Toroidal Rotation In Tokamak Plasmas

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Abstract. A comprehensive transport equation for the evolution of toroidal rotation in tokamak plasmas is developed from the two-fluid momentum equations taking account of the constraints imposed by faster time scale processes. In addition to the usual collision-induced and microturbulence-induced transport processes, the plasma toroidal rotation equation includes the effects of non-axisymmetric field errors produced by external fields and MHD-type instabilities in the plasma. Non-resonant field errors produce a toroidal torque throughout the plasma that relaxes the toroidal flow to an "intrinsic" ion-temperature-gradient diamagnetic-type flow in the direction counter to the plasma current. A resonant field error causes a toroidal torque localized near its rational surface. The combination of resonant and non-resonant field errors is found to predict scalings for error field penetration and mode locking thresholds that are in closer agreement with empirical data from tokamak plasmas.

1. Introduction

Determining the magnitude, radial profile and evolution of toroidal rotation in tokamak plasmas (and ITER) is an important issue — for $\mathbf{E}\times\mathbf{B}$ flow shear control of anomalous transport, prevention of locked modes, ELM control via RMPs etc. Many effects influence the evolution of toroidal rotation in tokamak plasmas. Momentum sources and radial plasma transport due to axisymmetric neoclassical and paleo-classical as well as microturbulence-induced anomalous processes are usually considered. In addition, the toroidal rotation can be affected by magnetic field errors, which this work concentrates on. Most of these plasma transport processes can also produce momentum pinch and intrinsic rotation effects.

2. Field Errors And Their Effects On Toroidal Plasma Rotation

Small, non-axisymmetric field errors (FEs) in tokamaks arise from coil irregularities, active control coils and magnetic field distortions caused by collective plasma instabilities (e.g., NTMs, RWMs). Non-resonant field errors cause transit-time magnetic pumping (TTMP), ripple-trapping and radial drifts of bananas; they lead to non-ambipolar radial particle fluxes and toroidal flow damping over the entire plasma. Resonant field errors cause localized electromagnetic torques on rational surfaces in toroidally rotating plasmas. Toroidal flow inhibits penetration of resonant field errors into the plasma by producing a shielding effect on rational surfaces. Sufficiently large resonant FEs can lock plasma rotation at rational surfaces to the wall and lead to magnetic islands and reduced plasma confinement or disruptions. Analysis of field error effects will be facilitated by assuming the 3-D field components are first order in the small gyroradius expansion which will separate the time scales for damping of poloidal and toroidal flows.

3. Plasma And Magnetic Field Models, Perturbation Procedure

The tokamak plasma will be described by two-fluid equations including transport-level sources and a neoclassical-based viscous force $\langle \mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}\rangle$. In particular, the momentum (force balance) equation for electron and ion plasma species will be written in the form (see, for example [1])

$$mn\, d\mathbf{V}/dt = nq(\mathbf{E}+\mathbf{V}\times\mathbf{B}) - \boldsymbol{\nabla} p - \boldsymbol{\nabla}\cdot\boldsymbol{\pi} + \mathbf{R} + \mathbf{S}_m, \quad d\mathbf{V}/dt \equiv \partial/\partial t + \mathbf{V}\cdot\boldsymbol{\nabla}\mathbf{V}. \tag{1}$$

Here, $\mathbf{R}$ ($\sim n_e e\, \eta \mathbf{J}$) is the Coulomb collision frictional force density, $-\boldsymbol{\nabla}\cdot\boldsymbol{\pi}$ is the viscous force density, $\mathbf{S}_m$ is the momentum source per unit volume due for example to energetic neutral beam injection (NBI) or radio frequency (RF) waves; the other notation is standard.

The lowest order axisymmetric equilibrium magnetic field $\mathbf{B}_0$ is composed of toroidal ($\mathbf{B}_t$) and poloidal ($\mathbf{B}_p$) components. It will be represented in terms of the equilibrium poloidal magnetic flux $2\pi\psi_0(\rho)$ by

$$\mathbf{B}_0 = \mathbf{B}_t + \mathbf{B}_p \equiv I\boldsymbol{\nabla}\zeta + \boldsymbol{\nabla}\zeta\times\boldsymbol{\nabla}\psi_0 = \boldsymbol{\nabla}\psi_0\times\boldsymbol{\nabla}(q\,\theta-\zeta), \quad I(\psi_0)\equiv RB_t. \tag{2}$$

Here, $\rho \equiv \sqrt{\psi_t/\psi_t(a)}$ is a dimensionless radial coordinate based on the toroidal magnetic flux ψ_t, ζ is the toroidal (axisymmetry) angle, $q(\psi_0)\equiv \mathbf{B}\cdot\boldsymbol{\nabla}\zeta/\mathbf{B}\cdot\boldsymbol{\nabla}\theta = d\psi_t/d\psi_0$ is the inverse of the rotational transform of the magnetic field and θ is the "straight-field-line" (on a flux suface) poloidal angle. The contravariant base vectors ($\mathbf{e}^i \equiv \boldsymbol{\nabla} u^i$) for the non-orthogonal $u^i = \rho,\theta,\zeta$ coordinate system are $\mathbf{e}^\rho \equiv \boldsymbol{\nabla}\rho$, $\mathbf{e}^\theta\equiv\boldsymbol{\nabla}\theta$, $\mathbf{e}^\zeta\equiv\boldsymbol{\nabla}\zeta$. The covariant base vectors ($\mathbf{e}_i \equiv \partial\mathbf{x}/\partial u^i$) are $\mathbf{e}_\rho = \sqrt{g}\,\boldsymbol{\nabla}\theta\times\boldsymbol{\nabla}\zeta$, $\mathbf{e}_\theta = \sqrt{g}\,\boldsymbol{\nabla}\zeta\times\boldsymbol{\nabla}\rho$, $\mathbf{e}_\zeta = \sqrt{g}\,\boldsymbol{\nabla}\rho\times\boldsymbol{\nabla}\theta$, in which the Jacobian is $\sqrt{g} = 1/\boldsymbol{\nabla}\rho\cdot\boldsymbol{\nabla}\theta\times\boldsymbol{\nabla}\zeta = (d\psi_0/d\rho)/\mathbf{B}_0\cdot\boldsymbol{\nabla}\theta = (d\psi_0/d\rho)(qR^2/I)$.

Because of toroidal axisymmetry, $\mathbf{e}^\zeta \equiv \boldsymbol{\nabla}\zeta = \hat{\mathbf{e}}_\zeta/R$ and $\mathbf{e}_\zeta = R^2\boldsymbol{\nabla}\zeta = R\,\hat{\mathbf{e}}_\zeta$ in which $R(\mathbf{x})$ is the major radius to $\mathbf{x}$ and $\hat{\mathbf{e}}_\zeta$ is a unit vector in the ζ direction.

