Negative Energy Waves and Stability of Rotating Plasmas

V.I. Ilgisonis 1), I.V. Khalzov 2), A.I Smolyakov 1), 3)

1) Russian Research Centre "Kurchatov Institute", Moscow, Russia
2) University of Wisconsin, Madison, USA
3) University of Saskatchewan, Saskatoon, Canada

e-mail contact of main author: vil@nfi.kiae.ru

Abstract. Eigen-mode analysis of ideal magnetohydrodynamic (MHD) systems with flows is performed. It is shown that energy of stable oscillatory modes (waves) can be both positive and negative. Negative energy waves always correspond to non-symmetric modes which are nonuniform along the direction of the flow. Coupling of negative and positive energy waves is shown to be a universal mechanism of non-symmetric MHD instabilities in flowing media. To study stability of non-symmetric modes, a new variational approach is developed based on Lyapunov theory. This approach provides sufficient and (under some assumptions) necessary stability condition.

1. Introduction

Stability study of rotating plasmas is of high current interest in many applied and fundamental problems. Rotation is a common phenomenon in modern fusion experimental devices (such as tokamaks) where it is believed to improve the overall plasma confinement by stabilizing kink and resistive wall modes and suppressing turbulence [1]. At the same time, plasma rotation in the presence of magnetic field may lead to destabilizing effects, e.g., magnetorotational instability (MRI) which is widely accepted now as a source of turbulence and angular momentum transport in accretion disks [2].

The behavior of many plasma systems is well described by ideal magnetohydrodynamics (MHD). The majority of stability studies in MHD are related to spectral method – analysis of eigenvalues of dynamic operator linearized near the equilibrium state. The methodological difficulty of correct spectral stability analysis is in the necessity of finding not only the eigenvalues of the linearized system but also the corresponding eigenvectors, which have to satisfy the particular boundary conditions. Besides, in the case of systems with plasma flows the linearized operator of dynamics becomes non-Hermitian (non-self-adjoint), therefore its eigenvalues are generally complex [3]. As a result, the spectral stability study of such systems constitutes a very challenging mathematical problem and is often restricted to simple geometries.

Another way to make a judgement about the stability of the equilibrium state is to use variational methods, e.g., Lyapunov theory. According to Lyapunov stability theorem the stability of equilibrium state of a dynamical system is guaranteed if there is a Lyapunov functional – an integral of motion which has a strict local minimum (maximum) at the equilibrium state. There is no regular way to construct a Lyapunov functional. For conservative systems (such as ideal MHD) a natural Lyapunov functional candidate is the total energy of the system. This choice results in the well-known energy principle, first realized in Ref. [4] for static MHD equilibrium: if the change of the total energy is positive for any small deviations of a conservative system from the equilibrium state, then such equilibrium state is stable. From a practical point of view it is important that in the case of static equilibrium, energy principle gives both sufficient and necessary conditions for spectral stability [5].

Contrary to the static case, in the presence of stationary plasma flow the energy principle gives only the sufficient stability condition, which is normally too restrictive and almost never can be satisfied [6]. Limited applicability of the energy principle to MHD systems with flows can be explained by the existence in such systems of negative energy waves (NEW) – stable oscillatory eigen-modes, excitation of which leads to decrease of total energy of the system [7]. Different attempts have been made to find a variational approach which generalizes the energy principle for the systems with flows [6, 8–11], however, this problem is still far from a complete solution. Approach proposed in the present paper is an extension of the variational method developed in [11].

The structure of the paper is following. In section 2, we investigate the energy of waves in MHD system and point out the important role of negative energy waves in instabilities of plasma flows. In section 3, we suggest a variational approach suitable for stability study of flowing plasma. Our approach is based on construction of Lyapunov functional for linearized MHD system, which is usually referred to as formal stability analysis. In section 4, the potential applications of our approach are discussed.

