A Numerical Matching Technique for Resistive MHD Stability Analysis

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Abstract: We have developed a new numerical matching technique for linear stability analysis of resistive magnetohydrodynamics (MHD) modes. Unlike the conventional asymptotic matching theory, our matching technique utilizes an inner layer with a finite width, which resolves practical difficulties in the asymptotic matching theory. The inner-layer solution is directly, not asymptotically, matched onto the outer-region solution. We found that this direct matching is accomplished by requiring the continuity of perturbed magnetic field and the smooth disappearance of the parallel electric field. Our matching method is successfully applied to single and double tearing, internal kink and interchange modes. Since the inner-layer width can be thinner when the resistivity is lower, and since the matching is also successful for multiple resonant surfaces, this technique can enable us to calculate the resistive MHD stability of high-temperature toroidal plasmas accurately in a short time.

1. Introduction

One of the standard methods for linear stability analysis of resistive magnetohydrodynamics (MHD) modes has been the asymptotic matching theory [1, 2]. A number of applications have been made on the stability calculations so far for cylindrical [3–7] and toroidal [8–11] plasmas. In this theory, the plasma is divided into two kinds of regions; i.e., the so-called outer regions and an inner layer. In the outer region, the plasma inertia and electrical resistivity do not play a role and thus we solve the inertia-less and ideal MHD equation or the so-called Newcomb equation [12]. In the inner layer, on the other hand, the plasma inertia and resistivity are taken into account. Since we assume that the inner layer is very thin, the inner-layer equation can be significantly simplified. Then the solutions in both kinds of regions are matched asymptotically, resulting in the dispersion relation of the mode.

The asymptotic matching theory itself is well established, and sophisticated methods in its application to numerical calculations have been developed [5, 7, 9, 11, 13, 14]. However, if we can resolve some essential and practical difficulties in its application listed below, it will be quite useful. The difficulties are as follows: First, this theory is inapplicable if the minimum of the safety factor $q_{\min}$ becomes a rational number and the magnetic shear vanishes there in reversed-magnetic-shear tokamak plasmas. The resonant surface becomes an irregular singular point in this case, and thus we cannot obtain a Frobenius series solution needed for the matching procedure. Since disruption can often occur when $q_{\min}$ tries to pass through the rational number in such an operation, the stability analysis of this situation is important for fusion research. Second, even if the magnetic shear does not vanish at the resonant surface, numerical implementation of the asymptotic matching theory requires careful treatment [5, 13]. The resonant surface becomes a regular singular point of the Newcomb equation in this case. Then, the asymptotic matching requires an accurate value of a ratio of the so-called small and big solutions around the resonant surface in the outer region. The numerical accuracy of the ratio is known to be sensitive to the local accuracy of the MHD equilibrium and mesh arrangement around the resonant surface [13]. In addition, the radial coordinate in the inner-layer equation is stretched by using the smallness of resistivity. The frequency is also stretched at the same time. Then, the asymptotic matching requires that the inner-layer solution coincides with the outer-region solution as going to the infinity in the stretched radial coordinate. We need to treat this infinity carefully in numerical computation [11].

In order to resolve these difficulties, a new matching technique was developed for ideal MHD stability analysis [15]. This technique utilizes the inner region with a finite width. We solve the Newcomb equation in the outer region, and the plasma inertia is taken into account in the inner region. However, the radial coordinate nor the frequency is not stretched as in the asymptotic matching theory. Then, the radial component of the plasma displacement and its radial derivative are matched directly, not asymptotically, across the boundaries, resulting in the dispersion relation. Since the Frobenius series solution is not required as in the asymptotic matching theory, the new matching technique is applicable to the reversed-magnetic-shear plasmas with $q_{\min}$ being a rational number. Also since we match the solutions at rather far location from the resonant surface, the sensitivity problem of the numerical accuracy does not arise. This numerical matching technique was also successfully applied to stability analysis of resistive wall mode in rotating plasmas [16]. If we try to apply the asymptotic matching to this problem, we find that it is impossible to know where to locate the infinitely thin inner layer prior to obtaining the solution, since the location of the Alfvén resonances is determined by the plasma rotation frequency and the real frequency of the resistive wall mode itself, of which latter is determined after obtaining the solution [16]. By using the finite-width inner region which covers a possible range of the locations of the Alfvén resonances, the stability of the resistive wall mode was successfully calculated.