We consider mainly the hot core region of tokamak plasmas, which will be assumed to be in the banana-plateau collisionality regime. Thus, to lowest order in the small gyroradius expansion the density n, temperature T and pressure $p \equiv nT$ of both plasma species will be [2] constant on the equilibrium flux surfaces $\psi_0(\rho)$. To first order we allow for "zero-average" non-axisymmetric (3-D) perturbations (instability-induced fluctuations or due to field errors, denoted by tilde) and poloidal variations in the average (denoted by overbar) plasma parameters. Thus, we expand n, T and p, for example, as

$$p(\mathbf{x},t) = p_0(\rho) + \delta\,[\bar{p}_1(\rho,\theta) + \tilde{p}_1(\rho,\theta,\zeta,t)] + \mathcal{O}\{\delta^2\}, \tag{3}$$

in which $\delta \sim \varrho\,\nabla_\perp \sim \varrho/L_\perp \ll 1$ is the small gyroradius expansion parameter. Here $\varrho = v_{Ts}/\omega_{cs}$ is the most probable gyroradius for a species s with thermal speed $v_{Ts} \equiv \sqrt{2T_{s0}/m_s}$ and gyrofrequency $\omega_{cs} \equiv q_s B_0/m_s$. The electric potential ϕ is expanded similarly. The magnetic field will be expanded as

$$\mathbf{B} = \mathbf{B}_0(\rho,\theta) + \delta\,[\tilde{\mathbf{B}}_\perp + \tilde{\mathbf{B}}_\parallel] + \mathcal{O}\{\delta^2\}, \tag{4}$$

The perpendicular (subscript $\perp$) and parallel ($\|$) directions are defined relative to the equilibrium magnetic field direction $\mathbf{B}_0$. We will assume that the poloidal magnetic flux surfaces $\psi_0(\rho)$ are nested; thus, while $\tilde{\mathbf{B}}_\perp$ will be allowed to have resonant components, they will not be allowed to be large enough to form magnetic islands in the plasma region being considered. However, the analysis presented below will be applicable to the non-resonant regions outside magnetic islands. The magnitude of the total magnetic field is given approximately by

$$B \equiv |\mathbf{B}| = \left(B_0^2 + 2\,\delta\,B_0\tilde{B}_\parallel + \delta^2\tilde{B}_\perp^2 + \delta^2\tilde{B}_\parallel^2\right)^{1/2} \simeq B_0(\rho,\theta) + \delta\,\tilde{B}_\parallel(\rho,\theta,\zeta,t) + \mathcal{O}\{\delta^2\}. \tag{5}$$

The main effects of non-resonant magnetic field errors (induced externally or by MHD-type instabilities) will be caused by $\tilde{B}_\parallel$, which will in general be written as

$$\tilde{B}_\parallel = \sum_{mn\neq 0}\left[B_{mnc}(\rho,t)\cos(m\theta - n\zeta) + B_{mns}(\rho,t)\sin(m\theta - n\zeta)\right]. \tag{6}$$

Perpendicular gradients of instability-induced fluctuations will be assumed to scale as $1/\delta$ to reflect the short radial scale length of drift-wave-type perturbations. Thus, for example, $\boldsymbol{\nabla}_\perp\tilde{p}_1 \sim (1/\delta)\,\delta \sim \delta^0$. In contrast, parallel gradients of fluctuations will be assumed to scale with the overall tokamak plasma dimensions, and hence as δ^0; thus, $\boldsymbol{\nabla}_\parallel\tilde{p}_1 \sim \delta^0\delta \sim \delta$. Gradients of average quantities and parallel gradients of perturbations will be assumed to scale as δ^0.

4. Successsive Time Scales, Processes

The toroidal plasma rotation evolves on the long, transport time scale ($\gtrsim 0.1$ sec). Its evolution arises from effects that are formally second order in the small gyroradius expansion. To obtain a toroidal flow evolution equation on this long time scale we must take account of faster processes and the constraints they impose on the plasma behavior [3] — MHD radial force balance equilibrium from compressional Alfvén waves ($\lesssim 0.1\,\mu$sec time scale), equilibration along field lines and incompressible flows from thermalization and sound wave effects (on $\gtrsim$ μsec time scales), and poloidal flow damping from ion collisional effects (on $\gtrsim$ msec time scale).

Summing the density and momentum equations over species, to zeroth order in δ we readily obtain the magnetohydrodynamic (MHD) plasma equations $\partial\rho_m/\partial t + \boldsymbol{\nabla\cdot}\rho_m\mathbf{V} = 0$ and $\rho_m\,d\mathbf{V}/dt = \mathbf{J}\times\mathbf{B} - \boldsymbol{\nabla}P$, in which $P = p_e + p_i$ is the total plasma pressure. Neglecting electron inertia, the lowest order electron momentum equation yields Ohm's law: $\mathbf{E} + \mathbf{V}\times\mathbf{B} = (\mathbf{J}\times\mathbf{B} - \boldsymbol{\nabla}p_e)/n_e e$. Adding the non-relativistic Maxwell equations $\boldsymbol{\nabla\cdot}\mathbf{B} = 0$, $\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J}$, $\partial\mathbf{B}/\partial t = -\boldsymbol{\nabla}\times\mathbf{E}$ and an isentropic equation of state $d/dt\,\ln P/\rho_m^\Gamma = \mathcal{O}\{\delta^2\} \longrightarrow 0$, which is derivable from the species energy balance equations, completes the ideal MHD plasma description. The fastest time scale MHD processes are compressional Alfvén waves, which propagate perpendicular to magnetic field lines. On time scales longer than their natural wave periods ($\tau_A \sim a/c_A \lesssim 0.1\,\mu$sec), together with the condition $\mathbf{B}\boldsymbol{\cdot\nabla}P = 0 \Longrightarrow P_0 = P_0(\psi_0)$ (sound waves plus viscous damping thereof cause the plasma pressure to be constant along field lines), they cause tokamak plasmas to come into a radial force balance equilibrium with $\mathbf{J}_0\times\mathbf{B}_0 = \boldsymbol{\nabla}P_0 = \boldsymbol{\nabla}\psi_0\,dP_0/d\psi_0$. Since $\boldsymbol{\nabla}P_0 = \boldsymbol{\nabla}p_{e0} + \boldsymbol{\nabla}p_{i0}$, using this relation in the ideal MHD Ohm's law yields

$$0 = n_{i0}q_i(\mathbf{E}_0 + \bar{\mathbf{V}}_i\times\mathbf{B}_0) - \boldsymbol{\nabla}p_{i0}, \tag{7}$$

in which the lowest order electric field is electrostatic: $\mathbf{E}_0 \equiv -\boldsymbol{\nabla}\Phi_0(\psi_0)$. This MHD equilibrium ion force balance equation can also be obtained directly from the equilibium limit ($d/dt \longrightarrow 0$) of the ion momentum equation in (1) — by neglecting the frictional ($\mathbf{R} \sim \delta$) and viscous ($\boldsymbol{\nabla\cdot\pi} \sim \delta$) forces, and momentum sources ($\mathbf{S}_m \sim \delta^2$), which are higher order in the gyroradius expansion. Taking the "radial" ($\mathbf{e}_\rho$) projection of (7) yields

$$\Omega_t \equiv \bar{\mathbf{V}}_i \boldsymbol{\cdot\nabla}\zeta = -\left(\frac{d\Phi_0}{d\psi_0} + \frac{1}{n_{i0}q_i}\frac{dp_{i0}}{d\psi_0} - q\,\bar{\mathbf{V}}_i \boldsymbol{\cdot\nabla}\theta\right) + \mathcal{O}\{\delta^2\}. \tag{8}$$