2. Energy of eigenmodes in MHD

A lot of important physical information can be revealed from the analysis of energy of eigen-modes in the frame of ideal MHD. To do this, we start from the well-known linearized dynamic equation for plasma displacement vector $\boldsymbol{\xi}$,

$$\rho\ddot{\boldsymbol{\xi}} + 2\rho(\mathbf{V}\cdot\nabla)\dot{\boldsymbol{\xi}} - \mathbf{F}[\boldsymbol{\xi}] = 0, \tag{1}$$

where dot denotes a partial time derivative, ρ and $\mathbf{V}$ are stationary values of fluid density and velocity, respectively. The general form of linearized force operator $\mathbf{F}[\boldsymbol{\xi}]$ in ideal MHD is

$$\begin{aligned} \mathbf{F}[\boldsymbol{\xi}] &= -\rho(\mathbf{V}\cdot\nabla)^2\boldsymbol{\xi} + \rho(\boldsymbol{\xi}\cdot\nabla)(\mathbf{V}\cdot\nabla)\mathbf{V} + \nabla\cdot(\rho\boldsymbol{\xi})(\mathbf{V}\cdot\nabla)\mathbf{V} \\ &- \nabla\delta P + \frac{1}{4\pi}(\nabla\times\delta\mathbf{B})\times\mathbf{B} + \frac{1}{4\pi}(\nabla\times\mathbf{B})\times\delta\mathbf{B}. \end{aligned} \tag{2}$$

Here, $\mathbf{B}$ is equilibrium magnetic field and $\delta\mathbf{B} = \nabla\times(\boldsymbol{\xi}\times\mathbf{B})$ is its perturbation. The perturbation of fluid pressure δP can be specified by thermodynamic properties of the system. For example, if the process is adiabatic with adiabatic index γ then $\delta P = -\boldsymbol{\xi}\cdot\nabla P - \gamma P\nabla\cdot\boldsymbol{\xi}$. In the case of incompressible MHD, such equation appears to be excessive, instead one has to impose the incompressibility condition $\nabla\cdot\boldsymbol{\xi} = 0$.

A number of formal properties of Eq. (1) can be established. Force operator $\mathbf{F}[\boldsymbol{\xi}]$ is Hermitian (self-adjoint) in the following sense,

$$\int \boldsymbol{\eta}\cdot\mathbf{F}[\boldsymbol{\xi}]\,d^3\mathbf{r} = \int \boldsymbol{\xi}\cdot\mathbf{F}[\boldsymbol{\eta}]\,d^3\mathbf{r}, \tag{3}$$

while the second term in Eq. (1) is antisymmetric:

$$\int \rho\boldsymbol{\eta}\cdot(\mathbf{V}\cdot\nabla)\boldsymbol{\xi}\,d^3\mathbf{r} = -\int \rho\boldsymbol{\xi}\cdot(\mathbf{V}\cdot\nabla)\boldsymbol{\eta}\,d^3\mathbf{r}. \tag{4}$$

Integration in Eqs. (3) and (4) is performed over the fluid volume under the assumption that displacements at the plasma boundary vanish to avoid consideration of boundary and vacuum region perturbations.

A normal-mode solutions to Eq. (1) has a form

$$\boldsymbol{\xi}(\mathbf{r},t) = \hat{\boldsymbol{\xi}}(\mathbf{r})e^{-i\omega t}. \tag{5}$$

Then Eq. (1) leads to eigen-value problem

$$\omega^2\rho\hat{\boldsymbol{\xi}} + 2i\omega\rho(\mathbf{V}\cdot\nabla)\hat{\boldsymbol{\xi}} + \mathbf{F}[\hat{\boldsymbol{\xi}}] = 0. \tag{6}$$

