Kinetic theory of Geodesic Acoustic Modes: radial structures and nonlinear excitations

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Abstract. Geodesic Acoustic Modes (GAM) are shown to constitute a continuous spectrum due to radial inhomogeneities. The existence of a singular layer causes GAM to mode convert to short-wavelength kinetic GAM (KGAM) via finite ion Larmor radii; analogous to kinetic Alfvén waves (KAW). KGAM are shown to propagate radially outward; consistent with experimental observations and numerical simulation results. The degeneracy of GAM/KGAM with Beta induced Alfvén Eigenmodes (BAE) is demonstrated and discussed. We show that energetic particle driven oscillations can be excited from the GAM continuum, similarly to the Energetic Particle Mode (EPM) case. Furthermore, it is shown that KGAM can be nonlinearly excited by drift-wave (DW) turbulence via 3-wave parametric interactions, and the resultant driven-dissipative nonlinear system exhibits typical prey-predator self-regulatory dynamics. KGAM are preferentially excited with respect to GAM because of the radial wave-number dependence of the parametric excitation process. Plasma non-uniformity effects on nonlinear KGAM excitations are discussed.

In this work, we show that Geodesic Acoustic Modes (GAM) [1] constitute a continuous spectrum due to radial inhomogeneities. The existence of a singular layer, thus, suggests GAM collisionless damping due to absorption at the GAM continuum resonance [2] and linear mode conversion to short-wavelength kinetic GAM (KGAM) via finite ion Larmor radii (FLR) and finite magnetic drift orbit widths (FOW) [3]. This result is demonstrated by derivations of the GAM/KGAM mode structure and dispersion relation in the singular layer, indicating that, typically, KGAM propagate radially outward [3]. The formal identity of the GAM/KGAM wave equation to that describing shear Alfvén wave (SAW) mode conversion to kinetic Alfvén Wave (KAW) near the SAW resonance [4], suggests a complete analogy of GAM/KGAM ⇔ SAW/KAW. Our analyses also confirm that GAM and Beta induced Alfvén Eigenmodes (BAE) [5, 6] are degenerate in the long wavelength limit, where diamagnetic effects are ignored, even when FLR and FOW corrections are accounted for. Besides the importance of its physics implications, the usefulness of this result on the BAE/GAM degeneracy is that we may straightforwardly derive the governing equations for GAM using the kinetic theory results on BAE developed earlier [7, 8, 9].

We show that dynamics of GAM/KGAM excitations should be addressed as initial value problem in the presence of a radial nonuniform source. Details of these analyses are reported in a separate work. Since GAM/KGAM are toroidally and (nearly) poloidally symmetric structures, the source term can be associated with either an anisotropic distribution of (fast) particles in velocity space [10, 11] or by nonlinear excitations due to drift-wave (DW) turbulence [3, 12, 13, 14]. In this work, we show that, while GAM/KGAM are linearly stable due to ion Landau damping, they can be excited from the GAM continuum by energetic particles (Section 2), in analogy to Energetic Particle Modes (EPM) [15], and by DW turbulence via 3-wave resonant parametric interactions (Section 3) [3]. GAM are important to turbulence transport studies, since their low frequency radial structures can scatter DW fluctuations to stable short-wavelength domain and, thereby, suppress the DW turbulence transport [12, 13]. In addition, the BAE/GAM degeneracy and the possibility of exciting structures with different characteristic scales at the same frequency by a variety of dynamic interactions have a peculiar important role in influencing cross-scale couplings of meso- and micro-scales and thereby long time-scale behaviors in burning plasmas [16, 17].

1. GAM radial structures, GAM continuous spectrum and BAE/GAM degeneracy.

The BAE/GAM degeneracy can be easily demonstrated considering the magnetic flux surface averaged quasineutrality condition for axisymmetric fluctuations (toroidal mode number $n = 0$), which reads

$$\overline{J^{-1}\partial_r\left(J|\nabla r|^2\delta J_r\right)} \simeq \partial_r\left(\overline{\delta J_r}\right) = 0 \ , \tag{1}$$