While this equation provides a relation between average (designated by overbar) toroidal flow ($\bar{\mathbf{V}}_i \boldsymbol{\cdot\nabla}\zeta \sim \bar{V}_t/R$), the lowest order radial electric field $E_{0\rho} \equiv -(d\Phi_0/d\psi_0)(d\psi_0/d\rho)$ and the average poloidal ion flow ($\bar{\mathbf{V}}_i \boldsymbol{\cdot\nabla}\theta \sim \bar{V}_p/r$), it does not specify any of these quantities. The toroidal and poloidal ion flows are first order in the gyroradius. Thus, flows within tokamak flux surfaces are first order in the small gyroradius expansion. Average transport flows in the radial direction (i.e., $\bar{\mathbf{V}}\boldsymbol{\cdot\nabla}\psi_0$) will be second order in the gyroradius expansion.

Coulomb collisions cause the electrons to thermalize (become Maxwellian) on the electron collision time scale ($\sim 1/\nu_e \sim 10\,\mu$sec) and the ions to thermalize on the ion collision time scale ($\sim 1/\nu_i \sim$ msec). In doing so they cause the corresponding species temperatures to equilibrate along magnetic field lines over distances of order the collision length $\lambda \sim v_T/\nu \gg Rq$, which causes [2] the species density and temperature to become constant on flux surfaces on the collision time scale of the species: $n_0 = n_0(\psi_0)$, $T_0 = T_0(\psi_0)$ for $t > 1/\nu$. On this same time scale the species flow velocity becomes a defineable and physically meaningful quantity.

Since the lowest order average flows lie within a flux surface, these first order electron and ion flow velocities can be written in terms of their poloidal ($\bar{\mathbf{V}}\boldsymbol{\cdot\nabla}\theta$) and toroidal ($\bar{\mathbf{V}}\boldsymbol{\cdot\nabla}\zeta$) components:

$$\bar{\mathbf{V}}_1 \equiv \mathbf{e}_\theta \bar{\mathbf{V}}\boldsymbol{\cdot\nabla}\theta + \mathbf{e}_\zeta \bar{\mathbf{V}}\boldsymbol{\cdot\nabla}\zeta = \bar{V}_\parallel\,\mathbf{B}_0/B_0 + \bar{\mathbf{V}}_\wedge. \tag{9}$$

Alternatively, as indicated, the flows within the flux surface can be represented by their components parallel to ($\parallel$) and cross ($\wedge$, perpendicular to $\mathbf{B}_0$ but within a flux surface) the equilibrium magnetic field $\mathbf{B}_0$. The first order average and perturbed cross flows in each species are obtained by taking the cross product of $\mathbf{B}_0$ with the first order momentum equation (1):

$$\bar{\mathbf{V}}_{s\wedge} = \frac{\mathbf{B}_0\boldsymbol{\times\nabla}\psi_0}{B_0^2}\left(\frac{d\Phi_0}{d\psi_0} + \frac{1}{n_{s0}q_s}\frac{dp_{s0}}{d\psi_0}\right), \quad \tilde{\mathbf{V}}_{s\wedge} = \frac{1}{B_0^2}\,\mathbf{B}_0\boldsymbol{\times}\left(\boldsymbol{\nabla}\tilde{\phi}_1 + \frac{1}{n_{s0}q_s}\boldsymbol{\nabla}\tilde{p}_{s1}\right). \tag{10}$$

The terms in parentheses are the usual equilibrium and perturbed $\mathbf{E}\boldsymbol{\times}\mathbf{B}_0$ and diamagnetic flows.

Since the "equilibrium" density n_0 only changes on the transport time scale ($\partial/\partial \sim \delta^2$) and the radial transport flow ($\mathbf{V}\boldsymbol{\cdot\nabla}\psi_0$) and density sources are second order in δ, the lowest order average density equation reduces to $\boldsymbol{\nabla\cdot}\bar{\mathbf{V}}_1 = 0 + \mathcal{O}\{\delta^2\}$. Because of axisymmetry ($\partial/\partial\zeta \rightarrow 0$) in the equilibrium, this incompressibility condition for the first order flows reduces to [4]

$$(\mathbf{B}_0 \boldsymbol{\cdot\nabla}\theta)\,\frac{\partial}{\partial\theta}\left(\frac{\bar{\mathbf{V}}_1\boldsymbol{\cdot\nabla}\theta}{\mathbf{B}_0 \boldsymbol{\cdot\nabla}\theta}\right) = 0 \implies U_\theta(\psi_0) \equiv \frac{\bar{\mathbf{V}}_1\boldsymbol{\cdot\nabla}\theta}{\mathbf{B}_0 \boldsymbol{\cdot\nabla}\theta} = \frac{\bar{V}_\parallel}{B_0} + \frac{I}{B_0^2}\left(\frac{d\Phi_0}{d\psi_0} + \frac{1}{n_{s0}q_s}\frac{dp_{s0}}{d\psi_0}\right). \tag{11}$$

Similar considerations and orderings of the energy and heat flux fluid moment equations yield [3, 4] analogous formulas for the first order equilibrium heat flows within a tokamak flux surface:

$$Q_\theta(\psi_0) \equiv \frac{\bar{\mathbf{q}}_1\boldsymbol{\cdot\nabla}\theta}{\mathbf{B}_0 \boldsymbol{\cdot\nabla}\theta} = \frac{\bar{q}_\parallel}{B_0} + \frac{\bar{\mathbf{q}}_\wedge\boldsymbol{\cdot\nabla}\theta}{\mathbf{B}_0 \boldsymbol{\cdot\nabla}\theta}, \quad \frac{\bar{\mathbf{q}}_\wedge\boldsymbol{\cdot\nabla}\theta}{\mathbf{B}_0 \boldsymbol{\cdot\nabla}\theta} = \frac{5}{2}\,\frac{n_{s0}T_{s0}\,I}{q_s B_0^2}\,\frac{d\,T_{s0}}{d\psi_0}. \tag{12}$$

The flux-surface-average of the parallel ($\mathbf{B}_0\boldsymbol{\cdot}$) component of the lowest order ion heat flux equation [4] yields $\langle \mathbf{B}_0\boldsymbol{\cdot}\mathbf{R}_{\mathbf{q}_i}\rangle = 0 + \mathcal{O}\{\delta^2\}$ in which $\mathbf{R}_{\mathbf{q}_i} \propto -\nu_i \mathbf{q}_i$ is the ion heat friction force. Thus, we must have $\langle q_{i\parallel} B_0\rangle = 0$. Solving (12) for $q_{i\parallel}$ and substituting it in this relation yields [4]

$$Q_{i\theta} = \frac{1}{\langle B_0^2\rangle}\left\langle B_0^2\,\frac{\mathbf{q}_{i\wedge}\boldsymbol{\cdot\nabla}\theta}{\mathbf{B}_0 \boldsymbol{\cdot\nabla}\theta}\right\rangle = \frac{5}{2}\,\frac{n_{i0}T_{i0}\,I}{q_i\langle B_0^2\rangle}\,\frac{d\,T_{i0}}{d\psi_0}. \tag{13}$$

