt{\overline{\Psi}+1}\left[(\pi/2)/K(\bar{k})\right]}{\left(\nu_D/2\varepsilon\right)/\ln\left(4/\sqrt{\Delta k^2}\right)}\right]^{1/2}. \tag{12}$$

The layer width Δk^2 defined in Eq.(12) is obtained by setting y = 1 in Eq.(12). Using the distribution in Eq.(11), we calculate the island magnetic surface averaged particle flux:

$$\langle \Gamma \bullet \nabla\Psi \rangle = -\sqrt{2\varepsilon}\,\frac{N\mathrm{v}_t^4}{\pi^{3/2}}\left(\frac{r_w}{R}\right)^2\left|\frac{B}{Ic\Phi'}\right|\left(\frac{I}{\Omega}\right)^2 \boldsymbol{n}\bullet\nabla\theta\frac{q_s' r_w}{q_s} m\left[\nu_{*dc}^t \ln\left(\frac{16}{\sqrt{\nu_{*dc}^t}}\right)\right]^{1/2} \times$$

$$\left[\frac{\pi}{\bar{k}K(\bar{k})}\right]^3 \sum_{n=1}^{\infty}\sqrt{n}\left(\frac{\mathrm{q}^n}{1+\mathrm{q}^{2n}}\right)^2\left[\lambda_1\left(\frac{p'}{p}+\frac{e\Phi'}{T}\right)+\lambda_2\frac{T'}{T}\right], \tag{13}$$

where $\lambda_j = (1/2)\int_0^\infty dx x^{5/2}(x-5/2)^{j-1}e^{-x}(\nu_D/\nu_t)$ for j = 1-3 and $\nu_{*dc}^t = [\nu_t/(2\varepsilon)]\times \left[(Ic|\Phi'|/B)(n\bullet\nabla\theta)(q_s' r_w/q_s)m\sqrt{\Psi+1}(\pi/2)K^{-1}(\bar{k})\right]^{-1}$. The normalized heat flux $\langle q\bullet\nabla\Psi/T\rangle$ has the same form as $\langle\Gamma\bullet\nabla\Psi\rangle$ except λ_1 is replaced by λ_2 and λ_2 by λ_3.

The perturbed distribution function inside the magnetic island is

$$f_{010} = \mp\frac{B}{c\Phi'\Omega}\frac{\mathrm{v}^2}{2}\left[\frac{2E(k)}{K(k)}-1\right]\frac{\partial f_M}{\partial\Psi}\frac{r_w}{R}\sqrt{2}\,\frac{2\pi}{K(\hat{k})}\times$$

$$\sum_{m=1}^{\infty}\frac{\mathrm{q}^{m-1/2}}{1+\mathrm{q}^{2m-1}}\left[\left(1-e^{-\sqrt{2m-1}z}\cos\sqrt{2m-1}z\right)\cos(2m-1)\xi - e^{-\sqrt{2m-1}z}\sin\sqrt{2m-1}z\sin(2m-1)\xi\right], \tag{14}$$

where $\xi = \pi u/2K(\hat{k})$, $u = \int_0^\varphi dx/\sqrt{\hat{k}^2-\sin^2 x}$, $\hat{k}=1/\bar{k}$, the stretch variable z is defined as

$$z = \left(1-k^2\right)\left[\frac{(Ic\,|\Phi'|/B)(\hat{\boldsymbol{n}}\bullet\nabla\theta)(q_s' r_w/q_s)\left(m/\sqrt{2}\right)\left[(\pi/2)/K(\hat{k})\right]}{(\nu_D/\varepsilon)/\ln\left(4/\sqrt{\Delta k^2}\right)}\right]^{1/2}. \tag{15}$$

The q in Eq.(14) is defined in terms of $K(\hat{k})$ and $K' = K(\sqrt{1-\hat{k}^2})$. The layer width Δk^2 can be estimated by setting z = 1 in Eq.(15). The island surface averaged particle flux is

$$\langle \Gamma \bullet \nabla\Psi \rangle = -\sqrt{2\varepsilon}\,\frac{N\mathrm{v}_t^4}{2\sqrt{2}\pi^{3/2}}\left(\frac{r_w}{R}\right)^2\left|\frac{B}{Ic\Phi'}\right|\left(\frac{I}{\Omega}\right)^2 \boldsymbol{n}\bullet\nabla\theta\frac{q_s' r_w}{q_s} m\left[\nu_{*dt}^t \ln\left(\frac{16}{\sqrt{\nu_{*dt}^t}}\right)\right]^{1/2} \times$$

$$\left[\frac{\pi}{K(\hat{k})}\right]^3 \sum_{n=1}^{\infty}\sqrt{n-\frac{1}{2}}\left(\frac{\mathrm{q}^{n-\frac{1}{2}}}{1+\mathrm{q}^{2n-1}}\right)^2\left[\lambda_1\left(\frac{p'}{p}+\frac{e\Phi'}{T}\right)+\lambda_2\frac{T'}{T}\right], \tag{16}$$

where $\nu_{*dc}^t = [\nu_t/(2\varepsilon)]\times \left[(Ic|\Phi'|/B)(n\bullet\nabla\theta)(q_s' r_w/q_s)m\left(\pi/2\sqrt{2}\right)K^{-1}(\hat{k})\right]^{-1}$. Replacing λ_1 by λ_2 and λ_2 by λ_3 in Eq(16) yield the normalized heat flux $\langle q\bullet\nabla\Psi/T\rangle$. The gradients of plasma pressure and temperature inside the island are not necessary zero due either to plasma fueling or to finite transport processes along the field line. Because the gradient scale length of the radial electric field is of the order of the island width, the turbulence fluctuations are suppressed and the plasma confinement is improved [8,24].

4 Bounce-Transit and Drift Resonance

In the low collisionality regime, there can be resonances between the bounce frequency of the trapped particles and the toroidal drift frequency [25] and between the transit frequency of the circulating particles and the toroidal drift frequency [26]. Because the mechanisms of the resonances involve bounce and transit motions they can only be described by the non-bounce averaged drift kinetic equation [26]. An Eulerian approach to solve the drift kinetic equation including the physics of these resonances has been developed to calculate the enhanced plasma viscosity [26]. Transport consequences, including the modification on the bootstrap current and plasma flows, have also been calculated [27].

5 Discussions and Concluding Remarks

We have developed a comprehensive theory for the neoclassical toroidal plasma viscosity for real tokamaks that have error fields or MHD activities present. The refinement of the theory to include the finite gradient B effects on the boundary layer analysis and the boundary effects on the superbanana plateau resonance is presented. The theory is also extended to the region in the vicinity of the magnetic island. The theory can be used in modeling the toroidal plasma rotation or the radial electric field in ITER.

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