where δJ_r is the fluctuating radial current and $\overline{(...)}$ denotes magnetic flux surface averaging. Here, we have considered radially localized fluctuations in a general axisymmetric toroidal equilibrium with straight field line flux coordinates (r, θ, ξ) and Jacobian $J^{-1} = \nabla r \times \nabla\theta \cdot \nabla\xi$. Meanwhile, the equilibrium magnetic field is given by the Clebsch representation, $\mathbf{B} = \nabla(\xi - q\theta) \times \nabla\psi_p$, with $q(\psi_p) = \mathbf{B}\cdot\nabla\xi/\mathbf{B}\cdot\nabla\theta = d\psi/d\psi_p$ and $\psi(\psi_p)$ the toroidal (poloidal) magnetic flux function. Equation (1) describes GAM as well as flute-like SAW, like BAE, near the $qR_0 k_\parallel = nq - m = 0$ surface, m being the poloidal mode number and R_0 the torus major radius. At $k_\parallel = 0$ and for $\omega_*/\omega \to 0$, with ω_* the diamagnetic frequency, the GAM and BAE dynamics must be identical since, for both fluctuations, the dynamic behavior is dominated by the particle response to the radial electric field, which reflects the particle radial magnetic drifts associated with geodesic curvature [3].

Defining $\Omega \equiv (\omega/\omega_{ti})$, with $\omega_{ti} = v_{ti}/(qR_0)$ the ion transit frequency and $v_{ti} = (2T_i/m_i)^{1/2}$, and using Eqs. (12) and (14) of [7], we readily cast Eq. (1) above in the following form:

$$\partial_r\left(N_0\Lambda_0^2(\Omega)\partial_r\delta\phi\right) = 0 \ , \tag{2}$$

$$\Lambda_0^2(\Omega) = 1 + \left(q^2/\Omega\right)\left(F(\Omega) - N^2(\Omega)/D(\Omega)\right) \ . \tag{3}$$

Here, $N_0 = N_0(r)$ is the plasma density, which we assumed the same for electrons and unit charge ions, Λ_0^2 is the Λ^2 function introduced in [7, 18] evaluated at $\omega_*/\omega = 0$ and renormalized by a factor $v_A^2/(qR_0\omega)^2$, with v_A the Alfvén speed. Meanwhile, $F(\Omega) = \Omega(\Omega^2 + 3/2) + (\Omega^4 + \Omega^2 + 1/2)Z(\Omega)$, $N(\Omega) = \Omega + (1/2 + \Omega^2)Z(\Omega)$, $D(\Omega) = Z(\Omega) + (1 + 1/\tau)(1/\Omega)$, $\tau = T_e/T_i$ and $Z(\Omega) \equiv \pi^{-1/2}\int_{-\infty}^{\infty} e^{-y^2}/(y - \Omega)dy$ is the plasma dispersion function. Equation (3), based on the results of [7] and on the proof that BAE and GAM spectra are degenerate for $\omega_*/\omega \to 0$ [16, 17, 19], is valid in the $k_r\rho_i q \ll 1$ limit (k_r is the radial wave vector and $\rho_i = v_{ti}/\omega_{ci}$ the ion Larmor radius, with $\omega_{ci} = (eB)/(m_i c)$) and coincides with the corresponding expressions given by Sugama et al. [20] and by Gao et al. [21] in the $T_e/T_i = 0$ limit. From Eq. (3), it is readily verified that Λ_0 depends on $T_e(r)$, $T_i(r)$ and $q(r)$, which are all functions of the radial position. Equation (2) is similar to that describing the SAW resonance [2] and, thus, demonstrates that GAM constitutes a continuous spectrum described by $\Lambda_0^2 = 0$, giving the solution $\omega = \omega_{GAM}(r)$. Note that the crucial feature of continuous spectra is not that of a space-dependent frequency: in particular, fluctuations of the (GAM) continuous spectrum consist of singular structures, whose time asymptotic behavior is quasi-exponential [22, 23] $\propto (1/t)\exp(-i\omega_{GAM}(r)t)$ due to collisionless absorption [24]. The singular nature of the fluctuations is embedded in the corresponding value of k_r, which increases in time as [25]