Parallel force balance in the plasma is obtained from the flux-surface-average of the parallel ($\mathbf{B}_0\boldsymbol{\cdot}$) component of the sum of the electron and ion momentum equations (1), neglecting $\mathbf{S}_m \sim \delta^2$:

$$\rho_m \frac{\partial\langle \mathbf{B}_0 \boldsymbol{\cdot}\bar{\mathbf{V}}_i\rangle}{\partial t} = -\sum_s \langle \mathbf{B}_0\boldsymbol{\cdot\nabla\cdot}\boldsymbol{\pi}_{s\parallel}\rangle \simeq -\langle \mathbf{B}_0\boldsymbol{\cdot\nabla\cdot}\boldsymbol{\pi}_{i\parallel}\rangle. \tag{14}$$

The last form results from the electron viscous force being a factor of $\sqrt{m_e/m_i} \sim 1/60$ smaller than that for the ions. The flux-surface-average of the equilibrium parallel neoclassical viscous force is [4]

$$\langle \mathbf{B}_0 \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_{i\|} \rangle \simeq m_i n_i \left[\mu_{00} U_{i\theta} + \mu_{01} \frac{-2}{5 n_i T_i} Q_{i\theta} \right] \langle B_0^2 \rangle. \tag{15}$$

Here, $\mu_{00}, \mu_{01} \sim \sqrt{\epsilon}\, \nu_i$ are damping frequencies for the poloidal ion flow. Thus, Eq. (14) is an evolution equation for the parallel ion flow $V_{i\|}$ or the poloidal ion flow function $U_{i\theta}$. The parallel and poloidal ion flows come into equilibrium [5] on the ion collision time scale $t > 1/\nu_i \sim$ msec. Then, the equilibrium poloidal ion flow is obtained by setting the parallel viscous force in (15) to zero:

$$U_{i\theta} \simeq - \frac{\mu_{01}}{\mu_{00}} \frac{-2}{5 n_i T_i} Q_{i\theta} = c_p \frac{I}{q_i \langle B_0^2 \rangle} \frac{d T_{i0}}{d\psi_0}. \tag{16}$$

The "poloidal" coefficient $c_p \equiv \mu_{01}/\mu_{00}$ is 1.17 in the banana collisionality regime ($\nu_{*i} \equiv \nu_i R q / \epsilon^{3/2} v_{Ti} \ll 1$). However, it depends on the ion collisionality regime; also, with impurities it depends on gradients of the impurity density and temperature. It is often evaluated numerically using the NCLASS code [6].

Physically, the ion parallel viscous force $\langle \mathbf{B}_0 \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_{i\|} \rangle$ is caused by collisional trapped particle drag on the untrapped particles that carry the parallel (poloidal) flow. This parallel ion viscous force damps the poloidal flow to an ion-temperature-gradient-driven diamagnetic-type flow because hotter ions are more collisionless than bulk ions and hotter regions at smaller radii damp less than colder ones at larger radii.

Analysis of the effects of viscous forces is simplified in tokamaks and quasi-symmetric stellarators compared to that in general stellarators because the variations in the magnetic field strength B occur predominantly in a single direction (poloidal in tokamaks). Thus, in tokamak plasmas the poloidal flow damping occurs on a much faster time scale ($\sim 1/|\bar{\mathbf{V}}_1 \cdot \boldsymbol{\nabla} \theta| \sim 1/\nu_i \sim 1/\delta$) than the viscous damping of the toroidal flow, which is demoted to the much longer transport time scale ($\sim 1/\delta^2$) — because $\tilde{B}_\| / B_0 \sim \delta \ll 1$. Also, since ions cause the predominant non-ambipolar particle fluxes in tokamaks, we will be concentrating on what in stellarators is usually referred to as the "ion root" of particle transport.

Having determined the poloidal ion flow $\bar{\mathbf{V}}_i \cdot \boldsymbol{\nabla} \theta = U_{i\theta} (\mathbf{B}_0 \cdot \boldsymbol{\nabla} \theta) = U_{i\theta} (I / q R^2)$, we substitute it into the toroidal flow relation (8) to obtain the more specific toroidal flow relation (for $t > 1/\nu_i \gtrsim 1$ msec)

$$\Omega_t(\rho, \theta, t) = - \left(\frac{d\phi_0}{d\psi_0} + \frac{1}{n_{i0} q_i} \frac{d p_{i0}}{d\psi_0} - \frac{c_p I^2}{q_i R^2 \langle B_0^2 \rangle} \frac{d T_{i0}}{d\psi_0} \right). \tag{17}$$

While this equation provides a relation between the toroidal rotation frequency $\Omega_t \equiv \bar{\mathbf{V}} \cdot \boldsymbol{\nabla} \zeta \simeq \bar{V}_t / R$ and the lowest order radial electric field $E_{0\rho} \equiv -(d\Phi_0 / d\psi_0)(d\psi_0 / d\rho)$, it does not specify either of these quantities. To proceed further we need to obtain an equation for the evolution of either the radial electric field or the toroidal rotation itself. We proceed by calculating the radial particle transport fluxes and hence net radial current in the spirit of trying to calculate the radial electric field.

5. Toroidal Rotation Equation

The second order radial particle flux for each species will be obtained by taking the cross product of the momentum equation in (1) with the equilibrium magnetic field $\mathbf{B}_0$ using the small gyroradius expansions indicated in (3) and (4), and averaging over fluctuations to yield

$$n_0 q \bar{\mathbf{V}}_{2\perp} = \sum \mathbf{F}_{\rm orces} \times \mathbf{B}_0 / B_0^2. \tag{18}$$

The net force densities on each plasma species, averaged over fluctuations, to lowest order [3] are

$$\sum \mathbf{F}_{\rm orces} = n_0 q \, (-\boldsymbol{\nabla} \bar{\phi}_1 + \bar{\mathbf{E}}^A + \overline{\tilde{\mathbf{V}}_1 \times \tilde{\mathbf{B}}}) - \boldsymbol{\nabla} \bar{p}_1 - \boldsymbol{\nabla} \cdot \bar{\boldsymbol{\pi}} + \bar{\mathbf{R}} + \bar{\mathbf{S}}_m - m n_0 \frac{\partial \bar{\mathbf{V}}_1}{\partial t} - m n_0 \overline{\tilde{\mathbf{V}}_1 \cdot \boldsymbol{\nabla} \tilde{\mathbf{V}}_1}. \tag{19}$$