Multiplying this equation by complex conjugate $\hat{\boldsymbol{\xi}}^*$ and integrating over the fluid volume, we arrive at quadratic equation for eigen-frequency ω,

$$A\,\omega^2 - 2B\,\omega - C = 0, \tag{7}$$

where $A = \int \rho|\hat{\boldsymbol{\xi}}|^2\, d^3\mathbf{r}$, $B = -\,i\int \rho\hat{\boldsymbol{\xi}}^*\cdot(\mathbf{V}\cdot\nabla)\hat{\boldsymbol{\xi}}\, d^3\mathbf{r}$ and $C = -\int \hat{\boldsymbol{\xi}}^*\cdot\mathbf{F}[\hat{\boldsymbol{\xi}}]\, d^3\mathbf{r}$ are real by definition. The solution to Eq. (7) is

$$\omega = \frac{B + s\sqrt{B^2 + AC}}{A}, \tag{8}$$

where either $s = 1$ or $s = -1$ for a particular eigen-mode. Therefore, eigen-mode is unstable only if $B^2 + AC < 0$.

The dynamics described by Eq. (1) provides conservation of energy

$$E = \frac{1}{2}\int \Big(\rho|\dot{\boldsymbol{\xi}}|^2 - \boldsymbol{\xi}^*\cdot\mathbf{F}[\boldsymbol{\xi}]\Big)\, d^3\mathbf{r}, \tag{9}$$

where the displacement $\boldsymbol{\xi}$ is assumed to be complex. Substituting $\boldsymbol{\xi}$ from Eq. (5), we obtain the energy of the eigen-mode

$$E = \frac{1}{2}\,(A\,|\,\omega|^2 + C)e^{2\gamma t}, \tag{10}$$

where $\gamma = \mathrm{Im}(\omega)$. Since energy is conserved, E in Eq. (10) cannot depend on time and must be equal to zero for any unstable eigen-mode with $\gamma \neq 0$. Energy of stable eigen-mode with $\gamma = 0$ is given by

$$E = s\omega\,\sqrt{B^2 + AC}, \tag{11}$$

and can be either positive (positive energy waves, PEW) or negative (negative energy waves, NEW). The latter is realized for eigen-modes with $-B^2/A < C < 0$ and $\mathrm{sign}(B) = -s$. All NEW are non-symmetric modes, i.e., they have spatial dependence along the equilibrium flow, so $B \neq 0$. As discussed in Ref [7], there is an interval of equilibrium parameters at which non-symmetric modes with positive and negative energies can coexist. When the frequencies of such modes are coincident (resonance condition), the energy can be transferred from NEW to PEW leading to instability. In fact, such coupling of NEW and PEW is a universal mechanism of any non-symmetric instability in ideal MHD system with flow.

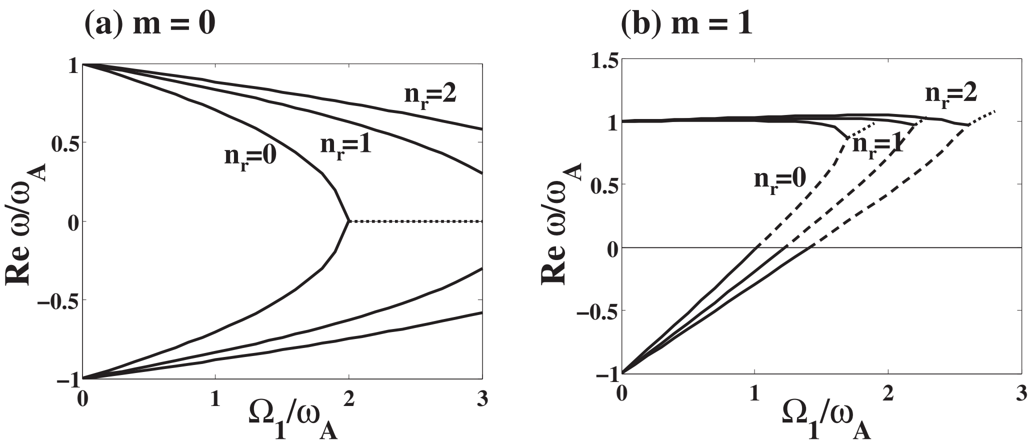

FIG. 1. Dependence of eigen-frequencies on normalized value of angular velocity Ω_1/ω_A for (a) axisymmetric modes with $m = 0$, and (b) non-axisymmetric modes with $m = 1$. Solid lines correspond to positive energy waves, dashed lines – to negative energy waves, dotted lines represent real part of frequency in unstable region. Value of n_r denotes radial wave-number, i.e., number of zeros in radial direction of corresponding eigen-function.