$$k_r \simeq -(d\omega_{GAM}(r)/dr)t \ . \tag{4}$$

Equation (4) can be viewed as a physical manifestation of phase mixing [24] of fluctuations belonging to the GAM continuous spectrum and, as such, it is an observable phenomenon: see e.g. Fig. 4 of Ref. [26] for a visualization of this effect from numerical simulations. Experimentally, when the system is globally perturbed with a broad band frequency spectrum at some initial time, phase mixing as described by Eq. (4) should be visible as radial spreading of the fluctuations belonging to the continuous spectrum with characteristic speed scaling as the phase velocity [27]. When $k_r\rho_i q \sim 1$ additional phenomena start becoming important as discussed below. The existence of continuous spectra should not be ignored for a correct description of the relevant dynamics. In fact, even when the continuous spectrum is analyzed in kinetic theory, when FLR/FOW effects may be invoked and discretize the continuum description, it is the intrinsic time scale of the process that matters and determines the system behavior, e.g. the inverse mode growth rate (γ_L^{-1}). This behavior has been demonstrated for EPM, which can be excited from the SAW continuum discretized as a collection of Kinetic Toroidal Alfvén Eigenmodes (KTAE): above the EPM excitation threshold, the cumulative effect of weakly driven KTAE on the EPM corresponds exactly to the continuum damping expression [28], since the discrete KTAE are indistinguishable on a $\approx \gamma_L^{-1}$ time scale. In Section 2, we demonstrate that this same qualitative behavior is expected with the GAM continuum, from which energetic particles can excite a discrete mode when the drive exceeds continuum damping.

In the long wavelength limit, the GAM/KGAM dielectric function ϵ_g is readily derived using BAE/GAM degeneracy and is given in the form [3]

$$\epsilon_g = b\left[\Lambda_0^2 - b\left(3/4 + (q^2/\Omega)S_0(\Omega)\right)\right] \quad , \tag{5}$$

where $b = k_r^2\rho_i^2/2$, $S_0(\Omega)$ is the function $S(\Omega)$, defined in Eq. (B28) of [8] and Eq. (21) of [9], evaluated at $\omega_*/\omega = 0$, i.e.

$$\begin{aligned} S_0(\Omega) = \frac{q^2}{2\Omega^2}&\left[L(\Omega) - 2L\left(\frac{\Omega}{2}\right) - \frac{2N(\Omega)}{D(\Omega)}\left(H(\Omega) - 2H\left(\frac{\Omega}{2}\right)\right) + \frac{N(\Omega)^2}{D(\Omega)^2}\right. \\ &\times\left.\left(F - 2F\left(\frac{\Omega}{2}\right)\right)\right] + T(\Omega) - \frac{2N(\Omega)}{D(\Omega)}V(\Omega) + \frac{N(\Omega)^2}{D(\Omega)^2}Z(\Omega) + \frac{q^2}{\Omega^2 D(\Omega/2)} \\ &\times\left[F\left(\frac{\Omega}{2}\right) - F(\Omega) - \frac{N(\Omega)}{D(\Omega)}\left(N\left(\frac{\Omega}{2}\right) - N(\Omega)\right)\right]^2 \quad , \end{aligned} \tag{6}$$

with $L(\Omega) = \Omega^7 + (5/2)\Omega^5 + (19/4)\Omega^3 + (63/8)\Omega + (\Omega^8 + 2\Omega^6 + 3\Omega^4 + 3\Omega^2 + 3/2)\,Z(\Omega)$, $H(\Omega) = \Omega^5 + 2\Omega^3 + 3\Omega + (\Omega^6 + (3/2)\Omega^4 + (3/2)\Omega^2 + 3/4)\,Z(\Omega)$, $T(\Omega) = \Omega^3 + (5/2)\Omega + (\Omega^4 + 2\Omega^2 + (3/2))\,Z(\Omega)$ and $V(\Omega) = \Omega + (\Omega^2 + 1)\,Z(\Omega)$. In the fluid limit, $|\Omega| = |\omega/\omega_{ti}| \gg 1$, one readily derives from Eq. (5) [3] the expression given in [29] and confirms that FLR/FOW effects strengthen GAM collisionless dissipation [20].