The flux-surface-average "radial" particle flux of each species (subscript s) induced by these forces is

$$\Gamma_\psi \equiv \langle n_0 \bar{\mathbf{V}}_{2\perp} \cdot \boldsymbol{\nabla} \psi_0 \rangle = \left\langle \frac{\mathbf{B}_0 \times \boldsymbol{\nabla} \psi_0}{q_s B_0^2} \cdot \sum \mathbf{F}_{\rm orces} \right\rangle = \frac{1}{q_s} \left\langle \left(\frac{I \, \mathbf{B}_0}{B_0^2} - \mathbf{e}_\zeta \right) \cdot \sum \mathbf{F}_{\rm orces} \right\rangle. \tag{20}$$

As indicated in the discussion following (14) above, the flux-surface-average of the parallel (to $\mathbf{B}_0$) forces vanish on the transport time scale. Thus, it is only the variations of parallel forces divided by B_0^2 within flux surfaces and the forces in the toroidal direction $\mathbf{e}_\zeta \propto \boldsymbol{\nabla} \zeta$ that contribute to the radial particle flux. Further, the radial particle fluxes can be split into ambipolar and non-ambipolar components. The ambipolar particle fluxes $\Gamma_\psi^{\rm a}$ are caused by [3]: $\mathbf{E}\times\mathbf{B}_0$ flow, classical diffusion [1] due to $\mathbf{R}$, Pfirsch-Schlüter diffusion due to the variation of the parallel Ohm's law within the flux surface [4], and neoclassical diffusion due to [4] $\langle \mathbf{B}_0\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\pi}\rangle$. Since for ambipolar flows the electron and ion flows are equal, these components yield no net radial current. The toroidal components of the rest of the forces in (19) yield non-ambipolar particle fluxes $\Gamma_\psi^{\rm na}$ due to [3]: the $\overline{\tilde{\mathbf{V}}\times\tilde{\mathbf{B}}}$ forces within resistive layers around rational surfaces, the viscous forces $-\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\pi}$, polarization flows induced by the $mn_0\,\partial\bar{\mathbf{V}}_1/\partial t$ inertial force, the Reynolds stress force $mn_0\overline{\tilde{\mathbf{V}}_1\boldsymbol{\cdot}\boldsymbol{\nabla}\tilde{\mathbf{V}}_1}$ due to micro-turbulence, and momentum sources.

When summed over species, the non-ambipolar particle fluxes can in principle produce a net radial current. However, the flux-surface-averaged charge conservation relation and Gauss' law yield $\partial\langle\rho_q\rangle/\partial t+\langle\boldsymbol{\nabla}\boldsymbol{\cdot}\,\mathbf{J}\rangle=\epsilon_0\langle\boldsymbol{\nabla}\boldsymbol{\cdot}\,\partial\mathbf{E}/\partial t\rangle+(1/V')(\partial/\partial\psi_0)(V'\langle\mathbf{J}\boldsymbol{\cdot}\boldsymbol{\nabla}\psi_0\rangle)=0$, in which $V'\equiv dV(\rho)/d\rho$ with $V(\rho)$ being the volume of the ρ flux surface. In order to obtain a steady-state radial electric field [8] we set the flux-surface-average of the net radial current (i.e., $\langle\mathbf{J}\boldsymbol{\cdot}\boldsymbol{\nabla}\psi_0\rangle$) to zero, which yields [7, 8] the transport-time-scale toroidal flow evolution equation ($\mathbf{e}_\zeta\equiv R^2\boldsymbol{\nabla}\zeta$):

$$\underbrace{m_in_{i0}\langle R^2\rangle\frac{\partial\Omega_t}{\partial t}}_{\text{inertia}}=\underbrace{\langle\mathbf{e}_\zeta\boldsymbol{\cdot}\overline{\tilde{\mathbf{J}}\times\tilde{\mathbf{B}}}\rangle}_{\text{res. FEs}}-\underbrace{\langle\mathbf{e}_\zeta\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\pi}_{\|i}\rangle}_{\text{due to }\tilde{B}_\|}-\underbrace{\langle\mathbf{e}_\zeta\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\pi}_{\perp i}\rangle}_{\text{cl, neo, paleo}}-\underbrace{\langle m_in_{i0}\mathbf{e}_\zeta\boldsymbol{\cdot}\overline{\tilde{\mathbf{V}}_i\boldsymbol{\cdot}\boldsymbol{\nabla}\tilde{\mathbf{V}}_i}\rangle}_{\text{fluctuations}}+\underbrace{\langle\mathbf{e}_\zeta\boldsymbol{\cdot}\sum_s\mathbf{S}_{ms}\rangle}_{\text{sources}}.\quad(21)$$

This equation can also be obtained directly from the flux-surface-average of the toroidal angular momentum projection (i.e., $\langle\mathbf{e}_\zeta\boldsymbol{\cdot}\,\rangle=\langle R^2\boldsymbol{\nabla}\zeta\boldsymbol{\cdot}\,\rangle$) of the momentum balance equation (1).

The first two terms on the right of (21) are discussed in the next two sections. The third term, $\langle\mathbf{e}_\zeta\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\pi}_{\perp i}\rangle$, represents the effects of perpendicular ion viscosity due to classical [1] and neoclassical [11] transport due to radial gyromotion and guiding center motion off of flux surfaces, plus paleoclassical transport processes [12] that result from transforming the drift-kinetic equation from laboratory ($\mathbf{x}$) to poloidal flux coordinates [13]. They can all be put in the form $(1/V')(\partial/\partial\rho)(V'\Pi_{i\zeta})$ with $\Pi_{i\zeta}=m_in_{i0}\langle R^2\rangle(-\chi_\zeta\,\partial\Omega_t/\partial\rho+\text{other terms})$, which indicates radial diffusion plus pinch-type and other effects on toroidal plasma rotation. However, these collisional transport effects are usually negligibly small: $\chi_\zeta\sim\nu_i\varrho_i^2(1+q^2)<0.1\,\text{m}^2/\text{s}$ for classical [1] and neoclassical [11] processes and $\chi_\zeta\sim\eta/\mu_0<1\,\text{m}^2/\text{s}$ (magnetic field diffusivity) for paleocassical processes [13]. The toroidal component of the Reynolds stress term $\langle m_in_{i0}\mathbf{e}_\zeta\boldsymbol{\cdot}\overline{\tilde{\mathbf{V}}_i\boldsymbol{\cdot}\boldsymbol{\nabla}\tilde{\mathbf{V}}_i}\rangle$ induced by plasma micro-turbulence can be put into the same form [14]. It is likely to be the dominant radial momentum transport process that balances large momentum sources $\langle\mathbf{e}_\zeta\boldsymbol{\cdot}\sum_s\mathbf{S}_{ms}\rangle$, e.g., due to NBI; its determination is an active area of research [14].

