

Magnetohydrodynamic Equilibrium with Flow in Toroidal Plasmas and its Relevance to Internal Transport Barriers

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Abstract. The appearance of transport barriers in plasmas is usually taken to be the result of bifurcations in the transport equations, in which the possible control parameters is the radial electric field, in association with shear flow, which may contribute to the reduction or suppression of turbulent fluctuations. Yet, most calculations start from the assumption of magnetohydrodynamic (MHD) equilibria calculated from the Grad-Shafranov equation, in which no flows are taken into account. The purpose of this work is to study the effect of toroidal and poloidal flows in the quantities related to the build up of internal transport barriers. The force balance equation, gives rise to a modified Grad-Shafranov equation, which must be solved along with a Bernoulli equation. In order to understand the role of the input power, it is useful to understand the behaviour of the MHD conserved quantities as well, which are related to the topology of the fields. The electrostatic field potential remains a surface quantity, so far as resistivity is ignored. It is found that, while most of the equilibrium quantities show small variations, the most important changes are observed in the electric field.

1. Introduction

Transitions between L-modes and H-modes, as well as the appearance of internal transport barriers (ITB) are complex phenomena which require a deep understanding of the relevant transport processes. They may be interpreted as bifurcations in the transport equations, as the result of radial electric fields associated to shear flows, which may contribute to the reduction or suppression of turbulent fluctuations [1]. However, it is worth exploring up to what extent an appropriate description can be achieved from the point of view of magnetohydrodynamics (MHD). Most calculations start from magnetohydrodynamic (MHD) equilibria calculated from the Grad-Shafranov equation, in which no flows are taken into account. The derivation of the modified equilibrium equations, and the identification of the relevant surface quantities when flows are taken into account, was made by Hameiri [2]. Almaguer et al. [3] showed that such equations could be obtained from a variational principle, when a Lyapunov functional is constructed from the relevant MHD conserved quantities, allowing the study of Lyapunov stability. More recently, other authors have studied the problem, providing analytic solutions for incompressible plasmas, both in cylindrical and toroidal geometries [4-6].

The purpose of this work is to study the effect of toroidal and poloidal flows in the quantities related to the build up of internal transport barriers. In this case, the force balance equation, gives rise to a modified Grad-Shafranov equation, which must be solved along with a Bernoulli equation. Pressure is no longer a surface quantity, but additional surface quantities arise, as well as new MHD conserved quantities, which are related to the topology of the fields. The electrostatic field potential remains a surface quantity, so long as resistivity is ignored. It is found that, while most of the equilibrium quantities show small variations, the most important changes are observed in the electric field.

On the other hand, it must be recognised that the approach described above, in the framework of ideal MHD with flow, does not take into account that both input power and dissipation are

important in real systems, and that they are what determine the nature of the bifurcation in the formation of transport barriers. Ball et al. [7-9], for instance, have studied the bifurcation problem using *economical* or *minimal* models, based on reduced MHD equations with phenomenological terms, which allow an optimization approach. Bifurcations are obtained by controlling such parameters as those related to the power source, damping and shear flow driving rate. Therefore, the functionals that are conserved quantities, or constrictions, in ideal MHD, actually evolve when input power and dissipation are considered. Since the bifurcations are, not just a change in the solution of a given equation, but a change in the nature of the equilibrium equation, and it is these functionals that determine the equation, it is in their evolution that bifurcations in the equilibrium are to be sought.

2. The model equations

We start from the standard ideal MHD equations [3]

$$\rho_t = -\nabla \cdot (\rho \mathbf{v}), \quad (1)$$

$$s_t = -\mathbf{v} \cdot \nabla s, \quad (2)$$

$$\mathbf{v}_t = -(\mathbf{v} \cdot \nabla \mathbf{v}) - \rho^{-1} \nabla p + \rho^{-1} \mathbf{j} \times \mathbf{B}, \quad (3)$$

$$\mathbf{B}_t = \nabla \times (\mathbf{v} \times \mathbf{B}), \quad (4)$$

(subscripts denote partial derivatives) with

$$\nabla \cdot \mathbf{B} = 0, \quad \mathbf{j} = \nabla \times \mathbf{B}, \quad p = \rho^2 e_\rho(\rho, s), \quad T = e_s(\rho, s), \quad (5)$$

where $\rho, s, \mathbf{v}, \mathbf{B}$, and $e(\rho, s)$ are the mass density, specific entropy, plasma fluid velocity, magnetic field, and the specific internal energy respectively. The plasma pressure p and the temperature T are determined as functions of ρ and s from a prescribed equation of state relation for the specific internal energy $e = e(\rho, s)$ through the first law of thermodynamics:

$$\begin{aligned} de &= e_\rho d\rho + e_s ds \\ &= \rho^{-2} p d\rho + T ds \end{aligned} \quad (6)$$