Since nonlinear excitation favors short KGAM radial wavelengths (see Section 3), we need to relax the $k_r\rho_i q \ll 1$ assumption in Eqs. (3) to (5) and derive corresponding expressions that are valid at short wavelength. It can be shown that a very compact expression for ϵ_g can be derived for for $1/q^2 \ll k_r\rho_i \ll 1$ [30], i.e., $\epsilon_g = \mathbb{R}\mathrm{e}\epsilon_g + i\mathbb{I}\mathrm{m}\epsilon_g$ with

$$\begin{aligned} \mathbb{R}\mathrm{e}\epsilon_g \;=\; & b\left\{1 - \left(\frac{7}{4} + \tau\right)\frac{q^2}{\Omega^2} + b\frac{q^2}{\Omega^2}\left(\frac{31}{16} + \frac{9}{4}\tau + \tau^2\right) - \frac{q^2}{\Omega^4}\left(\frac{23}{8} + 2\tau + \frac{\tau^2}{2}\right)\right. \\ & \left. - b\frac{q^4}{\Omega^4}\left(\frac{747}{32} + \frac{481}{32}\tau + \frac{35}{8}\tau^2 + \frac{1}{2}\tau^3\right)\right\} \quad , \end{aligned} \tag{7}$$

$$\mathbb{I}\mathrm{m}\epsilon_g = \sqrt{2}\frac{\omega}{|\omega|}\exp\{-|\Omega|/(\sqrt{2b}q)\}\left[1+\frac{\sqrt{2b}q}{|\Omega|}+\frac{2bq^2}{\Omega^2}(1+5\tau/4+\tau^2)\right.$$
$$\left. -2b+\frac{1}{24}\left(\frac{\Omega^2}{4b^2q^4}-\frac{\sqrt{2b}|\Omega|}{b^2q^3}\right)\right] \ . \qquad (8)$$

With this expression for ϵ_g, the GAM/KGAM collisionless damping rates are in excellent agreement with recent TEMPEST numerical simulations [30, 31].

With the formal substitution $\Lambda_0^2 \to b^{-1}\epsilon_g$, with ϵ_g provided by either Eq. (5) or (7) and (8), the structure of Eq. (2) is identical to that describing SAW mode conversion to KAW near the SAW resonance [4]. Thus, GAM mode conversion to short wavelength KGAM is expected near the singular layer ($\Lambda_0^2 = 0$), with the well-known Airy function behavior for the homogeneous solution of the modified Eq. (2) [4]. Equations (5) or (7) and (8) also confirm that GAM/KGAM propagate outwards, except for sufficiently low T_e/T_i at $q < 2.6$ [3]. The Airy function behavior for the homogeneous solution of Eq. (2), modified by FLR/FOW [4] as prescribed by Eqs. (5) or (7) and (8), does not allow solving for the GAM/KGAM frequency, which remains undetermined until the non-homogeneous problem is solved in the presence of a source term [4]. Global sources that excite the system with a broad band frequency spectrum at some initial time tend to excite the GAM continuum, while a more localized source with a narrow frequency spectrum tends to excite KGAM. The time coherence of the source is therefore an important factor as well, since it introduces some characteristic time and corresponding frequency in GAM/KGAM excitations. The initial value problem of GAM/KGAM excitation in the presence of a radial nonuniform source will be addressed in a separate work. Here, we discuss GAM excitation by energetic particles (Section 2) and by DW turbulence via 3-wave resonant parametric interactions (Section 3).

2. GAM excitation by energetic particles and connection with EPM.

As in the case of the SAW continuum, from which EPM are excited when wave-particle resonant drive exceeds continuum damping, energetic particles can excite modes from the GAM continuum. Energetic particles are readily included into Eq. (1), since it coincides with the flux surface averaged vorticity equation. This, in turn, can be taken in the reduced form of Eq. (13) of Ref. [32], which is appropriate for investigating fast ion excitations of low frequency waves. Combining the flux surface averaged Eq. (13) of Ref. [32] with Eq. (2) of Ref. [3], one has

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\beta_i\Omega^2}{q^2R_0^2}r\Lambda_0^2(\Omega)\frac{\partial}{\partial r}\delta\bar{\phi}\right) = \overline{\left\langle\frac{4\pi e_E}{c^2}J_0(k_\perp\rho_E)\omega\omega_{dE}\delta K_E\right\rangle} \ . \qquad (9)$$