6. Neoclassical Toroidal Viscosity (NTV)

In axisymmetric neoclassical theory there is no toroidal viscous torque [4] due to parallel stresses ($\boldsymbol{\pi}_\|$) in the plasma (i.e., $\langle R^2\boldsymbol{\nabla}\zeta\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\pi}_\|\rangle=0$) — because in an axisymmetric system the magnetic field strength does not vary in the toroidal direction and hence the flow is not viscously damped in this symmetry direction. However, non-axisymmetries due to small field errors (i.e., $\tilde{B}_\|$) can cause bounce-average radial drifts of particles and non-ambipolar (superscript na) radial particle fluxes ($\Gamma_\psi^{\rm na}$). The concomitant neoclassical toroidal viscous (NTV) torque is then determined from the relation $\langle\mathbf{e}_\zeta\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\pi}_{\|i}\rangle=q_i\Gamma_\psi^{\rm na}$. The physical effects that induce non-ambipolar radial ion particle fluxes are: magnetic pumping (TTMP) via non-axisymmetry effects on untrapped ions [9, 7], radial drifts of ions trapped in toroidally localized magnetic field wells due to the ripple induced by the finite number of toroidal field coils [10], and radial drifts of the centers of banana drift orbits of trapped ions induced by the non-axisymmetry [15]. Since these effects involve different physical processes and regions of velocity space (barely passing for TTMP, deeply trapped in ripple wells for ripple-trapped, and all particles trapped in the "global" variation of B_0 along field lines for banana center drifts), their effects can just be added.

We will illustrate the generic form of these effects by discussing the determination of the ion banana-drift effects [15, 16], which are often dominant in the hot core of tokamak plasmas. Here, we adopt an equilibrium magnetic field model $\mathbf{B}_0=\boldsymbol{\nabla}\psi_0\times\boldsymbol{\nabla}\beta$ in which $\beta\equiv q\theta-\zeta$ is the cross ($\sim$ poloidal) field line label for the $\mathbf{B}_0$ in (2). In the low ion collisionality regime $\nu_{*i}\equiv\nu_iRq/\epsilon^{3/2}v_{Ti}\ll1$ the lowest order bounce-averaged drift-kinetic equation is $\dot{\beta}\,\partial f_0/\partial\beta\equiv\langle\mathbf{v}_d\boldsymbol{\cdot}\boldsymbol{\nabla}\beta\rangle_b\,\partial f_0/\partial\beta=\langle\mathcal{C}\{f_0\}\rangle_b$, which is satisfied by a Maxwellian on the equilibrium flux surface: $f_0=f_{\rm M}(\psi_0)$. Here, $\dot{\beta}\equiv\langle\mathbf{v}_d\boldsymbol{\cdot}\boldsymbol{\nabla}\beta\rangle_b\simeq\omega_E\equiv d\Phi_0/d\psi_0$ is the bounce-averaged (subscript b) cross ($\boldsymbol{\nabla}\beta$ direction here) drift frequency of the ions, which is given approximately by the $\mathbf{E}\times\mathbf{B}_0$ drift frequency. The next order drift-kinetic equation is [15, 16]

$$\dot{\beta}\,\frac{\partial f_1}{\partial\beta}-\langle\mathcal{C}\{f_1\}\rangle_b=\langle\mathbf{v}_d\boldsymbol{\cdot}\boldsymbol{\nabla}\psi_0\rangle_b\frac{\partial f_M}{\partial\psi_0},\quad\frac{\partial f_M}{\partial\psi_0}=f_M\left[\frac{1}{p_{i0}}\frac{dp_{i0}}{d\psi_0}+\frac{q_i}{T_{i0}}\frac{d\Phi_0}{d\psi_0}+\left(\frac{\mathcal{E}}{T_{i0}}-\frac{5}{2}\right)\frac{1}{T_{i0}}\frac{dT_{i0}}{d\psi_0}\right].\quad(22)$$

The bounce-average radial drifts of particles at frequency $\langle \mathbf{v}_d \cdot \boldsymbol{\nabla}\psi_0\rangle_b \propto (\tilde{B}_\parallel/B_0)\, v_{d0}$ clearly cause the distribution function distortion f_1 which will lead to the ion non-ambiplolar flux

$$\Gamma_\psi^{\rm na} \equiv \langle n_i \mathbf{V}_i \cdot \boldsymbol{\nabla}\psi_0\rangle = \langle \int d^3v\, \mathbf{v}_d \cdot \boldsymbol{\nabla}\psi_0\, f_1 \rangle \sim -D^{\rm na}\, (dn_{i0}/d\psi_0)\, \langle |\boldsymbol{\nabla}\psi_0|^2\rangle, \tag{23}$$

with non-ambipolar diffusion coefficient $D^{\rm na}$ (m^2/s). The thermodynamic drive $\partial f_M/\partial\psi_0$ can also be written in terms of the toroidal rotation frequency Ω_t using its definition in (17):

$$\frac{\partial f_M}{\partial\psi_0} = \frac{q_i f_M}{T_{i0}}[\Omega_t - \Omega(\mathcal{E},\theta)], \quad \Omega(\mathcal{E},\theta) = \left[c_p \frac{I^2}{R^2\langle B_0^2\rangle} + \left(\frac{\mathcal{E}}{T_{i0}} - \frac{5}{2}\right)\right] \frac{1}{q_{i0}} \frac{dT_{i0}}{d\psi_0}. \tag{24}$$

Thus, the thermodynamic drive $\partial f_m/\partial\psi_0$ can either be thought of as being caused by (in the usual kinetic sense) the radial gradients of the potential, pressure and temperature gradients, or as being caused by (in the fluid moment sense) a toroidal flow velocity or rotation. After being averaged over particle kinetic energy $\mathcal{E}$ and a flux surface, $\Omega_* \equiv \langle\langle\Omega\rangle\rangle_{\mathcal{E}}$ will represent an "intrinsic" toroidal rotation frequency to which the neoclassical toroidal viscosity (NTV) will try to relax the plasma to restore ambipolarity.

Since derivations [15, 16] of the neoclassical toroidal viscosity are rather complicated, we will only sketch their derivation and indicate the dominant scaling factors in their results. In such analyses it is assumed that the poloidal variation of the magnetic field B is small: $\epsilon \equiv (B_{\max} - B_{\min})/(B_{\max} + B_{\min}) \sim r/R_0 \ll 1$ — the large inverse aspect ratio expansion. Only trapped ions, which are a fraction $\sim \sqrt{\epsilon} < 1$ of the ions, are involved in banana-drift effects; their effective collision frequency is [2, 4] ν_i/ϵ. Thus, for scaling purposes in solving (22), we use $\langle \mathcal{C}\{f_1\}\rangle_b \sim -(\nu_i/\epsilon) f_1$. In the $1/\nu$ regime, which is applicable for $\nu_i/\epsilon > \langle \mathbf{v}_d \cdot \boldsymbol{\nabla}\beta\rangle_b \simeq \omega_E \equiv d\Phi_0/d\psi_0 \simeq (1/RB_p)(d\Phi_0/dr) \simeq q\,(E_r/rB_t)$, trapped particles drift radially a distance (in poloidal flux ψ_0) $\Delta_{1/\nu} \sim \langle \mathbf{v}_d \cdot \boldsymbol{\nabla}\psi_0\rangle_b/(\nu_i/\epsilon)$ in an effective collision time ϵ/ν_i. Then, the solution of (22) is $f_1^{1/\nu} \sim [\Delta_{1/\nu}/(\nu_i/\epsilon)]\partial f_M/\partial\psi_0$, the integrand of the energy integral in the non-ambipolar particle flux $\Gamma_\psi^{\rm na}$ scales as $\mathcal{E}^4 e^{-\mathcal{E}/T_{i0}}$ and we obtain $D_{1/\nu}^{\rm na} \sim \bar{D}_{1/\nu}^{\rm na} \tilde{B}_{\rm eff}^2/B_0^2$, with $\bar{D}_{1/\nu}^{\rm na} \sim \epsilon^{3/2}(v_{d0}^2/\nu_i)$. Here, $v_{d0} = T_{i0}/(q_i B_p R_0)$ is a reference radial ion drift speed and $\tilde{B}_{\rm eff}^2/B_0^2$ is an effective magnitude of the square of the field errors (summed over all B_{mnc} and B_{mns} components with various weighting factors [15, 16]) whose precise definition depends on the specific collisionality regime and radial drift process. Using the relation $\langle \mathbf{e}_\zeta \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_{i\parallel}\rangle = q_i \Gamma_\psi^{\rm na}$ yields the neoclassical toroidal viscous torque in the generic form