Boundary conditions are chosen as

$$\mathbf{v} \cdot \hat{\mathbf{n}}|_{\partial D} = 0 = \mathbf{B} \cdot \hat{\mathbf{n}}|_{\partial D}. \quad (7)$$

Assuming axisymmetry $\partial/\partial \vartheta = 0$ in cylindrical coordinates (r, ϑ, z) , and proposing functions $\psi = \psi(r, z, t)$, $\phi = \phi(r, z, t)$, the magnetic and velocity fields can be represented as

$$\mathbf{B} = \nabla \psi \times \nabla \vartheta + b(\psi) \nabla \theta, \quad (8)$$

$$\mathbf{v} = \nabla \phi \times \nabla \vartheta + u \nabla \theta + \nabla \xi. \quad (9)$$

It is well known that (1) to (4) yield the conservative equation for energy

$$\partial_t (1/2 \rho v^2 + 1/2 B^2 + \rho e) = -\nabla \cdot [(v^2/2 + h + B^2/\rho) \rho \mathbf{v} - \mathbf{B} \cdot (\mathbf{v} \times \mathbf{B})], \quad (10)$$

so taking into account the boundary conditions (7),

$$H = \int_D (1/2\rho v^2 + 1/2B^2 + \rho e) d^3x \quad (11)$$

is a conserved quantity. It has furthermore been shown that the following quantities are also conserved:

$$C_0 = \int_D \rho F_0(\psi) d^3x, \quad (12)$$

associated with the conservation of mass,

$$C_1 = \int_D (\mathbf{v} \cdot \mathbf{B}) F_1(\psi) d^3x, \quad (13)$$

which represents the conservation of cross-helicity,

$$C_2 = \int_D r^{-2} b F_2(\psi) d^3x, \quad (14)$$

related to the conservation of magnetic-helicity, and

$$C_3 = \int_D \rho u F_3(\psi) d^3x, \quad (15)$$

representing the conservation of angular momentum. $F_0, F_1, F_2,$ and F_3 are in principle arbitrary constants of ψ .

The equilibrium equations can thus be found upon minimising the functional

$$H_c = H + (C_0 + C_1 + C_2 + C_3), \quad (16)$$

yielding the following set, which generalises to the Grad-Shafranov equation:

$$\mathbf{v} = -\rho^{-1} F_1(\psi) \mathbf{B} - r F_3(\psi) \hat{\theta}, \quad (17)$$

$$\left[F_1^2(\psi) / 2\rho^2 \right] \mathbf{B} - r F_3(\psi) + h(\rho, s) = F_0(\psi), \quad (18)$$

$$[1 - F_1^2(\psi) / \rho] b = r^2 F_1(\psi) F_3(\psi) - F_2(\psi), \quad (19)$$

$$\begin{aligned} \nabla \cdot \left\{ [1 - F_1^2(\psi) / \rho] r^{-2} \nabla \psi \right\} \\ = \mathbf{v} \cdot \mathbf{B} F_1'(\psi) + b r^{-2} F_2'(\psi) + \rho u F_3'(\psi) + \rho F_0'(\psi). \end{aligned} \quad (20)$$

This set of equations is consistent with the generalization of the Grad-Shafranov equation done by Hameiri Ref [2], and has been used in the past to study the Lyapunov stability of an axisymmetric toroidal plasma, by performing the second variation of functional (16)[3]. It is akin to that found independently by Siminitzis et al. [4]. The square of the Alfvén Mach number can be defined as $M^2 = F_1^2(\psi) / \rho$.

It can be easily seen that in the absence of flow, (17) vanishes, while (18) and (19) yield $F_0(\psi) = h(\rho, s)$, and $F_2(\psi) = b$. Substituting these simplifications in (20) the usual Grad-Shafranov equation is recovered.

3. Criticality and Bifurcations

Once the appropriate equations for the plasma equilibrium with flow have been established, a strategy can be developed to understand the nature of changes in their solutions, and relate them to the build-up of transport barriers. In order to do so, it is important to clarify a few concepts, which can be illustrated in a very elementary way [10].