Here, we have separated the flux surface averaged response in the scalar potential fluctuation, $\delta\phi = \delta\bar{\phi} + \delta\tilde{\phi}$ and $\beta_i = 8\pi N_0 T_i/B_0^2$. Subscripts E indicate fast ion quantities, characterized by electric charge e_E, mass m_E, Larmor radius $\rho_E = m_E c v_\perp/(e_E B_0)$ and equilibrium distribution function F_{0E}. The Bessel function $J_0(k_\perp\rho_E)$ accounts for energetic ions FLR, $k_\perp$ is the perpendicular wave vector, ω_{dE} is the energetic particle magnetic drift frequency, $\omega_{dE} = m_E c/(e_E B_0)(\mathbf{k}\times\mathbf{b})\cdot\kappa(v_\perp^2/2+v_\parallel^2)$, $\kappa = \mathbf{b}\cdot\nabla\mathbf{b}$ is the magnetic curvature, $\mathbf{b} = \mathbf{B_0}/B_0$ and the non-adiabatic fast ion response is given by

$$\left[v_\parallel\nabla_\parallel - i\left(\omega-\omega_d\right)\right]_E \delta K_E = i\left(\frac{e}{m}\right)_E QF_{0E}J_0(k_\perp\rho_E)\left[\left(\delta\phi-\delta\tilde{\psi}\right)+\left(\frac{\omega_d}{\omega}\right)_E\delta\tilde{\psi}\right] \ . \qquad (10)$$

By definition, we have $QF_{0E} = 2\omega\partial_{v^2}F_{0E}+m_Ec/(e_EB_0)(\mathbf{k}\times\mathbf{b})\cdot\nabla F_{0E}$ and $\delta\tilde{A}_\parallel \equiv -i\left(c/\omega\right)\mathbf{b}\cdot\nabla\delta\tilde{\psi}$, having considered that the flux surface averaged parallel vector potential is negligible for

GAM. In deriving Eq. (9), terms containing $\delta\tilde{\psi}$ are negligible in the flux surface averaging. The term on the r.h.s. describes the GAM excitation by energetic particles via geodesic curvature coupling, which requires F_{0E} velocity space anisotropy since GAM are nearly $(n, m) = (0, 0)$ toroidally and poloidally symmetric modes, while we dropped the fast ion residual contribution to the polarization current as it is typically negligible [32].

When Eq. (9) is solved as a boundary value problem, global GAM structures can be found [33], that are destabilized by the fast ions [34]. Furthermore, in the small fast ion orbit limit, the r.h.s. of Eq. (9) has the form $-\partial_r\left(\beta_i/(q^2R_0^2)\Delta\Omega_E^2\partial_r\delta\bar{\phi}\right)$; thus, the local GAM continuum frequency [3] is shifted according to $\Omega^2 = \Omega_{GAM}^2 + \Delta\Omega_E^2$ [35]. This result follows the same idea of [36] for evaluating the effect of energetic ions on the accumulation point of the TAE frequency gap. Generally, however, energetic ion orbits are not small and give negligible contribution in the inertial layer region. Equation (9) is structurally the same as Eq. (23) of [37], where it was shown that bound states can be obtained with localized energetic ion radial profiles, which drive the mode where the source term is strongest. For energetic ions localized at r_0, we assume $\mathrm{r.h.s.} \propto \exp(-(r-r_0)^2/L_E^2) \simeq 1-(r-r_0)^2/L_E^2$ [37], with L_E the characteristic scale of the energetic ion radial profile. Summarizing the results of [37], localized solutions (bound states), which are affected by an exponentially small continuum damping, exist if $|L_C| \gg L_E$ with $L_C^{-1} = \partial_{r_0}\ln\Lambda_0^2(\Omega_0^2; r_0)$, where Ω_0 is the mode frequency and we have explicitly maintained the dependence of Λ_0 on r. More detailed discussions of this case and of the analogy [37] with EPM [15] will be given in a separate work. Here, we want to focus on the opposite limit, $|L_C| \ll L_E$, when coupling of the mode with the GAM continuum becomes of order unity. Since the mode structure is $\partial_r\delta\bar{\phi} \simeq C/\left(r_0\Lambda_0^2(\Omega_0^2; r_0)\right)/(1+(r-r_0)/L_C)$ in the inertial layer at $r \simeq r_0$, one can readily construct the following dispersion relation (variational principle)

$$\frac{i\pi\beta_i\Omega^2\mathrm{sgn}(\Omega)}{|r_0\partial_{r_0}\Lambda_0^2(\Omega^2; r_0)|} = \delta W_E(\Omega) = q^2(r_0)R_0^2|C|^{-2}\int r dr\delta\bar{\phi}^*\mathrm{RHS}[\mathrm{Eq.}\ (9)] \quad , \tag{11}$$

where the l.h.s. represents GAM continuum damping and δW_E plays the role of potential energy due to fast ion contributions in the ideal regions. Equation (11) is the exact counterpart of the EPM dispersion relation [15]. In fact, near marginal stability,