$$\langle \mathbf{e}_\zeta \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_{i\parallel}\rangle \simeq m_i n_{i0} \langle R^2\rangle\, \mu_{i\,t} \frac{\tilde{B}_{\rm eff}^2}{B^2}(\Omega_t - \Omega_*), \quad \Omega_* \equiv \langle\langle \Omega(\mathcal{E},\theta)\rangle\rangle_{\mathcal{E}} = \frac{c_p + c_t}{q_i} \frac{dT_i}{d\psi_0}, \tag{25}$$

in which we used $\langle I^2/(R^2\langle B_0^2\rangle)\rangle = 1 + \mathcal{O}\{\epsilon^2\} \to 1$. In general the toroidal viscosity frequency scales as $\mu_{i\,t} \sim \bar{D}^{\rm na}/\varrho_{ip}^2$ in which $\varrho_{ip} \equiv v_{Ti}/(q_i B_p/m_i)$ is the ion gyroradius in the poloidal magnetic field. In the $1/\nu$ regime the toroidal viscosity frequency scales as $\mu_{i\,t}^{1/\nu} \sim \epsilon^{3/2} q^2 \omega_{ti}^2/\nu_i$ in which $\omega_{ti} \simeq v_{Ti}/R_0 q$ is the ion transit frequency. The toroidal coefficient $c_t^{1/\nu} \simeq 2.4$; it is larger than unity because of the $\mathcal{E}^4 e^{-\mathcal{E}/T_{i0}}$ weighting in the integral over energy in the evalution of $\Gamma_\psi^{\rm na}$ in the $1/\nu$ regime.

In the ν regime ($\nu_i/\epsilon \ll \omega_E$) the radial motion is limited by the "poloidal" $\mathbf{E}\times\mathbf{B}$ drift motion of the banana centers: $\Delta_\nu \sim \langle \mathbf{v}_d \cdot \boldsymbol{\nabla}\psi_0\rangle/\omega_E$. Then, the lowest order solution of (22) is $f_{1,0} \sim -\Delta_\nu\, \partial f_m/\partial\psi_0$, which yields no radial particle flux — because the drift motion of the ion banana centers is oscillatory here. The next order solution is $f_{1,1}^\nu \sim -(\nu_i/\epsilon\,\omega_E) f_{1,0} \sim -[(\nu_i/\epsilon)/\omega_E^2]\langle \mathbf{v}_d \cdot \boldsymbol{\nabla}\psi_0\rangle_b \partial f_M/\partial\psi_0$, which leads to a diffusion coefficient $\bar{D}_\nu^{\rm na} \sim \nu_i v_{d0}^2/\epsilon^{1/2}\omega_E^2$, toroidal viscosity frequency $\mu_{i\,t}^\nu \sim (\nu_i/\epsilon^{1/2})(q\,\omega_{ti}/\omega_E)^2$ and toroidal coefficient $c_t \simeq -0.24$, which is negative because of the smaller energy weighting factor of $\mathcal{E} e^{-\mathcal{E}/T_{i0}}$ in the energy integral involved in $\Gamma_\psi^{\rm na}$ in the ν regime. It should also be mentioned that in the ν regime the usual analysis [15] yields a (logarithmically) divergent integral at the trapped-passing boundary. A boundary layer analysis that scales as $\sqrt{\nu_i}$ has recently been developed [17] to remove this singularity; the transport level gets modified slightly near the transition to the $1/\nu$ regime.

Present tokamak plasma experiments often operate with $\nu_i/\epsilon \sim \omega_E$, i.e., near the transition between the $1/\nu$ and ν collisionality regimes. Thus, it is desirable to have expressions for the NTV damping frequency $\mu_{i\,t}$ and the toroidal coefficient c_t that go smoothly from one regime to the other. We have developed [16] an energy smoothing procedure similar to that developed [18] for smoothing the transition from the banana through plateau to Pfirsch-Schlüter collisionality regime in axisymmetric neoclassical transport theory [2, 4]. The key physical point is that since the collisional processes involved in NTV are predominantly pitch angle scattering and the collision frequency varies dominantly as $1/\mathcal{E}^{3/2}$, individual ions are in the $1/\nu$ regime if $\mathcal{E} < \mathcal{E}_c \equiv T_{i0}(\nu_i/\epsilon\,\omega_E)^{2/3}$ and in the ν regime for $\mathcal{E} > \mathcal{E}_c$. Thus, in performing the energy integrals involved in determining $\Gamma_{\psi}^{\rm na}$ in (23), we use $f_1^{1/\nu}$ for $\mathcal{E} < \mathcal{E}_c$ and $f_{1,1}^{\nu}$ for $\mathcal{E} > \mathcal{E}_c$. The results are provided in [16].

In addition to the usually dominant banana-drift effects, transit-time magnetic pumping (TTMP) [9, 7] causes radial non-ambipolar particle fluxes and toroidal torques that can be put into the form given in (25). The regime of validity of the TTMP analysis [7] is $(\tilde{B}_{\|}/B_0)^{3/2} \ll \nu_i/\omega_{ti}$. The scalings of the key parameters for TTMP neoclassical toroidal viscosity (NTV) processes are: reference diffusivity $\bar{D}_{\rm TTMP}^{\rm na} \sim q\,\omega_{ti}\varrho_{ip}^2$, toroidal viscous frequency $\mu_{i\,t}^{\rm TTMP} \sim q\,\omega_{ti}$, and toroidal coefficient $c_t \simeq -0.67$

In addition, a non-ambipolar particle flux [10] is produced by radial drifts of ripple-trapped ions. The toroidal torque they cause is similar in form to (25). The standard theory for the $\Gamma_{\psi}^{\rm na}$ they cause [10] is valid for $\nu_i \ll (\tilde{B}_{\|}/B_0)^{3/2} N\omega_{ti}$ in which N is the number of discrete toroidal field coils. If the toroidal field ripple is large enough to create local magnetic field wells along the magnetic field $\mathbf{B}$ [i.e., if $\alpha \equiv \epsilon \sin\theta/(Nq\tilde{B}_{\|}/B_0) \ll 1$], the ripple-trapped ions induce a diffusivity $D_{\rm ripple}^{\rm na} \sim (v_{d0}^2/\nu_i)G(\alpha)(\tilde{B}_{\|}/B_0)^{3/2}$, and $\mu_{i\,t}^{\rm ripple} \sim (q^2\omega_{ti}^2/\nu_i)G(\alpha)(B_0/\tilde{B}_{\|})^{1/2}$, with $c_t \simeq 3.5$. The function $G(\alpha)$ takes account of the variability of ripple well depths. It has the properties [10] that it is unity at $\alpha = 0$ but steeply decreases with α: $G(1) \simeq 0.05$, $G(2) \simeq 0.005$. Pragmatically, while ripple trapping effects can be dominant near the plasma edge, they are usually negligible in the core of tokamak plasmas where typically $\alpha > 1$.