In the present paper, we extend the idea of Ref. [15] to linear resistive MHD modes [17]. Then we solve the Newcomb equation in the outer region and take into account the plasma inertia and resistivity in the inner region. Since the order of radial derivative of the governing equation in the inner region is four, instead of two as in the case of ideal MHD modes, we need to select two of the four independent solutions which can match onto the outer-region solution. We have found the appropriate boundary conditions; the continuity of perturbed magnetic field and the smooth disappearance of the parallel electric field [17]. The newly developed numerical matching technique has been applied to various resistive MHD modes such as single and double tearing, internal kink and interchange modes. Here we show numerical results of the double tearing mode and the corresponding non-resonant modes in finite-beta reversed-magnetic-shear plasmas in a cylindrical geometry.

The present paper is organized as follows. In Sec. 2., the formulation of the numerical matching technique is presented. The numerical results of the double tearing modes are shown in Sec. 3.. Summary and conclusions are given in Sec. 4..

2. Formulation

Let us consider a cylindrical plasma with the radius a and the length $2\pi R_0$. The cylindrical coordinates (r, θ, z) is used. The high-beta reduced MHD equation [18] is adopted here.

The magnetic and velocity fields are given by $\mathbf{B} = B_0\hat{\mathbf{z}} + \nabla\psi \times \hat{\mathbf{z}}$ and $\mathbf{v} = \hat{\mathbf{z}} \times \nabla\varphi$, respectively, where $\hat{\mathbf{z}}$ denotes the unit vector in the z direction. The perturbed quantities are assumed to be proportional to $\mathrm{e}^{\mathrm{i}\,(m\theta+n\zeta)+\gamma t}$ where m and n are the poloidal and toroidal mode numbers, respectively, $\zeta := z/R_0$ is the toroidal angle and γ is the growth rate. Quantities are normalized by using the length a, the magnetic field B_0, the velocity $v_{\mathrm{A}} := B_0/\sqrt{\mu_0\rho_0}$ with ρ_0 being the mass density, and the time $\tau_{\mathrm{A}} := a/v_{\mathrm{A}}$. The inverse aspect ratio is denoted by $\varepsilon := a/R_0$. The linearized high-beta reduced MHD equation is written as

$$\gamma\nabla_\perp^2\varphi = -\mathrm{i}\,F\nabla_\perp^2\psi - \frac{\mathrm{i}\,mJ_0'}{r}\psi + \frac{\mathrm{i}\,m\kappa_{0r}}{r}p, \tag{1}$$

$$\gamma\psi = -\mathrm{i}\,F\varphi + \eta\nabla_\perp^2\psi, \tag{2}$$

$$\gamma p = \frac{\mathrm{i}\,m\beta'}{r}\varphi. \tag{3}$$

where φ, ψ and p are the perturbed electrostatic potential, the magnetic flux function and normalized pressure, respectively. The perpendicular Laplacian is defined by $\nabla_\perp^2 := \frac{1}{r}\frac{\partial}{\partial r}r + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$. The equilibrium current is given by $J_0 := \nabla_\perp^2\psi_0$ with ψ_0 being the equilibrium component of the magnetic flux function, and the equilibrium normalized pressure is denoted by β, where the prime denotes the r derivative of the equilibrium quantities. The equilibrium magnetic curvature is defined by using the poloidal magnetic field as $\kappa_{0r} := -B_{0\theta}^2/r$. The normalized resistivity is denoted by η which is the inverse of the Lundquist number. The quantity F is related to the parallel wave number and is defined by $F := m\varepsilon\left(\frac{n}{m} + \frac{1}{q}\right)$, where $q := -\varepsilon r/\psi_0'$ is the safety factor. Note that the equilibrium plasma flow is dropped in this study.