$$\frac{\gamma}{|\omega_0|} = (-\Omega_0\partial_{\Omega_0}\mathbb{R}\mathrm{e}\delta W_E)^{-1}\left(\mathrm{sgn}(\Omega_0)\mathbb{I}\mathrm{m}\delta W_E - \frac{\pi\beta_i\Omega_0^2}{|r_0\partial_{r_0}\Lambda_0^2(\Omega_0^2; r_0)|}\right) \quad , \tag{12}$$

expressing the mode excitation when the mode drive by fast ions exceeds continuum damping. Meanwhile, the real mode frequency is given by $\mathbb{R}\mathrm{e}\delta W_E(\Omega_0) \simeq 0$, i.e. it is determined by fast ions nonresonant response. For consistency, we must also have $|L_C| \ll L_E$, i.e. $\Lambda_0^2(\Omega^2; r_0)$ must be sufficiently small, otherwise the $|L_C| \gtrsim L_E$ limit applies. In this way, the mode frequency driven by the localized energetic ion source is not too different from the local GAM continuum frequency [3].

3. GAM/KGAM nonlinear excitation by drift-wave turbulence.

Here, we follow the approach of Ref. [38] and assume that a pump wave in DW turbulence spectrum (e.g. an ITG mode) is characterized by frequency ω_0 and wave-vector $\mathbf{k}_0$, while the corresponding scalar potential in toroidal geometry is

$$\delta\phi_0 = A_0 e^{-in_0\xi}\sum_m e^{im\theta - i\omega_0 t}\phi_0(n_0 q - m) + \mathrm{c.c.} \quad , \tag{13}$$

where A_0 is the mode amplitude and $\phi_0(n_0q-m)$ provides the radial structure of the single poloidal harmonics m for fixed toroidal mode number n_0. It has been recently demonstrated [3] that the pump DW can spontaneously decay into a zonal mode (KGAM), given by $\delta\phi_r = A_r e^{ik_\zeta r - i\omega_\zeta t} + \text{c.c.}$ and characterized by $(\omega_\zeta, \mathbf{k}_\zeta)$, and a lower-sideband DW (ITG)

$$\delta\phi_- = A_- e^{in_0\xi + ik_\zeta r - i\omega_- t} \sum_m e^{-im\theta} \phi_0^*(n_0q-m) + \text{c.c.} \quad , \tag{14}$$

with $\omega_- = \omega_\zeta - \omega_0^*$ and $\mathbf{k}_0 + \mathbf{k}_- = \mathbf{k}_\zeta = \hat{\mathbf{r}} k_\zeta$. Following Ref. [38] and averaging on the fast radial variations $\propto |\phi_0(n_0q-m)|^2$, associated with the local structures of the DW poloidal harmonics, the zonal mode evolution equation in the local radial limit becomes [3]

$$\partial_t \epsilon_g A_\zeta = -(c/2B)\alpha_i k_\theta k_\zeta k_\zeta^2 \rho_i^2 \left\langle\left\langle |\hat{\phi}_0|^2 \right\rangle\right\rangle A_0 A_- \quad , \tag{15}$$

where [38] $\alpha_i = \delta P_{\perp i0}/(eN_0\delta\phi_0) + 1$, $\delta P_{\perp i0}$ is the perpendicular ion pressure fluctuation due to the pump DW (ITG) and $\left\langle\left\langle |\hat{\phi}_0|^2 \right\rangle\right\rangle = \sum_m \int_{m-1/2}^{m+1/2} |\phi_0(n_0q-m)|^2 d(n_0q) = \int_{-\infty}^{+\infty} |\hat{\phi}_0(\eta)|^2 d\eta$, with $\hat{\phi}_0(\eta) \equiv (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \exp[i(n_0q-m)\eta]\phi_0(n_0q-m)d(n_0q-m)$. Given the pump DW (ITG) and the zonal mode (KGAM) and denoting with D_- the sideband dispersion function, the lower-sideband DW (ITG) obeys the same evolution equation discussed in Ref. [38], i.e.

$$D_- A_- = (i/\omega_0)(c/B) k_\theta k_\zeta (T_i/T_e) A_\zeta A_0^* \quad , \tag{16}$$

$$D_- \simeq (\partial D_{0r}/\partial\omega_0)\left(\Delta - \omega_\zeta - i\gamma_d\right) \quad . \tag{17}$$