In order to illustrate these results, we calculate the energies and frequencies of eigen-modes of incompressible conducting fluid rotating in a uniform transverse magnetic field $\mathbf{B} = B_0\mathbf{e}_z$. The equilibrium velocity profile used in our calculations corresponds to the electrically driven flow in circular channel and has a form

$$\mathbf{V} = r\Omega(r)\mathbf{e}_\varphi, \quad \Omega(r) = \frac{\Omega_1 r_1^2}{r^2} \tag{12}$$

in cylindrical system of coordinates $\{r, \varphi, z\}$. Here, r_1 and r_2 are inner and outer radii of the channel (we take $r_2/r_1 = 5$), respectively, and Ω_1 is the angular velocity at r_1. A detailed stability analysis of such flow has been performed in Ref. [7] and [12], assuming normal modes in the form $\boldsymbol{\xi}(\mathbf{r}, t) = \boldsymbol{\xi}(\mathbf{r}) \exp(-i\omega t + im\phi + ik_z z)$.

In Fig. 1 the dependences of frequencies of axisymmetric ($m = 0$) and non-axisymmetric modes ($m = 1$) on the equilibrium parameter Ω_1/ω_A are shown. In the axisymmetric case [Fig. 1(a)], only positive energy waves can be excited in the system. The merging point of two branches in Fig. 1(a) corresponds to $\Omega_1/\omega_A \approx 2.0$ which is the threshold of magnetorotational instability for $m = 0$. The nature of axisymmetric MRI is not related to the subject of negative energy waves and can be explained by the mechanism similar to one of Rayleigh-Taylor instability [13].

For $m = 1$ modes [Fig. 1(b)], both positive and negative energy waves can coexist in the system when $\Omega_1/\omega_A > 1$. The threshold of instability in this case is $\Omega_1/\omega_A \approx 1.7$ (it corresponds to radial mode with $n_r = 0$), when frequencies of NEW and PEW are coincident, which is in agreement with the above discussion. Energy of symmetric eigen-modes (modes corresponding to static equilibrium or modes without spatial dependence along equilibrium flow) is never negative, that is why their stability is successfully investigated by use of standard energy principle. In a case of non-symmetric modes, the energy principle fails if NEW can be excited in the system, therefore modified approach should be used. Such approach is developed below.

3. Lyapunov stability criterion for plasma flows

As shown in Ref. [11], the linearized system (1) has an infinite set of exact invariants:

$$E_n = \frac{1}{2}\int \left(\rho|\boldsymbol{\xi}^{(n+1)}|^2 - \boldsymbol{\xi}^{*(n)} \cdot \mathbf{F}[\boldsymbol{\xi}^{(n)}]\right) d^3\mathbf{r}, \tag{13}$$

where $\boldsymbol{\xi}^{(n)}$ is the n-th time derivative. Generally, these integrals are independent. E_0 corresponds to the energy, integral E_1 – to the invariant similar to cross helicity [6, 9]. Higher order invariants (13) have no obvious nonlinear analogues. Using recurrence relation, which follows immediately from Eq. (1),

$$\boldsymbol{\xi}^{(n+2)} = -2(\mathbf{V}\cdot\nabla)\boldsymbol{\xi}^{(n+1)} + \frac{\mathbf{F}[\boldsymbol{\xi}^{(n)}]}{\rho}, \tag{14}$$

all integrals (13) can be expressed in terms of initial perturbations $\dot{\boldsymbol{\xi}}_0 = \dot{\boldsymbol{\xi}}|_{t=0}$ and $\boldsymbol{\xi}_0 = \boldsymbol{\xi}|_{t=0}$. In particular,

$$E_1(\dot{\boldsymbol{\xi}}_0, \boldsymbol{\xi}_0) = \frac{1}{2}\int \left(\frac{1}{\rho}\left|\mathbf{F}[\boldsymbol{\xi}_0] - 2\rho(\mathbf{V}\cdot\nabla)\dot{\boldsymbol{\xi}}_0\right|^2 - \dot{\boldsymbol{\xi}}_0^* \cdot \mathbf{F}[\dot{\boldsymbol{\xi}}_0]\right) d^3\mathbf{r}. \tag{15}$$