In the following, we present the formulation to obtain the growth rate by the numerical matching technique. As mentioned above, the plasma is divided into two kinds of regions; outer and inner regions. In the present paper, let us assume that the mode-resonant surface(s) is located in the middle of the plasma, and an inner region with a finite width, spanning from $r = r_{\mathrm{L}}$ to r_{R}, covers the resonant surface(s). Outside the inner region is the outer region. There are two outer regions; one is at $0 \le r < r_{\mathrm{L}}$ and the other is at $r_{\mathrm{R}} < r \le 1$.

2.1. Outer region

In the outer region, we solve the inertia-less ideal MHD equation, which is obtained by combining the vorticity equation (1) with its inertia term dropped, the parallel Ohm's law (2) with its resistivity term dropped and the pressure equation (3) as

$$\mathcal{N}\psi = 0, \tag{4}$$

$$\mathcal{N} := \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) - \left[\left(\frac{m}{r}\right)^2 - \frac{m}{rF}\left(J_0' + \frac{m\kappa_{0r}\beta'}{rF}\right)\right]. \tag{5}$$

Let us express ψ in the outer regions as $\psi = \psi_l G_{\mathrm{out},l}(r)$ where l stands for L or R corresponding to the regions $0 \le r < r_{\mathrm{L}}$ and $r_{\mathrm{R}} < r \le 1$, respectively. Here $G_{\mathrm{out},l}$ is obtained by solving $\mathcal{N}G_{\mathrm{out},l} = 0$ with boundary conditions $G_{\mathrm{out,L}}(0) \propto r^m$ around $r = 0$ and $G_{\mathrm{out,L}}(r_{\mathrm{L}}) = 1$, and $G_{\mathrm{out,R}}(r_{\mathrm{R}}) = 1$ and $G_{\mathrm{out,R}}(1) = 0$. The constant coefficient ψ_l express the amplitudes, which will be determined by the matching condition explained below.

2.2. Inner region

In the inner region, we solve Eqs. (1) – (3) by giving a guess value γ. Let us denote the solution as $(\varphi, \psi, p)^{\mathrm{T}} = \psi_{\mathrm{L}} \mathbf{G}_{\mathrm{in,L}}(r) + \psi_{\mathrm{R}} \mathbf{G}_{\mathrm{in,R}}(r)$. The boundary condition for the ψ component of $\mathbf{G}_{\mathrm{in},l}$, or $G_{\mathrm{in},\psi,l}$, is given by $G_{\mathrm{in},\psi,\mathrm{L}}(r_{\mathrm{L}}) = 1$ and $G_{\mathrm{in},\psi,\mathrm{L}}(r_{\mathrm{R}}) = 0$, and $G_{\mathrm{in},\psi,\mathrm{R}}(r_{\mathrm{L}}) = 0$ and $G_{\mathrm{in},\psi,\mathrm{R}}(r_{\mathrm{R}}) = 1$. Then ψ becomes continuous across the boundaries. The boundary condition for p is not necessary in this case. If we include a thermal conductivity term in Eq. (3), a boundary condition for p is required [17].

The boundary condition for the φ component of $\mathbf{G}_{\mathrm{in},l}$, or $G_{\mathrm{in},\varphi,l}$, is obtained as follows. Since the inner-region solution needs to be matched onto the outer-region solution or the ideal MHD solution, $G_{\mathrm{in},\varphi,l}$ should be obtained so that it satisfies the ideal parallel Ohm's law. However, if we solve the parallel Ohm's law for φ and impose it as a boundary condition, it leads to a solution where the parallel electric field $E_{\|}$ suddenly becomes zero at the boundaries. If the boundaries are located well in the ideal MHD region, this might give an accurate solution. However, if we take the width of the inner region finite but thin, $E_{\|}$ is finite at the boundaries actually, and we may fail to obtain the accurate solution. Therefore, an appropriate boundary condition seems to be smooth disappearance of $E_{\|}$ as approaching the outer region. We have found that this boundary condition is obtained by combining the ideal Ohm's law and its radial derivative as

$$\frac{\mathrm{d}G_{\mathrm{in},\varphi,l}}{\mathrm{d}r} = \left(\frac{1}{G_{\mathrm{in},\psi,l}} \frac{\mathrm{d}G_{\mathrm{in},\psi,l}}{\mathrm{d}r} - \frac{F'}{F} \right) G_{\mathrm{in},\varphi,l}. \tag{6}$$