Here, D_{0r} is the Hermitian part of the DW (ITG) dispersion function, γ_d is the sideband damping, $\Delta = (k_\zeta^2/2)(\partial D_{0r}/\partial\omega_0)^{-1}(\partial^2 D_{0r}/\partial k_{0r}^2) = \omega_0 - \omega_1$ and ω_1 is the solution of $D_{0r}(\omega_1, \mathbf{k}_{0\theta} \pm \mathbf{k}_\zeta) = 0$ [38]. From Eqs. (15) and (17), the frequency resonance condition for the resonant decay gives

$$\begin{cases} \omega_\zeta = \omega_{gr} + i\gamma_\zeta \\ \omega_{gr} = \Delta \end{cases} \quad , \tag{18}$$

where the real KGAM zonal mode frequency, ω_{gr}, is such that $\epsilon_{gr}(\omega_{gr}) = 0$. Given Eq. (18), we readily have $\epsilon_g \simeq (i/\omega_{gr}) k_\zeta^2 \rho_i^2 (\gamma_\zeta + \gamma_g)$, with γ_g the KGAM damping, and $D_- \simeq -i(\partial D_{0r}/\partial\omega_0) \times (\gamma_\zeta + \gamma_d)$ from Eq. (17). In this way, Eqs. (15) and (16) become, respectively,

$$\left(\gamma_\zeta + \gamma_g\right) A_\zeta = -(c/2B)\alpha_i k_\theta k_\zeta \left\langle\left\langle |\hat{\Phi}_0|^2 \right\rangle\right\rangle A_0 A_- \ , \tag{19}$$

$$\left(\gamma_\zeta + \gamma_d\right) A_- = -(c/\omega_0 B)(\partial D_{0r}/\partial\omega_0)^{-1}(T_i/T_e) k_\theta k_\zeta A_\zeta A_0^* \ . \tag{20}$$

Denoting $c_s^2 = T_e/m_i$, $\rho_s^2 = c_s^2/\omega_{ci}^2$ and

$$\gamma_{RD}^2 = \alpha_i (T_i/T_e)/(2\omega_0)(\partial D_{0r}/\partial\omega_0)^{-1}(k_\theta\rho_s k_\zeta c_s)^2 \left\langle\left\langle |\hat{\Phi}_0|^2 \right\rangle\right\rangle |eA_0/T_e|^2 \quad , \tag{21}$$

from Eqs. (19) and (20) we readily derive the excitation rate of the KGAM zonal mode, γ_ζ,

$$\left(\gamma_\zeta + \gamma_g\right)\left(\gamma_\zeta + \gamma_d\right) = \gamma_{RD}^2 \quad . \tag{22}$$

At threshold, Eq. (22) gives $\gamma_{RD,th}^2 = \gamma_g\gamma_d$, while the wavelength of the KGAM can be estimated from Eq. (18), i.e. $\omega_{gr} = \Delta \approx \omega_0(k_\zeta^2/k_{0r}^2)$, yielding $k_\zeta\rho_i \approx |\omega_{gr}/\omega_0|^{1/2} k_{0r}\rho_i$. The result of Eq. (22) should be compared with the zero-frequency ZF generation rate, Γ_ζ [38] ,

$$\left(\Gamma_\zeta + \nu_\zeta\right)\left(\Delta^2 + (\Gamma_\zeta + \gamma_d)^2\right) = \gamma_M^2(\Gamma_\zeta + \gamma_d) \quad . \tag{23}$$

Here, ν_ζ is the ZF collisional dissipation rate [39], while [38, 40]

$$\gamma_M^2 = 2\alpha_i(r/R_0)^{1/2}/(1.6q^2\omega_0)(T_i/T_e)(\partial D_{0r}/\partial\omega_0)^{-1}(k_\theta\rho_s k_\zeta c_s)^2 \left\langle\left\langle|\hat{\Phi}_0|^2\right\rangle\right\rangle |eA_0/T_e|^2 \quad . \tag{24}$$