Following Arnold's approach [14], we incorporate integrals of motion (13) into a Lyapunov functional candidate by means of Lagrange multipliers λ_n:

$$U(\dot{\boldsymbol{\xi}}_0, \boldsymbol{\xi}_0) = \sum_{n=0}^{N} \lambda_n E_n(\dot{\boldsymbol{\xi}}_0, \boldsymbol{\xi}_0). \tag{16}$$

Theorem 1 gives sufficient condition for formal stability of system described by Eq. (1).

Theorem 1 *If there exist such real numbers λ_n and integer $N \in [0, \infty]$ that the form (16) is positively definite for all $\dot{\boldsymbol{\xi}}_0$ and $\boldsymbol{\xi}_0$, then the form (16) is a Lyapunov functional, and the equilibrium state is formally (spectrally) stable.*

Theorem 1 under certain assumptions also provides necessary condition for spectral stability, i.e., if the system is stable then there are such λ_n which make functional U non-negative for any perturbations. To demonstrate this, we prove the following theorem first.

Theorem 2 *Energy of oscillatory eigen-modes in linearized ideal MHD is additive.*

Proof. Consider a superposition of two eigen-modes with different real eigen-frequencies, $\omega_1 \neq \omega_2$,

$$\boldsymbol{\xi} = c_1\boldsymbol{\xi}_1 + c_2\boldsymbol{\xi}_2. \tag{17}$$

The total energy of this perturbation is

$$\begin{aligned} E &= \frac{1}{2}\int (\rho|\dot{\boldsymbol{\xi}}|^2 - \boldsymbol{\xi}^* \cdot \mathbf{F}[\boldsymbol{\xi}])\, d^3\mathbf{r} = |c_1|^2 E(\boldsymbol{\xi}_1) + |c_2|^2 E(\boldsymbol{\xi}_2) \\ &+ \frac{1}{2}c_1c_2^*\int (\omega_1\omega_2\rho\hat{\boldsymbol{\xi}}_1 \cdot \hat{\boldsymbol{\xi}}_2^* - \hat{\boldsymbol{\xi}}_1 \cdot \mathbf{F}[\hat{\boldsymbol{\xi}}_2^*]) + \frac{1}{2}c_1^*c_2\int (\omega_1\omega_2\rho\hat{\boldsymbol{\xi}}_1^* \cdot \hat{\boldsymbol{\xi}}_2 - \hat{\boldsymbol{\xi}}_1^* \cdot \mathbf{F}[\hat{\boldsymbol{\xi}}_2]). \end{aligned} \tag{18}$$

The last two integrals in this equation are zero. Indeed, consider eigen-value problems for eigen-modes $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$:

$$\omega_1^2\rho\hat{\boldsymbol{\xi}}_1 + 2i\omega_1\rho(\mathbf{V}\cdot\nabla)\hat{\boldsymbol{\xi}}_1 + \mathbf{F}[\hat{\boldsymbol{\xi}}_1] = 0, \tag{19}$$

$$\omega_2^2\rho\hat{\boldsymbol{\xi}}_2^* - 2i\omega_2\rho(\mathbf{V}\cdot\nabla)\hat{\boldsymbol{\xi}}_2^* + \mathbf{F}[\hat{\boldsymbol{\xi}}_2^*] = 0, \tag{20}$$

Multiplying the first equation by $\omega_2\hat{\boldsymbol{\xi}}_2^*$ and the second one by $-\omega_1\hat{\boldsymbol{\xi}}_1$, integrating them over the plasma volume and summing them up, we obtain:

$$(\omega_1 - \omega_2)\int(\omega_1\omega_2\rho\hat{\boldsymbol{\xi}}_1\cdot\hat{\boldsymbol{\xi}}_2^* - \hat{\boldsymbol{\xi}}_1\cdot\mathbf{F}[\hat{\boldsymbol{\xi}}_2^*]) = 0. \tag{21}$$

If $\omega_1 \neq \omega_2$ then integral in Eq. (21) is zero, q.e.d. A similar statement is valid for any of the integrals given by Eq. (13).