Let us first consider the following algebraic cubic equation:

$$G(y) = y^3 - (\lambda - \lambda_c)y = 0, \quad (21)$$

where λ_c is a constant, and λ is a parameter which defines a family of equations. It is clear that upon varying λ , it is possible to find three different sets of solutions;

$$\lambda > \lambda_c; \quad y_1 = 0, \quad y_2 = \sqrt{\lambda - \lambda_c}, \quad y_3 = -\sqrt{\lambda - \lambda_c}, \quad (22)$$

$$\lambda = \lambda_c; \quad y_1 = 0, \quad y_2 = 0, \quad y_3 = 0, \quad (23)$$

$$\lambda < \lambda_c; \quad y_1 = 0, \quad y_2 = i\sqrt{\lambda - \lambda_c}, \quad y_3 = -i\sqrt{\lambda - \lambda_c}, \quad (24)$$

so that depending on the value of λ the cubic function $G(y)$ can have three real roots, a degenerate one, or only one. From the geometric point of view, the nature of the solutions changes with λ . From the first and second derivatives of $G(y)$ extremal points are found for the first case, when they are real, which disappear for the second and third case. Therefore, there is a bifurcation at the critical value λ_c , when $G''(y) = 0$ i.e.: a change from one solution branch to another. Criticality is a qualitative change in the equation properties when a parameter is varied, and can lead to a bifurcation which leads to new solutions or to loss of solutions. However, a bifurcation can only happen if different solutions exist in the first place.

3.1. Criticality and bifurcations, and differential operators

When we have a differential operator $L(\psi)$, defining an equation $L(\psi) + J(\psi) = 0$, such as the case of Grad-Shafranov equation or its generalised system with flow, it may be possible to extend the criticality ideas above by writing

$$G(\psi) = L(\psi) + J(\psi) = 0, \quad (25)$$

and defining the Gateaux derivative in terms of the function ψ

$$G_L(\mu)|_{\psi} = \lim_{\varepsilon \rightarrow 0} \frac{G(\psi + \varepsilon\mu) - G(\psi)}{\varepsilon} \equiv \mu G_{\psi}(\psi). \quad (26)$$

Following the analogy with the elementary example presented above, a sufficient condition for criticality at a solution ψ of (25), will be that in a given direction μ a nontrivial solution of

$$G_L(\mu)|_{\psi} = 0 \quad (27)$$

exists somewhere in the domain of G . In that case G is critical and ψ is a critical solution.

3.2 Bifurcations of equilibrium equations and transport barriers

It has been proposed by Solano [10] that transport barriers can be interpreted as bifurcations in the solutions of the Grad-Shafranov equation. In order to allow the existence of solutions to which the equation can bifurcate, it would be necessary that $J(\psi)$ in (25) had the right nonlinearity, such as $b(\psi)b'(\psi) = \alpha_o(\psi - \psi_c)^3$, where α_o is a constant and ψ_c is a critical solution. Although this may be a reasonable approach from the mathematical point of view, it is necessary to take into account two things: An important driving parameter for transport barriers is the external power sources, which modify the current density profile, and therefore the safety factor shear. This might be taken into account in the form of $J(\psi) = b(\psi)b'(\psi) + p'(\psi)$. However, transport barriers are associated to the existence of shear flow, for which the pressure p is no longer a surface quantity. Generalised equilibrium equations such as those derived in section 2 are likely to be more appropriate, where the flow is induced as a radial electric field is produced over a layer. While the work in Refs. [4-6] is extremely valuable, the extra bonus of identifying the relevant conserved quantities helps to control the parameters which induce bifurcations, such as those clearly determined in Refs. [7-9]; The power source which modifies the safety factor and consequently the pressure gradient, the flow source, which presumably is the radial electric field, induced by the power source, and the damping originated by the turbulence fluctuations. While the model proposed in section 2 is ideal, proper tuning of the conserved quantities can be used in order to investigate the existence of new branches of solutions, and therefore bifurcations.

Conclusions

In spite of their physical complexity, the build-up of transport barriers show the main features of bifurcations in simpler equations, which is encouraging to explore the possibility of finding them from the MHD point of view, by tuning its conserved quantities. While the Grad-Shafranov equations is not sufficient to describe the phenomena which are observed in the experiments, use of a proper generalised equilibrium set of equations such as that presented in section 2 may be more suitable. In fact, Siminitzis et al. [4] have found solutions to their generalised model in terms of a parameter, such that when it is zero the Solovév equilibrium is recovered, and when it is greater than zero separatrix solutions, such as those found in spherical tokamaks can be found, and trianguarity depends on the flow. Different solutions are found when the flow parameter is smaller than zero. Therefore, although this is not what we are searching for, it is a good example of bifurcations in generalised equilibrium equations.

Two important matters remain to be studied: (a) To find solutions in which the poloidal flow and radial electric field can be prescribed on thinner layers, such as those found experimentally in H modes. (b) Although promising analytical studies can be found in the literature which show behaviour of solutions similar to those observed in the experiments, they still need to include the effects of power input and dissipation. To understand the nature of relaxation of to equilibrium states from the variational point of view, when dissipative effects are included. Experimental devices are actually not in a real equilibrium, but in a steady state out of equilibrium. This is essential for understanding their *self-organisation* properties.

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