2.3. Matching condition

As explained above, we solve the inner-region equation by giving a guess value γ. By imposing the continuity of $\mathrm{d}\psi/\mathrm{d}r$ across the boundaries, we can obtain the true growth rate γ. This condition is expressed as

$$\begin{pmatrix} G'_{\mathrm{in},\psi,\mathrm{L}}(r_{\mathrm{L}}) - G'_{\mathrm{out,L}}(r_{\mathrm{L}}) & G'_{\mathrm{in},\psi,\mathrm{R}}(r_{\mathrm{L}}) \\ G'_{\mathrm{in},\psi,\mathrm{L}}(r_{\mathrm{R}}) & G'_{\mathrm{in},\psi,\mathrm{R}}(r_{\mathrm{R}}) - G'_{\mathrm{out,R}}(r_{\mathrm{R}}) \end{pmatrix} \begin{pmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{R}} \end{pmatrix} = 0. \tag{7}$$

Since $(\psi_{\mathrm{L}}, \psi_{\mathrm{R}})^{\mathrm{T}}$ is not zero, the determinant of the matrix in front of it must be zero. This gives us the true growth rate γ.

3. Application to reversed magnetic shear plasmas

In this section, numerical results obtained by our numerical matching technique are presented. We have succeeded to apply our technique to various kinds of resistive MHD modes so far [17]. Here we focus on stability analysis of reversed magnetic shear plasmas, including a plasma with its minimum safety factor being a rational number. As explained in Introduction, this is an important situation for fusion plasma research since disruption can often occur when $q_{\min}$ tries to pass through the rational number in a reversed magnetic shear plasma. We assumed the equilibrium safety factor q and normalized pressure β as

$$q(r) = q_{\min} \left[(\alpha - 1) \left(\frac{r}{r_{\min}} \right)^4 - 2(\alpha - 1) \left(\frac{r}{r_{\min}} \right)^2 + \alpha \right], \tag{8}$$

$$\beta(r) = \beta_0 (1 - r^2). \tag{9}$$

The q profile has a minimum at $r = r_{\rm min}$. Figure 1 shows a typical example of the q and β profiles, where $q_{\rm min} = 2$, $\alpha = 3$ and $\beta_0 = 10^{-2}$ were used.

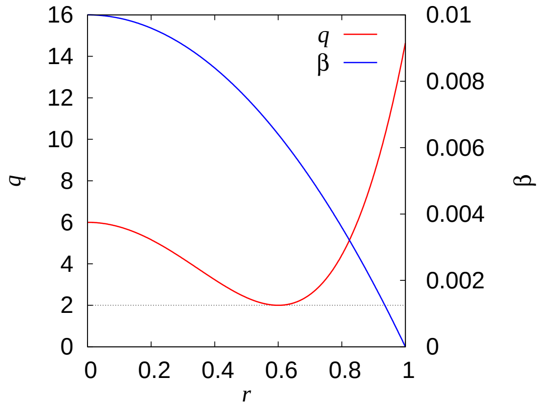


FIG. 1. An example of equilibrium profiles of safety factor q and normalized pressure β with $q_{\rm min} = 2$, $\alpha = 3$ and $\beta_0 = 10^{-2}$ in Eqs. (8) and (9).