Now we consider a stable system (all eigen-frequencies ω_j are real). The general solution to Eq. (1) can be written as

$$\boldsymbol{\xi} = \sum_j c_j\boldsymbol{\xi}_j. \tag{22}$$

For any given initial conditions (i.e. $\boldsymbol{\xi}|_{t=0} = \boldsymbol{\xi}_0$, $\dot{\boldsymbol{\xi}}|_{t=0} = \dot{\boldsymbol{\xi}}_0$), coefficients c_j are uniquely specified. According to Theorem 2, integrals in the form of Eq. (13) can be expressed as

$$E_n(\boldsymbol{\xi}) = \sum_j \omega_j^{2n}|c_j|^2E(\boldsymbol{\xi}_j), \tag{23}$$

where $E(\boldsymbol{\xi}_j)$ is the energy of the j-th eigen-mode. Substituting Eq. (23) into Lyapunov functional candidate (16), we obtain:

$$U(\boldsymbol{\xi}) = \sum_{n=0}^{N}\lambda_n\sum_j\omega_j^{2n}|c_j|^2E(\boldsymbol{\xi}_j) = \sum_j|c_j|^2E(\boldsymbol{\xi}_j)\sum_{n=0}^{N}\omega_j^{2n}\lambda_n. \tag{24}$$

In order to make this form positively definite for all initial perturbations, we have to ensure that it is positive for every eigen-mode independently. This results in conditions

$$\lambda_0 + \omega_k^2\lambda_1 + \omega_k^4\lambda_2 + \ldots \quad > \quad 0 \quad \text{for every PEW}, \tag{25}$$

$$\lambda_0 + \omega_l^2\lambda_1 + \omega_l^4\lambda_2 + \ldots \quad < \quad 0 \quad \text{for every NEW}. \tag{26}$$

If these inequalities are satisfied simultaneously by some choice of $\{\lambda_n\}$ then Theorem 1 also gives necessary stability condition. We have to emphasize again that Eq. (21) is valid for oscillatory modes only. For unstable or decaying modes functional U given by Eq. (16) cannot be reduced to the form (24) and has no definite sign for any choice of $\{\lambda_n\}$.

It should be noted here that the form (24) cannot be made strictly positively definite if the energy of some eigen-mode is zero, i.e., $E(\boldsymbol{\xi}_j) = 0$ for some j. As follows from Eq. (11), such situation is realized either when $\omega_j = 0$ (neutral eigen-mode) or when $B_j^2 + A_jC_j = 0$ (marginal stability condition). To separate these two possibilities, one can find the value of potential energy proportional to quantity C from Eq. (7). The trial function, which minimizes U at the stability threshold, normally corresponds to $C \neq 0$ for moving medium ($B \not\equiv 0$), while neutral mode with $\omega = 0$ always provides $C = 0$. There is no such difference when $B \equiv 0$, so the energy principle can be applied [$N = 0$ in Eq. (16)].