When the toroidal viscous force in (25) is used in (21), it tries to relax the plasma rotation not to the laboratory frame but rather to the intrinsic toroidal rotation frequency Ω_* — so the ion radial non-ambipolar flux vanishes. For the usual case with a radially decreasing ion temperature profile, Ω_* is negative. Thus, it represents an intrinsic rotation that is usually in the "counter" direction — opposite to the plasma current direction, as embodied in the derivative with respect to the poloidal flux ψ_0. The numerical coefficient $c_p + c_t$ in the formula for the intrinsic toroidal rotation Ω_* in (25) depends on the specific processes that are dominant and the collisionality regime. The poloidal numerical coefficient c_p represents the effect of the poloidal ion flow $\bar{\mathbf{V}}_i\cdot\boldsymbol{\nabla}\theta$; as discussed in Section 4 above, standard axisymmetric neoclassical theory gives $c_p \simeq 1.17$ in the banana collisionality regime. The toroidal coefficient c_t is positive when hotter ions drift radially more rapidly than colder ones in the presence of an ion temperature gradient; it ranges from -0.67 to about 2.4, depending on which non-axisymmetry process is involved. Thus, the intrinsic toroidal rotation numerical coefficient $c_p + c_t$ typically ranges from about 0.5 to 3.6.

Recent experiments on DIII-D in which large (but still $\sim \delta \ll 1$) static non-axisymmetric field errors were deliberately applied have confirmed that [19] "The observed magnitude, direction and radial profile of the offset rotation are consistent with neoclassical theory predictions [24]." The offset rotation referred to is the Ω_* defined in (25). However, neither the theory nor the experiment are currently precise enough to determine the magnitude of the numerical coefficient $c_p + c_t$. Also, the torque exerted by the vacuum field errors on the plasma seems a bit larger than the theoretical predictions; thus, perhaps some plasma amplification effects [20] on the field error amplitude and spectrum [21] in the core need to be taken into account to obtain detailed quantitative agreement on the torque induced by field errors.

7. Resonant Field Error Penetration

Next we consider the $\langle \mathbf{e}_\zeta\cdot\tilde{\mathbf{J}}\times\tilde{\mathbf{B}}\rangle$ term in (21). This toroidal torque vanishes for ideal MHD perturbations throughout the hot core of tokamak plasmas [22]. However, in the vicinity of a low order rational surface (e.g., $q = 2/1$) non-ideal effects (e.g., due to resistivity [22] and two-fluid diamagnetic flows [23]) can allow an externally imposed resonant field error to induce a finite parallel current $\tilde{\mathbf{J}}_{\|}$ and nonzero $\tilde{\mathbf{B}}_{\perp}$ within the thin non-ideal boundary layer around the low order rational surface. This produces a toroidal torque $\langle \mathbf{e}_\zeta\cdot\tilde{\mathbf{J}}_{\|}\times\tilde{\mathbf{B}}_{\perp}\rangle$ near a rational surface in a toroidally rotating plasma. Toroidal flow inhibits penetration of resonant field errors into the plasma by producing a shielding effect on the rational surface. Above a critical field error amplitude — termed the *penetration threshold* — the plasma rotation can no longer suppress the resonant torque, and plasma rotation at the rational surface rapidly (in a few ms) locks to the wall (laboratory frame) [22]. After locking, often a magnetic island driven by the resonant field error emerges and leads to either plasma confinement degradation or disruption.

8. Resonant Field Error Penetration With NTV

Recently, the toroidal flow damping effects of neoclassical toroidal viscosity on resonant penetration thresholds in tokamaks have been considered [24, 25]. Unlike the localized resonant field error torque, which tries to lock plasma rotation at the resonant surface to the wall, the NTV generates a global torque that attempts to rotate the plasma at the rate Ω_* which is in the "counter" (to the plasma current) direction and depends on the ion temperature gradient, as indicated in (21) and (25) above. The key element in analyzing resonant field error effects is to determine the radial structure of the toroidal flow V_t in the near vicinity of the rational surface. A model equation developed from (21) that includes radial momentum diffusion, NTV effects and the $\langle \mathbf{e}_\zeta \cdot \tilde{\mathbf{J}}_\parallel \times \tilde{\mathbf{B}}_\perp \rangle$ effect of a resonant field error is [24, 25]

$$\frac{\partial V_t}{\partial t} - \chi_\zeta \nabla^2 V_t + \mu_{i\,t} \frac{\tilde{B}^2_{\text{eff}}}{B_0^2} (V_t - V_*^{\text{nc}}) = F_{\text{em}} \, r \, \delta(r - r_{mn}) + F_0. \tag{26}$$

Here, χ_ζ is the (likely anomalous) toroidal momentum diffusivity, F_{em} represents the amplitude of the $\langle \mathbf{e}_\zeta \cdot \tilde{\mathbf{J}}_\parallel \times \tilde{\mathbf{B}}_\perp \rangle$ torque at the rational surface defined by $q(r_{mn}) = m/n$ and F_0 represents the toroidal momentum source $\langle \mathbf{e}_\zeta \cdot \sum_s \mathbf{S}_m \rangle$, e.g., due to NBI. The equilibrium limit of this equation is solved [24] for $V_t(r)$ in the vicinity of the rational surface at r_{mn} using a WKB procedure when the NTV damping effects are dominant in the bulk plasma outside the singular layer: $\Gamma_s \equiv [r_{mn}^2 \mu_{i\,t} (\tilde{B}^2_{\text{eff}}/B_0^2)/\chi_\zeta]^{1/2} \gg 1$. Adding NTV flow damping effects to a standard resonant error field penetration model [22] increases the locking threshold for $\tilde{B}^2_{mn}/B_0^2$ by a factor of Γ_s [24] — because the NTV tries to keep the plasma rotating toroidally at Ω_* outside the layer. It also predicts penetration threshold scalings for the resonant field error $\tilde{B}_{mn}$ that agree better [25] with experimental mode locking results from a wide variety of ohmic-level tokamak plasmas. A particular success of the new theory is its linear scaling with plasma density [25] (with the caveats that $\tau_E \propto n_e$ and that χ_ζ has no significant scaling with n_e); previous theories [23] had a weaker density dependence. However, detailed quantitative comparisons remain to be made.

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