Figure 2 shows the growth rate γ of $m/n = 2/1$ mode as a function of $q_{\rm min}$ for $\eta = 10^{-6}$. The equilibrium beta was assumed to be $\beta_0 = 0$ or $\beta_0 = 10^{-2}$. The width of the inner region was chosen to be $\Delta r = 0.2$, which is wide enough to obtain a good convergence. The radial grid number was 800 and was accumulated around $r = r_{\rm min}$. The symbols indicated by "global" means the growth rate obtained by solving Eqs. (1) – (3) in the whole domain without using the matching technique. The grid number and the size are same as the numerical matching solutions. The growth rate obtained by the numerical matching technique, indicated by $\Delta r = 0.2$, overlap well the data obtained by the global calculations for both $\beta_0 = 0$ and 10^{-2}. For $q_{\rm min} < 2$, the $m/n = 2/1$ mode obtained here is the double tearing mode. The double tearing mode in the zero beta equilibrium becomes stable for $q_{\rm min} \geq 2$. On the other hand, the $m/n = 2/1$ mode is still unstable for $q_{\rm min} \geq 2$ if the equilibrium beta is finite. Even for $q_{\rm min} = 2$, our numerical matching technique succeeds to calculate the growth rate, which is impossible by the asymptotic matching method. The modes with $q_{\rm min} > 2$ are non-resonant modes destabilized by the equilibrium pressure gradient. This destabilization easily occurs because the magnetic shear vanishes or becomes very small around $r = r_{\rm min}$.

Figure 3 shows eigenfunctions of a $m/n = 2/1$ mode for a $q_{\rm min} = 2$ and $\beta_0 = 10^{-2}$ finite-beta and reversed magnetic shear plasma. The eigenfunction indicated by "global" means that it is obtained by solving Eqs. (1) – (3) in the whole domain. The parallel electric field $E_{||}$ was calculated by using φ and ψ after the eigenvalue and the eigenfunctions are obtained. We observe that the eigenfunctions φ, ψ and p obtained by the numerical matching technique overlap well the global solution. As for $E_{||}$, the outer solution does not overlap the global solution since we solve the Newcomb equation in the outer region where $E_{||}$ is zero. On the other hand, however, the inner solution overlap well the global solution.

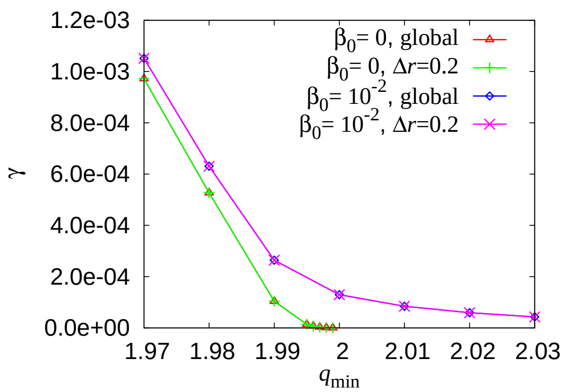
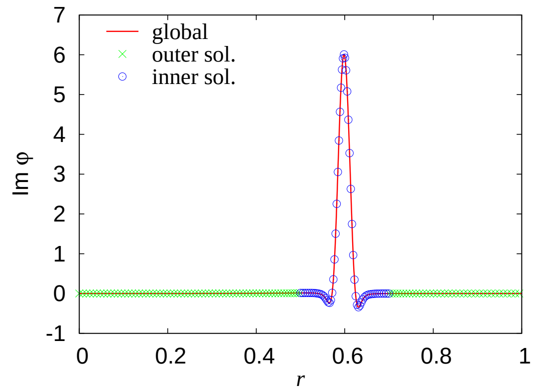
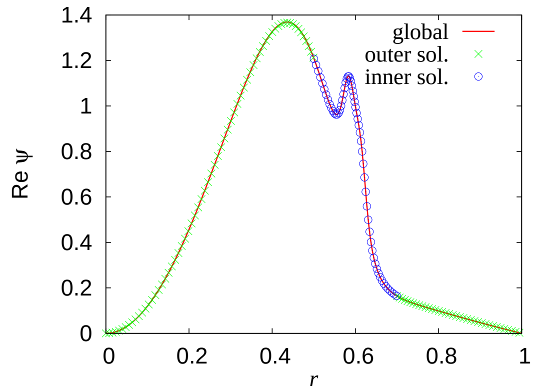