For illustration of the developed method, we study the stability of a cold (pressure $P = 0$), constant-density non-magnetized gas rotating around gravitational center with potential $\Phi(r)$. All equilibrium quantities are assumed to depend only on the radius r in the cylindrical system of coordinates $\{r, \varphi, z\}$. The equilibrium velocity is then

$$\mathbf{V} = r\Omega(r)\mathbf{e}_\varphi, \quad r\Omega^2(r) = \frac{\partial\Phi}{\partial r}. \tag{27}$$

We look for solution to Eq. (1) in the form of a Fourier series, considering perturbations in the reference frame that rotates around z-axis with equilibrium angular velocity $\Omega(r)$,

$$\boldsymbol{\xi}(t,\mathbf{r}) = \sum_{m,k_z} \boldsymbol{\xi}_{m,k_z}(t,r)\exp\{im(\varphi - \Omega(r)t) + ik_z z\}.$$

Eq. (1) yields the equation for the dynamics of each Fourier mode (we omit subscripts)

$$\ddot{\boldsymbol{\xi}} + 2\Omega\hat{\mathbf{A}}\,\dot{\boldsymbol{\xi}} - \hat{\mathbf{B}}\,\boldsymbol{\xi} = 0, \tag{28}$$

where operators $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ are matrices:

$$\hat{\mathbf{A}} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{\mathbf{B}} = \begin{pmatrix} -r\partial(\Omega^2)/\partial r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{29}$$

The stability condition for Eq. (28) is easily established by the spectral method. Taking $\boldsymbol{\xi} \sim \exp(i\omega t)$, we arrive at the well-known Rayleigh criterion (necessary and sufficient condition for spectral stability)

$$4\Omega^2 + r\frac{\partial\Omega^2}{\partial r} \geq 0. \tag{30}$$

Now we apply to Eq. (28) our variational method. Note that in this case all invariants in Eq. (13) are local, i.e., corresponding integrands are conserved for every spatial point (r,z), so the first two invariants E_0 and E_1 are

$$\begin{aligned} E_0 &= \frac{1}{2}\left(|\dot{\boldsymbol{\xi}}|^2 - \boldsymbol{\xi}^{*T}\hat{\mathbf{B}}\boldsymbol{\xi}\right) = \frac{1}{2}\left(|\dot{\xi}_r|^2 + |\dot{\xi}_\varphi|^2 + |\dot{\xi}_z|^2 + r\frac{\partial\Omega^2}{\partial r}|\xi_r|^2\right), \\ E_1 &= \frac{1}{2}\left(|\hat{\mathbf{B}}\boldsymbol{\xi} - 2\Omega\hat{\mathbf{A}}\dot{\boldsymbol{\xi}}|^2 - \dot{\boldsymbol{\xi}}^{*T}\hat{\mathbf{B}}\dot{\boldsymbol{\xi}}\right) = \frac{1}{2}\left(\left|r\frac{\partial\Omega^2}{\partial r}\,\xi_r - 2\Omega\dot{\xi}_\varphi\right|^2 + \left(4\Omega^2 + r\frac{\partial\Omega^2}{\partial r}\right)|\dot{\xi}_r|^2\right). \end{aligned} \tag{31}$$

If we choose $U = E_1$, we arrive at the spectral stability condition, which is exactly the Rayleigh criterion (30). The energy principle ($U = E_0$) gives more restrictive sufficient stability condition: $r\partial(\Omega^2)/\partial r \geq 0$. This confirms the fruitfulness of the approach.

4. Summary

We have demonstrated the physical difference between instabilities of symmetric modes (all modes in static equilibria and modes which are uniform along the equilibrium flows) and non-symmetric modes. Our results show that coupling of waves with positive and negative energy is a universal mechanism for any non-symmetric MHD instabilities of flowing plasma.

Energy of symmetric eigen-modes is never negative, so the energy principle can be successfully applied to study their stability. To investigate the stability of flowing plasma with respect to non-symmetric modes, we developed a variational method (Theorem 1) based on inclusion of a new set of invariants into Lyapunov functional. Under certain assumptions this method can provide both sufficient and necessary conditions for stability.

The method is verified for a simple analytical example; the obtained stability condition is shown to be both necessary and sufficient. The relative simplicity of the analysis in considered example is due to the simple form of dynamic operators, which are represented as finite dimensional matrices. In more general case, to find the adequate stability criterion other integrals from the set (13) can be included into analysis.

This work is supported in part by the Russian Foundation for Basic Research (project 07-02-00441), the National Science Foundation, and the Natural Sciences and Engineering Research Council of Canada.

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