Significantly above threshold, $\gamma_\zeta \simeq \gamma_{RD}$ and $\Gamma_\zeta \simeq \gamma_M$. Thus, both γ_ζ and Γ_ζ scale linearly with A_0 and k_ζ. That nonlinear excitation favors zonal modes at short radial wavelengths justifies our suggestion that KGAM are preferentially excited by the 3-wave parametric interactions described here. Meanwhile, $\gamma_M^2/\gamma_{RD}^2 = 4(r/R_0)^{1/2}/(1.6q^2)$; i.e. the KGAM and zero-frequency ZF generation rates have similar scalings, so that their relative importance may be ultimately determined by the threshold condition, which for KGAM reads $\gamma_{RD,th}^2 = \gamma_g\gamma_d$ and was derived above, while for zero-frequency ZF is given by $\gamma_{M,th}^2 = (\nu_\zeta/\gamma_d)(\Delta^2 + \gamma_d^2)$ [38].

Denote $A_0 e^{-i\omega_0 t} = a_0(t)e^{-i\omega_0 t}$, $A_- e^{-i\omega_- t} = a_-(t)e^{i\omega_0 t - i\omega_{gr} t}$ and $A_\zeta e^{-i\omega_\zeta t} = a_\zeta(t)e^{-i\omega_{gr} t}$. Since ω_{gr} is independent of k_ζ in the lowest order, it is obvious to expect that k_ζ will have a toroidal mode number dependence via the wave frequency and number matching conditions; thus, $a_\zeta(t)e^{ik_\zeta r} = \sum_n a_{\zeta n}(t)e^{ik_{\zeta n} r}$. Similarly, the KGAM zonal mode damping will reflect the n dependence via $k_{\zeta n}$, typically increasing with $k_{\zeta n}\rho_i$, as suggested by Eq. (8). In this way, we can rewrite the nonlinear dynamic system given by Eqs. (19) and (20) as

$$(\partial_t + \gamma_{gn})\, a_{\zeta n} = -(c/2B)\alpha_i k_{\theta n} k_{\zeta n} \left\langle\left\langle|\hat{\Phi}_0|^2\right\rangle\right\rangle a_{0n} a_{-n}, \tag{25}$$

$$(\partial_t + \gamma_{dn})\, a_{-n} = -(c/\omega_{0n}B)(T_i/T_e)(\partial D_{0r}/\partial\omega_{0n})^{-1} k_{\theta n} k_{\zeta n} a_{\zeta n} a_{0n}^*, \tag{26}$$

$$(\partial_t - \gamma_{0n})\, a_{0n} = (c/\omega_{0n}B)(T_i/T_e)(\partial D_{0r}/\partial\omega_{0n})^{-1} k_{\theta n} k_{\zeta n} a_{\zeta n} a_{-n}^*. \tag{27}$$

Here, we have written Eq. (27) following the same derivation used for Eq. (26) and γ_{0n} is the linear growth rate of the pump DW (ITG) with toroidal mode number n. In addition, we have considered all possible pump DW (ITG) toroidal mode numbers, extending Eqs. (19) and (20). This driven-dissipative system based on 3-wave couplings exhibits limit-cycle behaviors, period-doubling and route to chaos as possible indication of the existence of strange attractors [41]. The 3-wave nonlinear system is characterized by prey-predator self-regulation, where KGAM are preferentially excited with respect to GAM because of the radial wave-vector dependence of the parametric excitation process. In fact, Eqs. (25) to (27) obey the following (plasmon) energy conservation laws

$$(\partial_t - 2\gamma_{0n})\, |a_{0n}|^2 = -\,(\partial_t + 2\gamma_{dn})\, |a_{-n}|^2 \quad , \tag{28}$$

$$(\partial_t - 2\gamma_{0n})\, |a_{0n}|^2 = -(2/\omega_{0n}\alpha_i)(T_i/T_e)(\partial D_{0r}/\partial\omega_{0n})^{-1} \left\langle\left\langle|\hat{\Phi}_0|^2\right\rangle\right\rangle^{-1} (\partial_t + 2\gamma_{gn})\, |a_{\zeta n}|^2 \tag{29}$$

These general properties are consistent with recent experimental observations on HL-2A [27], which show that electric field and density fluctuation radial envelopes are modulated by GAM via an energy-conserving triad interaction; this is further confirmed by cross- and auto- bicoherence analyses for interactions between GAM and turbulent fluctuations that reflect the resonant nature of GAM-DW nonlinear coupling [27]. In nonuniform plasmas, it can be shown that KGAM growth is limited by finite DW pump width and/or k_ζ mismatch [42].

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