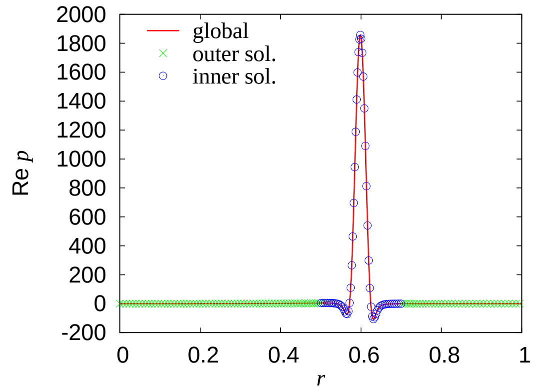
FIG. 2. Growth rate γ of $m/n = 2/1$ mode in reversed magnetic shear plasmas for $\eta = 10^{-6}$ as a function of $q_{\rm min}$. The inner region width was $\Delta r = 0.2$. The resultant growth rate γ overlap well those obtained by solving Eqs. (1) – (3) in the whole domain indicated by "global".



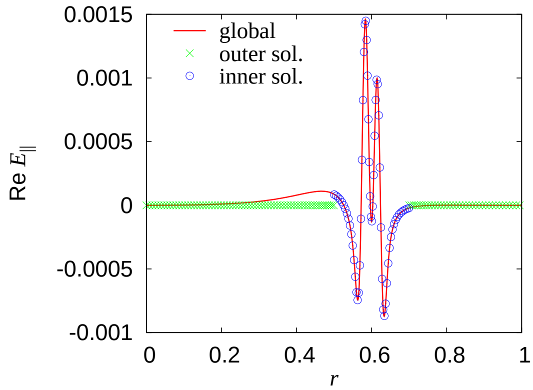
(a) Im φ.



(b) Re ψ.



(c) Re p.



(d) Re $E_{\parallel}$.

FIG. 3. Eigenfunctions of a $m/n = 2/1$ mode for a $q_{\min} = 2$ and $\beta_0 = 10^{-2}$ finite-beta and reversed magnetic shear plasma. The eigenfunction indicated by "global" means that it is obtained by solving Eqs. (1) – (3) in the whole domain. The matched solution overlaps well the global solution.

4. Summary and conclusions

We have developed a new numerical matching technique for linear stability analysis of resistive magnetohydrodynamics (MHD) modes. The conventional asymptotic matching theory has several difficulties in its numerical implementation in practice because of the singularity of the problem; i.e., (i) it is inapplicable to a case where the minimum safety factor being a rational number in reversed-magnetic-shear tokamak plasmas since the rational surface becomes irregular singularity, (ii) the numerical implementation is not so straightforward because of the sensitive behavior of the outer solution on the local equilibrium accuracy and grid arrangements near the resonant surface and (iii) the inner layer solution needs to be obtained in the infinite space expanded by the smallness of the resistivity, which is also not straightforward to treat numerically. Unlike the conventional asymptotic matching theory, our numerical matching technique utilizes an inner layer with a finite width, which resolves those practical difficulties in the asymptotic matching theory. Neither the radial coordinate nor the growth rate is expanded by using smallness of the electrical resistivity in the inner layer. The inner-layer solution is directly, not asymptotically, matched onto the outer-region solution. We found that this direct matching is accomplished by requiring the continuity of perturbed magnetic field and the smooth disappearance of the parallel electric field. Although our matching technique has been successfully applied to various resistive MHD modes, we focused on $m/n = 2/1$ modes in reversed magnetic shear plasmas in the present paper. We have obtained accurate growth rates of double tearing modes for $q_{\min} < 2$ for zero and finite betas, and also for the $m/n = 2/1$ modes with $q_{\min} = 2$ and higher for finite beta. The eigenfunctions agree well with those obtained by the corresponding global calculations.

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